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THE VALUE OF INFORMATION IN A COOPERATIVE ENVIRONMENT

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Abstract

In this paper we analyze the value of the information in a cooperative game. There is a player, the innovator, having a know how or relevant information which is not useful for himself but it can be sold to some potential buyers. The n *potential users of the information are involved in a market having all them the same characteristics. The expected utility of each of them can be improved by obtaining the information. The whole situation is modeled as a* (n + *1)–person game. The Shapley Value is the cooperative solution studied.*
We deal with a game in characteristic form function, where this function can be non-superaditive. Supearditivity have been a usual assumption in cooperative games, but we show that under a weak version of superaditivity it is still possible to use the Shapley Value as a cooperative solution. We give conditions for the weak superaditivity and study the implications of those conditions on the resulting market.
We also compare the Shapley Value with the outcomes obtained in a noncooperative approach by Quintas (1995). Finally we arrive to the conclusion that the innovator prefers the noncooperative outcome and the users prefer the cooperative outcomes.

Resumen

En este artículo se analiza el valor de la información en un juego cooperativo. Existe un jugador, el innovador, quien tiene una idea o información relevante que no le es útil, pero que puede ser vendida a algunos potenciales compradores. Los potenciales n *usuarios de la información están involucrados en un mercado donde todos tienen las mismas características. La utilidad esperada de cada uno de ellos puede aumentar obteniendo la información. La situación se modela como un juego con (*n + *1) personas. El Valor de Shapley es la solución cooperativa que se utiliza y estudia en este problema.*

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Se trabaja con un juego que tiene una forma funcional característica, donde esta función puede ser no-superaditiva. La superaditividad ha sido un supuesto usual empleado en juegos cooperativos, pero se demuestra que utilizando una versión débil de superaditividad aún es posible usar el Valor de Shapley como la solución cooperativa. Se da condiciones para la superaditividad débil y se estudia las implicancias de estas condiciones sobre el mercado.
Además se compara el Valor de Shapley con los resultados obtenidos en un entorno no cooperativo desarrollado por Quintas (1995). Finalmente, se concluye que el innovador prefiere el resultado no cooperativo y los usuarios prefieren el resultado cooperativo.

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1. Introduction

In this paper we consider the following problem: There are n firms with similar characteristics. They produce a unique good. They interact in a market. There exist an agent who posses relevant information for the firms. This information can be sold to the firms. The innovator is not going to use the information for himself and the firms acquiring the information will be better than before obtaining it.

A typical example would be n farmers which are willing to obtain information about the upcoming weather. Thus they could know what is the best seed they should use for the upcoming season. They could pay for this information to an agent having it, in order to improve their expectations of choosing the right seed. Another example is the case of an innovator offering a new technology to n firms. They can then reduce their production costs. The information holder is not part of the market, but he can act strategically in order to maximize his utilities by selling the information.

There exist many papers studying the value of the information when the information holder acts strategically. Kamien (1992) surveys most of these studies.

The problem can be modeled as a $n + 1$ players game. This game can be cooperative or noncooperative. Quintas (1995) considered in a noncooperative framework under what conditions it was optimal and stable to sell the information to all the firms. However the situation was not so appealing from the buyers point of view. The information should be bought for all the buyers, but the utilities they obtained corresponded to the case when each of them were the unique uniformed player. Nevertheless they couldn't ignore the existence of the information. Thus it was concluded that nevertheless they should buy the technology. This result reflects many real situations where the introduction of a new technology produces serious damage in the local market. On the other hand it might be expected that in some cases the firms could act in a cooperative way in order to prevent the general damage mentioned above. We study the problem in a cooperative characteristic form game of $n + 1$ players.

The cooperative notion we use is the Shapley Value (Shapley (1953)). It gives to each player an average of the marginal contribution to any possible coalition he can form. It is the power measure assigned to each player. This power index is characterized by simple axioms (Shapley(1953)). It has no existence problem (as in the case of the Core solution, for instance) and it provides a vector of the agent utilities which can be compared in the several different approaches.

When we consider a game in characteristic form function superaditivity is usually assumed and the Shapley Value can be computed and it gives an imputation. However in our approach can also arise cooperative games that are not superaditive. Thus we show that under a weak version of superaditivity it is still possible to use the Shapley Value as a cooperative solution. We give conditions for the weak superaditivity and study the implications of those conditions on the resulting market.

2. The Information Market

We consider a market with n firms ($n \geq 2$) and an innovator who posses a patent or an information.

The set of agents will be denoted by: $N = \{1, 2, ..., n + 1\}$, where:

$I = \{1\}$ (the innovator) is the agent having a new information and $U = \{2, ..., n + 1\}$ (users) are the firms who could be willing to obtain the new information.

The n users or firms, interact in the same market, producing or performing the same activity, with the same technology or the same information. Thus all the users have the same incentives for the acquisition of the new information or technology. We will make certain assumptions about the problem we want to study:

2.1. The n users of the information are the same before and after the information holder offers the new technology.

It indicates that there are no exits or incoming agents in the market.

