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# Monte Carlo Option Pricing

Cecilia Maya

## **Monte Carlo Option Pricing**

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*Lecturas de Economía*, 61 (julio-diciembre 2004), pp. 53-70

**Resumen:** *El método Monte Carlo se aplica a varios casos de valoración de opciones financieras. El método genera una buena aproximación al comparar su precisión con la de otros métodos numéricos. La estimación que produce la versión Cruda de Monte Carlo puede ser aún más exacta si se recurre a metodologías de reducción de la varianza entre las cuales se sugieren la variable antitética y la variable de control. Sin embargo, dichas metodologías requieren un esfuerzo computacional mayor por lo cual las mismas deben ser evaluadas en términos no sólo de su precisión sino también de su eficiencia.*

**Palabras clave:** *método Monte Carlo, valoración de opciones, opciones financieras, métodos numéricos. Clasificación JEL: C15, G12.*

**Abstract:** *The Monte Carlo method is applied to various cases of financial option pricing. Its performance is satisfactory in terms of accuracy when it is compared to other numerical methods. The precision of the estimates provided by Crude Monte Carlo can be improved by implementing variance reduction techniques such as antithetic variate and control variate. However, the use of these techniques implies a greater computational effort; thus, there is a trade-off between a lower variance estimator and a higher computational requirement which demands us to check not only for the accuracy of the estimator but also for its efficiency.*

**Keywords:** *Monte Carlo Method, Option Pricing, Financial Options, Numerical Methods. JEL: C15, G12.*

**Résumé:** *La méthode de Monte Carlo est appliquée à de divers cas de l'évaluation financière d'option. Son exécution est satisfaisante en termes d'exactitude quand elle est comparée à d'autres méthodes numériques. La précision des évaluations fournies par la version Brut Monte Carlo peut être plus précise si on applique des techniques de réduction de la variance telles que la variable antithétique et la variable de contrôle. Cependant, l'utilisation de ces techniques implique un plus grand effort informatique; ainsi, on doit évaluer non seulement l'exactitude de l'estimateur mais également son efficacité.*

**Mots-clés:** *Méthode De Monte Carlo, Évaluation des Options, Options Financières, Méthodes Numériques*

# Monte Carlo Option Pricing

Cecilia Maya\*

**Introduction. -I. Overview of Option Valuation Methods. - II. Valuing European Call Options Using Monte Carlo. -III. Variance Reduction Techniques. -Conclusions. -References.**

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## Introduction

The theory of option pricing continues under construction even though more than thirty years have passed since the publication of the groundbreaking work by Black, Scholes, and Merton. One of the reasons for such a great interest in this subject is the wide range of its applications, which go from financial derivatives to capital budgeting, and more recently, to corporate valuation. In the beginning, options were thought as useful instruments to hedge risk, offering an infinite upside potential and a floor for losses equivalent to the premium or cost of the option. Later on, this concept has been extended to strategic investments under the name of real options. This type of options recognize the flexibility investment decision-makers have to undertake, defer, or abandon an investment, once more information about the project is known.

The critical factor in option pricing is "the precise description of the stochastic process governing the behavior of the basic asset" (Cox and Ross, 1976, p.146). The Black-Scholes formula is exact when the underlying follows a lognormal distribution. However, in real life, that is not the case and numerical methods shall be used instead. Frequently, asset prices follow non lognormal processes such as stochastic volatility or jump-diffusion ones.

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In search of an appropriate method to value options in the aforementioned cases, firstly, I present an overview of the different methods that have been suggested by the literature on option pricing. The conclusion from this overview is that the Monte Carlo method has appealing characteristics as an option valuation method, such as flexibility to function with different distributions, even empirical distributions of the underlying variables. Additionally, it can incorporate discontinuities such as those that arise from jump processes. The next step will be showing that it is also an accurate and efficient method. In what follows next, I test the Monte Carlo method to value European options with and without dividends. Once variance reduction techniques are implemented, this method proves to be efficient and accurate to value options.

### **I. Overview of Option Valuation Methods**

Many option valuation methods have been developed through the years after Black and Scholes published their work in 1973. Most of these methods being devoted to overcome the limitations of the Black and Scholes model, mainly that it can be used to value European options only and that this model demands the underlying asset to follow a lognormal distribution. Also, its solution is exact for a non-dividend paying stock or a stock which pays a continuous dividend proportional to the stock price only. For other cases, numerical methods must be used.

Trigeorgis (1991) classifies numerical techniques for option valuation in two groups:

—The first group comprises those approximating the underlying stochastic process directly, such as the binomial method (Cox, Ross and Rubinstein, 1979), the Log-transformed binomial (Trigeorgis, 1991), and the Monte Carlo simulation by Boyle (1977).

—The second group includes those approximating the resulting partial differential equations, such as Parkinson's (1977), and the finite differences schemes used by Brennan and Schwartz (1977, 1978).

Among these methods, the binomial model is probably the most simple and widely known. As Black and Scholes, they also start with a hedged portfolio where the value of the option "can be obtained by discounting the expected maturity value of the option at the risk free rate. The distribution of the maturity value of the option can be obtained from the distribution of the terminal stock value. Thus if the distribution of the terminal stock value is known the value of the option can be obtained by integration" (Boyle, 1977).

Both methods, binomial model and Black-Scholes, are simple and exact and that is probably the reason for their popularity. Compared to them, the Monte Carlo method will prove to be simple as well, offering very good approximations to the exact value of the option once variance reduction techniques are applied and, at the same time, allowing for greater flexibility which is greatly helpful at the time of valuing options. This flexibility comes from the fact that the distribution of terminal stock prices is determined by the process generating future stock price movements, a process that can be simulated on a computer. Once the terminal prices are obtained, the value of the option can be determined as well as the standard deviation of the estimation in order to determine the accuracy of the results.

The Monte Carlo method allows us to value options even when the underlying stochastic process is not a continuous one as it is required by the Black-Scholes model or when it does not follow a discrete binomial process as required by the binomial model. Jump processes, or mixtures of continuous with jump processes, even processes where what we know is just the empirical distribution, could also be modeled using Monte Carlo.

Next, I test the Monte Carlo method to value European options with and without dividends following the methodology first described by Boyle (1977). I will

apply Monte Carlo to the same cases that have been used to test some other methods in order to compare its performance relative to these other methods.

## II. Valuing European call options using Monte Carlo

In this section I apply the Monte Carlo methodology for valuing options to both, non-dividend and dividend paying stocks. This method works in the following way:

If the price of the stock,  $S$ , follows a geometric Brownian motion, then,

$$dS/S = \alpha dt + \sigma dz \tag{1}$$

$\alpha$  is the drift rate,  $\sigma$  is the periodic standard deviation, and  $dz$  is a Wiener process. Let  $G = \ln S$ . According to Ito's lemma, the process followed by  $G$  is an arithmetic Brownian motion as follows:

and

$$\tag{2}$$

then

where  $dG$  are i.i.d. and normally distributed with mean:  $(\alpha - \sigma^2/2)dt$  and variance:  $\sigma^2 dt$

Since it is possible to build a perfectly hedged portfolio comprised i.e. of a long position in one unit of the underlying asset,  $S$ , and a short position of  $n$  call options on it, in a way that the investor will be indifferent to the event of increasing or decreasing asset prices, the hedged portfolio should offer a rate of return equal to the risk-free rate,  $r$  (Cox and Ross, 1976). Derivative assets can be valued under an assumption of risk-neutrality since preferences do not matter in this circumstance. Thus, under this assumption of risk neutrality,  $dG$  are i.i.d. and normally distributed with mean:  $(r - \sigma^2/2)dt$  and variance:  $\sigma^2 dt$ .

Based on the properties of the lognormal distribution, a risk-neutral or "equivalent martingale" distribution of asset prices can be generated in the following way:

$$dG = d \ln S \cong \ln \frac{S_{t+1}}{S_t} = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

$$dz = \varepsilon \sqrt{dt} \tag{3}$$

then

$$S_{t+1} = S_t * e^{[(r - \frac{\sigma^2}{2})dt + \sigma \varepsilon \sqrt{dt}]}$$

where  $\varepsilon$  is a normally distributed random variable with a zero mean and unit variance.