2.2. All the players that acquire the new information make use of it.

This is a natural assumption in a noncooperative environment, and it is assumed in a cooperative model to avoid the formation of monopolies.

2.3. From the point of view of the players that acquire the information, their utilities will be computed under a conservator point of view, assuming that the noninformed agents take the right decision.

Before a noninformed agent buys the information he will compute how much is willing to pay for the information. Thus he will compute the utility that he will obtain interacting with a group of S informed players and $N - S$ noninformed players.

We also assume that:

2.4. All the users have the same previous information level (For instance, in the case of the farmers, they all have the same knowledge about the upcoming weather, or in the case of firms producing a good, they all have the same technology).

2.5. The utility they obtain depends on how many players take a right decision no matter the identity of them.

Thus if r players make a right decision (for instance, if they are farmers choosing the right seed for the upcoming weather) each utility function will be $a(r)$.

Remark 1: $a(r)$ is a decreasing function, because when more agents take the right decision, each agent obtains a lower utility level:

(1) $$\text{If } r \leq k \text{ then } a(r) \geq a(k)$$

We also assume that:

2.6. $a(1) = 1$ and $a(r) \geq 0$.

2.7. The agents making a wrong decision obtain no utility.

If an agent is uniformed, the probability of making a right decision (or success) can be described by a binomial probability distribution:

Definition 1 *A random binomial experiment consists of n repeated tries, fulfilling:*

All the tries are independent.

1. Each one has two possible outcomes, "success" or "failure".
2. The probability of success is denoted by p. It is constant.
   It defines a binomial experiment.

The random variable x gives the number of success in a binomial distribution with parameters p and n.

The probability function x is given by:

$$p(x;p,n) = \binom{n}{x} p^x (1-p)^{n-x}$$

In our case:

2.8. All the users are similar. Thus their actions represent the n tries.

2.9. Success indicates a right decision and Failure otherwise.

2.10. For each $j \in U$ there exists a probability p_j of having success. For instance, to choose the right seed for the upcoming weather. We also assume that this probability is the same for all players:

(2) $$p_j = c \quad \forall j \in U$$

Thus we can model the problem as a binomial experiment. The probability that k among n players take the right decision is:

$$p(x = k, c, n) = \binom{n}{k} c^k (1-c)^{n-k}$$

The utility obtained by the agents in U is:

$$p(x = k, c, n)a(k)$$

The success of each agent is independent of the remaining agents. Thus the aggregated utility of k players which succeeded is:

$$k\,p(x = k,\ c,\ n)a(k) = k\binom{n}{k}c^k(1-c)^{n-k}a(k) \tag{3}$$

We will introduce now some definitions and notations of cooperative games:

A coalition is a subset of player willing to cooperate among themselves. If a subset $S \subseteq N$ of players form a coalition, the total utility they obtain is denoted by $v(S)$.

Definition 2 *An n person game in characteristic function form is given by* (N, v), *where* $N = \{1, 2, ..., n+1\}$ *is the set of players and* $v : 2^N \to \Re$ *is the characteristic function.*

Thus in our case we have:

2.11. If the agents in a coalition S don't have the information (the innovator is not a member of S), we have that:

$$v(S) = w(S)\ \sum_{k=0}^{s} k\binom{s}{k}c^k(1-c)^{s-k}a(n-s+k) = \text{with } s = |S|. \tag{4}$$

It is so, because if k players have succeeded, each one has an expected utility: $p(x = k,\ c,\ n)a(n-s+k)$, because we assumed that the $(n-s)$ players outside the coalition also success.

2.12. If the agents in S have the information (the innovator belongs to S):

$$v(S) = u(S) = (s-1)\ a(n) \text{ with } s = |S|. \tag{5}$$

It is so because s players have succeeded and their expected utility will be $a\ (n)$.

By (4) and (5) we have:

Definition 3 *The characteristic function* $v : 2^N \to \Re$ *is defined by:*

$$v(S) = \begin{cases} u(S) & \text{if} \quad 1 \in S \\ \\ w(S) & \text{if} \quad 1 \notin S \end{cases} \quad \textit{for all } S \subseteq N.$$

It is immediate to observe that:

Remark 2:

1. $v(\phi) = 0$.

2. $v(\{1\}) = 0$, because when the innovator don't sell the information, he obtains no utility by the use of it.

3. From (4) and (5) we deduce that the function v depends only on the fact that the innovator belongs (or not) to it, and the number of agents in the coalition. Thus we will keep the notations $v(S)$, $w(S)$ and $u(S)$, but in all these cases they are functions depending only on the cardinality s of the set S.

4. The more important remark about the definition $v(S)$, is that condition 2.2 (that the players acquiring the information always make use of it) avoids monopolies. It corresponds to real situations. For instance in the decade of ´30 in the United States there were subsidies for the farmers to keep production under a given level in order to avoid a collapse in the prices. Moreover, condition 2.2 can eventually produce non superaditive games. Superaditivity is a usual assumption in a cooperative environment, however in our analysis we show that the Shapley Value can still be used under a weaker form of superaditivity which will be used in Lemma 1 and Lemma 2. It reinforces the role of Theorem 2.