In order to apply the Monte Carlo method to the valuation of a call option on a non-dividend paying stock, firstly, a set of stock prices ( $S_{t_1}, \dots, S_T$ ) is generated through a series of simulation trials. Secondly, the expected payoff from the option is computed as the risk-neutral mean — $E^*$ — of the maxima of the underlying asset's values at the expiration of the option minus the exercise price  $X$  or zero:

$$\tag{4}$$

The risk-neutral mean is then discounted at the risk-free rate to get the estimated value of the call option,  $C_0$ :

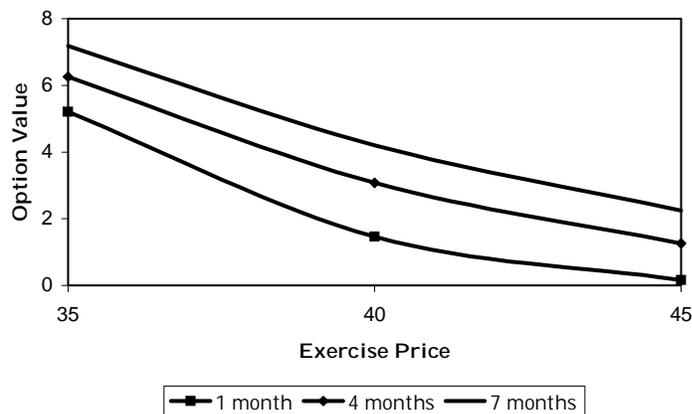
$$C_0 = e^{-rt} * E_0^* [\text{Max}(S_T - X, 0)] \tag{5}$$

The confidence interval of the estimate is  $C_0 \pm |-(2 * s / \sqrt{n})|$ , where  $s$  is the standard deviation of  $C_0$  and  $n$  is the number of paths used in the Monte Carlo simulation. Contrary to an analytical method such as Black-Scholes which gives an

exact value for the option, the Monte Carlo gives us a distribution for  $C$ . In this case of a call option on a non-dividend paying stock, there is an analytical solution, but in many other cases there is no analytical solution and a numerical method such as Monte Carlo should be used.

In order to analyze Monte Carlo's performance compared to other methods, I apply it to a call option on an asset with initial value of 40 and volatility of 30%, and different exercise prices at 35, 40, 45; the risk-free rate is 5%, and different time to expiration of 1, 4, and 7 months. By using all these parameter values I will be able to conclude how the method behaves for a wide range of cases when the option is in, at, out of the money and when it has different expiration times. Monte Carlo option values are shown in figure 1. As expected, the deeper the option is in the money, i.e.  $X = 35$ , the greater the value of the option. Also, longer maturity adds value to the option.

Figure 1. *Monte Carlo Option Value*



The parameter values used to analyze the performance of Monte Carlo are the same suggested by Geske and Sastri (1985) to compare a variety of numerical techniques for valuing options when analytic solutions do not exist. They focus both on the approximation theory and on the efficiency of the computation algorithms. I focus only on the ability of Monte Carlo to provide a good approximation since based on current development of computer hardware and software, computational efficiency is not an issue any more.

Table 1 shows the results obtained by using Monte Carlo compared to the results obtained by six other methods. These methods are Black-Scholes which provides an analytical solution, and five numerical methods which are the Binomial (Cox, Ross, and Rubinstein, 1979) and different versions of the finite differences methods by Brennan and Schwartz (1978): Finite Difference Explicit #1 (FDE1), Finite Difference Explicit #2 (FDE2),<sup>1</sup> Finite Difference Implicit #1 (FDI1), and Finite Difference Implicit #2 (FDI2).<sup>2</sup>

It can be seen how the Monte Carlo method offers a good approximation to the exact option value which is the one computed using Black-Scholes. The Monte Carlo value is, in most cases, equivalent to the mean of the values obtained using the other six methods, with the exception of the case where the option is in the money with an exercise price of 35 and one month to maturity. In every case, the benchmark value given by the Black-Scholes method is in the confidence interval. The standard errors of the estimation are in the range of a very low value of .0023 for the out-of-the-money option —  $S/X = 40/45$  with one month to maturity— up to .025 for the in-the-money options —  $S/X = 40/35$  with seven months to maturity— but still, a low standard error.<sup>3</sup> However, the accuracy of these results can be improved recurring to methods of variance reduction that will be addressed soon.

Table 1. *Numerical Methods for Option Valuation. A Comparison*

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1 Logarithmic transform of FDE1 by Brennan and Schwartz (1978).

2 Logarithmic transform of FDI1 by Brennan and Schwartz (1978).

3 There is no single rule about which is the acceptable level for the error of the estimation. I will consider a level of .05 as acceptable which is the same level required by Boyle (1977).

Table 1. *Continuation*

Exercise price \$	Solution Technique	Time to Maturity (months)		
		1	4	7
40	Montecarlo	1,4622	3,0751	4,2018
	<i>error</i>	<i>0,069</i>	<i>0,0148</i>	<i>0,0205</i>
	Conf, Interval*	[1,4486 1,4758]	[3,0461 3,1042]	[4,1616 4,2421]
	Black- Scholes	1,46	3,07	4,19
	Binomial	1,46	3,07	4,19
	FDE1	1,46	3,08	4,20
	FDE2	1,47	3,08	4,20
	FDI1	1,46	3,08	4,20
FDI2	1,46	3,08	4,20	
45	Montecarlo	0,1606	1,2627	2,2480
	<i>error</i>	<i>0,0023</i>	<i>0,0098</i>	<i>0,0156</i>
	Conf, Interval*	[0,1562 0,1651]	[1,2435 1,2819]	[2,2175 2,2785]
	Black- Scholes	0,16	1,25	2,24
	Binomial	0,16	1,25	2,24
	FDE1	0,16	1,26	2,24
	FDE2	0,17	1,26	2,25
	FDI1	0,16	1,26	2,24
FDI2	0,17	1,26	2,25	

\*95 confidence level

Next, following Boyle (1977), I compute the value of a European call option on a discrete dividend paying stock.<sup>4</sup> Since there is no analytical solution in this case, a numerical method should be used. As a benchmark for a good approximation to this option value I will use a numerical integration approach based on a trapezoidal rule by Chen (1969).<sup>5</sup> In this case, to simulate the stock price possible paths,  $S_t$  is assumed to be the price of the stock immediately after the quarterly dividend  $D_t$  has been paid. Then, if  $S_{t+1}$  is greater than  $D_{t+1}$ ,  $(S_{t+1} - D_{t+1})$  is used as the initial value at the start of the second period and the procedure is continued until  $S_T$ , the final value, is obtained. If at some point  $S_{t+m}$  ( $m = 1, 2, \dots, (T-t-1)$ ) is less than or equal to the dividend payment  $D_{t+m}$ , the process stops and another simulation trial is started. The rest is similar to the process described in the case of a non-dividend paying stock where the risk-neutral expected value of  $Max [S_T - X, 0]$  is then discounted at the risk-free interest rate to obtain the option value  $Co$ . Table 2 shows the results obtained with one hundred thousand trials and compares those values with the accurate values obtained by the benchmark method.

4 When the dividend is continuous, Black-Scholes gives an exact solution replacing the drift rate  $r$  by  $(r-\delta)$ , where  $\delta$  is the dividend yield.

5 Cited by Boyle (1977).

Table 2. *European Call Option without Dividends. Monte Carlo Valuation*

S/X	No-periods to maturity	Option values by Montecarlo 5000 trials	St.deviation of Montecarlo estimates	Option values by Montecarlo 100000 trials	St.deviation of Montecarlo estimates	Acurate option values by num, Integration
25/50	2	0,0040	0,0020	0,0031	0,0004	0,0030
	4	0,0870	0,0140	0,0757	0,0030	0,0750
	8	0,5390	0,0470	0,5436	0,0106	0,5410
	16	1,8970	0,1080	1,9552	0,0269	11,9570
50/50	2	5,1210	0,1140	5,0233	0,0248	5,0280
	4	7,4270	0,1700	7,2566	0,0374	7,2510
	8	10,7930	0,2580	10,5140	0,0571	10,4930
	16	15,0790	0,3890	15,0953	0,0903	15,0900
75/50	2	26,4450	0,2400	26,4047	0,0530	26,3690
	4	28,2500	0,3330	27,8246	0,0734	27,8190
	8	31,2570	0,4700	30,9003	0,1052	30,7130
	16	35,5030	0,6660	35,5712	0,1542	35,5580

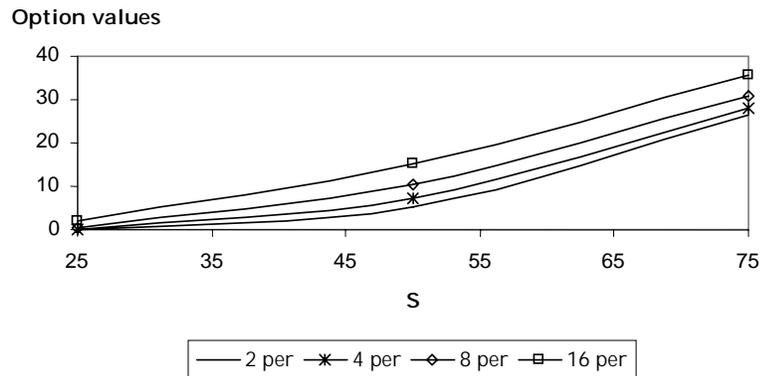
Boyle (1977) run five thousand trials and the estimates obtained then are also given to show how the accuracy of the method increases when a greater number of trials is performed. The number of trials,  $n$ , is a critical issue in applying the Monte Carlo method since the standard deviation of the estimation is inversely proportional to the squared root of  $n$ . In this case, as it is shown in table 2, although increasing the number of trials improves the accuracy notably, still the standard deviation of the estimation reaches unacceptable levels greater than .05, specially for those options that are deep in-the-money.

Figure 1 shows how the option values are higher when the option is deeper in the money or the number of periods to expiration increases. When the option is out-of-the-money, the estimation obtained by what can be called crude Monte Carlo<sup>6</sup> is close to the accurate value given by the benchmark up to two decimal places. However, as the option gets at or in-the-money or the time to maturity increases, the method becomes less precise, and confidence intervals widen. I will address this problem by resorting to different methods oriented to reduce the variance of the estimations.

### III. Variance Reduction Techniques

The precision of the estimates provided by crude Monte Carlo is in many cases below an acceptable level, therefore, variance reduction techniques shall be used

<sup>6</sup> Crude Monte Carlo may be defined as the applying this estimation technique without any variance reduction adjustment.

Figure 2. *European Call Option without dividends Monte Carlo Option Values*

to improve the results. However, the use of these techniques implies a greater computational effort; thus, there is a trade-off between a lower variance estimator and computational requirements. I will require, then, a measure of efficiency for the estimators that properly portraits this trade-off.

Suppose there are two estimators requiring computer time  $b_1$  and  $b_2$ , respectively, in a period of time  $t$ . The number of replications that can be performed is  $t/b_1$  or  $t/b_2$ . Using Monte Carlo, these two estimators will be:

$$\frac{b_1}{t} \sum_{i=1}^{t/b_1} \hat{\theta}_{1i}; \frac{b_2}{t} \sum_{i=1}^{t/b_2} \hat{\theta}_{2i}$$

For large  $t$ , these estimators are approximately distributed with mean  $q$  and standard deviations:  $\sigma_1 \sqrt{b_1/t}$  and  $\sigma_2 \sqrt{b_2/t}$  then, is the lower variance estimator if:

$$\sigma_1^2 * b_1 < \sigma_2^2 * b_2 \quad (6)$$

thus, in terms of efficiency, the lower variance estimator will be preferred to only if the variance ratio is smaller than the work ratio  $b_2/b_1$  (Boyle *et al.*, 1997). In what follows next, I attempt to improve the estimations obtained in the previous section by implementing different variance reduction techniques and compare the estimators used for this purpose with the crude Monte Carlo estimator in terms of efficiency.



variate method, the standard errors of the estimates are, in most cases, below the acceptable level of .05.

From an efficiency point of view, the work required to generate the antithetic variate estimator doubles compared to the crude Monte Carlo method. Thus, for the antithetic variate estimator to be more efficient

$$2 * \sigma_{av}^2 < \sigma_c^2 \quad (7)$$

where  $\sigma_{av}$  is the standard deviation of the antithetic variate estimator and  $\sigma_c$  is the standard deviation of the crude Monte Carlo estimator. Table 4 shows how the antithetic variate estimator is in deed more efficient, especially for deep in-the-money options for which the variance is above an acceptable level.

Table 4. Comparison of Variance Reduction Methods in Terms of Efficiency

	Variance Crude MonteCarlo	Two times Antithetic Variance	Two times control Variate Variance
S/X= 25/50			
<i>No. of periods</i>			
2	0,000000	0,000000	0,000000
4	0,000009	0,000008	0,000000
8	0,000117	0,000113	0,000007
16	0,000729	0,000707	0,000072
S/X= 50/50			
<i>No. of periods</i>			
2	0,000610	0,000359	0,000000
4	0,001376	0,000865	0,000003
8	0,003318	0,002191	0,000020
16	0,008046	0,005919	0,000104
S/X= 75/50			
<i>No. of periods</i>			
2	0,002809	0,000204	
4	0,005432	0,000959	0,000000
8	0,010899	0,003613	0,000012
16	0,023994	0,012262	0,000077

#### **A. The Control Variate Method.**

According to Hull (2000) this method is applicable when there are two similar derivatives, **A** and **B**. **A** is the derivative under consideration and **B** is another derivative that has an analytic solution available. In this case, **A** will be the

European call on a dividend paying stock and **B** will be an European call on a non-dividend paying stock which has an exact solution given by Black-Scholes. If we assume that **A** will pay dividends only after the exercise date and, thereafter, it performs exactly like a non-dividend paying stock, an investor should be indifferent between **A** and **B**—assuming same maturity and expiration date; then **B** will be a suitable control variate for **A**. After finding a proper control variate, two simulations are run using the same innovation  $\epsilon$  and the same number of time-steps.

From the first simulation I obtain an estimate, of the value of **A**; from the second simulation I estimate of the value of **B**. Compared to crude Monte Carlo, a better estimate of the value of **A**,  $F_A$ , is obtained by computing:

$$F_A = (\hat{F}_A - \hat{F}_B) + F_B \tag{8}$$

where  $F_B$  is the Black-Scholes value of the European call option on a non-dividend paying stock that is the known true value of **B**. The variance of this estimate is:

$$\text{Var}(F_A) = \text{Var}(\hat{F}_A) + \text{Var}(\hat{F}_B) - 2\text{Cov}(\hat{F}_A, \hat{F}_B) \tag{9}$$

Therefore, the effectiveness of this method will depend on a large covariance between the simulated estimates of **A** and **B**.

Table 5 shows the Monte Carlo estimates of the option value,  $F_A$ , the estimation error and the 95% confidence interval using the control variate method. Also, the accurate option values obtained by numerical integration are displayed (Boyle, 1977).

*Table 5. Control Variate Technique*

S/X	No periods to maturity	Option values			Accurate option values		
		Control Variate method	St. Dev. of estimate w/control variate	Confidence interval 95%	by numerical integration		
	2	0,0029	0,0000	0,0028	0,0030	0,003	ok
	4	0,0751	0,0004	0,0743	0,0759	0,075	ok
	8	0,5411	0,0019	0,5374	0,5448	0,541	ok
	16	1,9530	0,0060	1,9412	1,9648	1,957	ok
25/50	2	5,0280	0,0004	5,0272	5,0288	5,028	ok
	4	7,2519	0,0013	7,2494	7,2544	7,251	ok
	8	10,4884	0,0032	10,4821	10,4947	10,493	ok
	16	15,0887	0,0072	15,0746	15,1028	15,090	ok
50/50	2	26,3691	0,0002	26,3687	26,3695	26,369	ok
	4	27,8193	0,0008	27,8177	27,8209	27,819	ok
	8	30,7097	0,0024	30,7050	30,7144	30,713	ok
	16	35,5481	0,0062	35,5359	35,5603	35,558	ok
75/50	2	50,0000	0,0000	50,0000	50,0000	50,000	ok
	4	50,0000	0,0000	50,0000	50,0000	50,000	ok
	8	50,0000	0,0000	50,0000	50,0000	50,000	ok
	16	50,0000	0,0000	50,0000	50,0000	50,000	ok

The control variate estimates are completely satisfactory in terms of accuracy and efficiency. Although computer time doubles, the reduction in the variance is much greater than one half, thereby increasing efficiency (See Table 4). The accuracy of the estimation improves providing precise estimates up to three decimal places for options with maturities up to one year and two decimal points for longer maturities. The standard error of the estimates is greatly reduced, being just .007 the largest one. Finally, in all sixteen cases, the 95 percent confidence interval contains the accurate option value obtained by numerical integration.

This methodology for variance reduction can be improved in a way suggested by Boyle et al. (1997). Instead of (2,8) I may use:

$$F_A^\beta = \hat{F}_A + \beta(F_B - \hat{F}_B) \quad (10)$$

Where

$$Var(F_A^\beta) = Var(\hat{F}_A) + \beta^2 Var(\hat{F}_B) - 2\beta^* Cov(\hat{F}_A, \hat{F}_B) \quad (11)$$

The variance-minimizing  $\beta^*$  is:

$$\beta^* = Cov(\hat{F}_A, \hat{F}_B) / Var(\hat{F}_B)$$

$\beta^*$  can take a value different than one which is the value implicit in (8). A lower variance may be obtained by using (10) with  $\beta^*$  instead. By applying (10) it is possible to get a lower variance estimator, never a higher variance one.

In order to estimate  $\beta^*$ , I run a regression of  $\hat{F}_A$  on  $\hat{F}_B$  to get  $\hat{\beta}$  since I do not know the actual  $\beta^*$ . However,  $\hat{\beta}$  will introduce a bias in the estimator equal to:

$$\frac{1}{n} \sum F_{Ai} + \hat{\beta} \left( F_B - \frac{1}{n} \sum F_{Bi} \right) \quad (12)$$

Still, this methodology can be applied because this possible bias decreases as  $n$  increases and the estimator of  $\beta^*$  does not require to be very precise to achieve variance reduction.

Table 6. Control Variate Technique Adjusted for Beta

S/X	No periods to maturity	Beta	Option values Control Variate method-beta	St. Dev. of estimate w/control variate	Confidence interval 95%		Accurate option values by numerical integration	
	2	0,9315	0,003	0,0000	0,0029	0,0030	0,003	ok
	4	0,8999	0,075	0,0003	0,0747	0,0757	0,075	ok
	8	0,8728	0,541	0,0012	0,5387	0,5434	0,541	ok
25/50	16	0,8388	1,959	0,0035	1,9524	1,9661	1,957	ok

Continue...

Table 6. *Continuation*

	2	0,9836	5,028	0,0003	5,0271	5,0284	5,028	ok
	4	0,9667	7,252	0,0010	7,2498	7,2535	7,251	ok
	8	0,9476	10,495	0,0022	10,4910	10,4996	10,493	ok
50/50	16	0,9273	15,093	0,0048	15,0841	15,1029	15,090	ok
	2	0,9929	26,369	0,0002	26,3683	26,3691	26,369	ok
	4	0,9830	27,822	0,0008	27,8203	27,8235	27,819	ok
	8	0,9697	30,712	0,0022	30,7078	30,7164	30,713	ok
75/50	16	0,9541	35,556	0,0046	35,5465	35,5645	35,558	ok

The results obtained by adjusting the estimator for  $\beta$  are positive in the sense that a large reduction in variance is achieved compared to the variance obtained when  $\beta$  is one as in (8). The significant reduction in the standard deviation of the estimates is shown in Table 7. Since this methodology's only additional requirement in computer time is the calculation of  $\beta$ , and the reduction in variance is significant, I conclude that the beta adjusted control variate technique increases the efficiency of the Monte Carlo estimator.

Table 7. *Estimation Errors for Different Control Variate Techniques*

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As Boyle (1977) remarks, the introduction of an appropriate control variate provides a very efficient variance reduction technique in this case; however, in some other cases it may be difficult to find a suitable control variate. In those cases, other methods such as the antithetic variate must be used.

In summary, the Monte Carlo method can be successfully used to value options since this numerical method provides a good approximation to the correct value of the option once adequate variance reduction techniques are applied.

### Conclusions

In conclusion, Monte Carlo has appealing characteristics as an option valuation method, such as flexibility to function with different distributions, even empirical distributions of the underlying variables. Additionally, it can incorporate discontinuities such as those that arise from jump processes. For these cases, there are no analytical solutions thus requiring the use of a numerical method such as Monte Carlo.

The performance, in terms of accuracy, of the Monte Carlo method in comparison to other numerical methods is satisfactory. Its estimate is, in most cases, equivalent to the mean of the values obtained by using the other methods to value options which are in, at, and out-of-the-money and with different time to expiration. In every case, the benchmark value given by the Black-Scholes method is in the confidence interval of the Monte Carlo estimation.

The precision of the estimates provided by crude Monte Carlo can be improved by implementing variance reduction techniques. However, the use of these techniques implies a greater computational effort; thus, there is a trade-off between a lower variance estimator and computational requirements.

One of such techniques is the antithetic variate method. By using this method, both the accuracy and the efficiency of the Monte Carlo estimates are improved compared to crude Monte Carlo. The other variance reduction method is the control variate which estimates are completely satisfactory in terms of accuracy and efficiency. Although computer time doubles in comparison to crude Monte Carlo, the reduction in the variance is much greater than one half, thereby increasing efficiency. As Boyle (1977) remarks, the introduction of an appropriate control variate provides a very efficient variance reduction technique, however, in some cases it may be difficult to find a suitable control variate. In those cases, other methods such as the antithetic variate must be used.

In summary, the Monte Carlo method can be successfully used to value options since this numerical method provides a good approximation to the correct value of the option once adequate variance reduction techniques are applied.

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