Lemma 1: If the innovator is not in the coalition S (*i.e.*, $1 \notin S$), $T = \{i\}$, with $i \neq 1$, and $T \cap S = \emptyset$, then:

$$w(S \cup T) \geq w(S) + w(T)$$

Proof: By 2.11, and the hypothesis we have that $1 \notin S \cup T$ with $|S \cup T| = s + 1$ By the definition of w and splitting the last term of the sum we have:

$$w(S \cup T) = \sum_{j=0}^{s+1} j \binom{s+1}{j} c^j (1-c)^{s+1-j} a(n-s-1+j) =$$

$$\sum_{j=0}^{s} j \binom{s+1}{j} c^j (1-c)^{s+1-j} a(n-s-1+j) + (s+1)c^{s+1} a(n).$$

Using the following property

$$\binom{s+1}{j} = \binom{s}{j} + \binom{s}{j-1}$$

we obtain

$$w(S \cup T) = \sum_{j=1}^{s} j \binom{s}{j} c^j (1-c)^{s+1-j} a(n-s-1+j) +$$

$$\sum_{j=1}^{s} j \binom{s}{j-1} c^j (1-c)^{s+1-j} a(n-s-1+j) + (s+1)c^{s+1} a(n). \tag{6}$$

2.17. If in the first sum we use Observation 1, which indicates that $a(r)$ is a decreasing function, we have:

$$a(n - s - 1 + j) \geq a(n - s + j) \text{ with } 1 \leq j \leq s$$

and splitting the last term in the sum we have:

$$\begin{aligned} &\sum_{j=1}^{s} j \binom{s}{j} c^{j} (1-c)^{s+1-j} a(n-s-1+j) \\ &\geq (1-c)\left[\sum_{j=0}^{s-1} j \binom{s}{j} c^{j} (1-c)^{s-j} a(n-s+j) + sc^{s} a(n)\right] \end{aligned} \tag{7}$$

2.18. In the second sum we change the variable $k = j - 1$

$$\begin{aligned} &\sum_{j=1}^{s} j \binom{s}{j-1} c^{j} (1-c)^{s+1-j} a(n-s-1+j) \\ &= \sum_{k=0}^{s-1} (k+1) \binom{s}{k} c^{k+1} (1-c)^{s-k} a(n-s+k). \end{aligned} \tag{8}$$

$$\begin{aligned} &\geq c\left[\sum_{k=0}^{s-1} k \binom{s}{k} c^{k} (1-c)^{s-k} a(n-s+k)\right] \\ &\quad + c\left[\sum_{k=0}^{s-1} \binom{s}{k} c^{k} (1-c)^{s-k}\right] a(n) \end{aligned} \tag{9}$$

Last inequality results of distributing $(k + 1)$ and using Observation 1.
Using the Newton binomial form, the second sum we have

$$c\left[\sum_{k=0}^{s-1} \binom{s}{k} c^{k} (1-c)^{s-k}\right] a(n) = c(1-c^{s}) a(n). \tag{10}$$

Replacing (10) in (9) and making a new change of variable, $k = j$ we have:

$$\begin{aligned} &\sum_{j=1}^{s} j \binom{s}{j-1} c^{j} (1-c)^{s+1-j} a(n-s-1+j) = \\ &\sum_{j=0}^{s-1} j \binom{s}{j} c^{j+1} (1-c)^{s-j} a(n-s+j) + c(1-c^{s}) a(n) \end{aligned} \tag{11}$$

Arranging the last sum with (7) and using the definition of w we have:

$$w(S \cup T) \geq$$

$$(1-c)\left[\sum_{j=0}^{s-1} j\binom{s}{j} c^{j}(1-c)^{s-j} a(n-s+j)+s c^{s} a(n)\right]+$$

$$c\left[\sum_{j=0}^{s-1} j\binom{s}{j} c^{j}(1-c)^{s-j} a(n-s+j)\right]+c(1-c^{s}) a(n)+(s+1) c^{s+1} a(n)=$$

$$\sum_{j=0}^{s-1}\binom{s}{j} c^{j}(1-c)^{s-j} a(n-s+j)+[(1-c) s c^{s}+(s+1) c^{s+1}+c(1-c^{s})] a(n)$$

$$=\sum_{j=0}^{s} j\binom{s}{j} c^{j}(1-c)^{s-j} a(n-s+j)+c a(n)=w(S)+w(T)$$

Now we present a condition in order that the users have incentives for buying the information.

Lemma 2: If the innovator is not in the coalition S ($1 \notin S$) and he belongs to T ($1 \in T$) such that $S \cap T = \varnothing$, then $u(S \cup T) \geq w(S) + u(T)$ if and only if $w(S) \leq u(S \cup \{1\})$.

Proof:

By the definition of u and v we have:

(12) $$\mathrm{u}(S \cup T) = (s + t - 1)\, a(n)$$

(13) $$w(S)=\sum_{k=0}^{s} k\binom{s}{k} c^{k}(1-c)^{s-k} a(n-s+k)$$

(14) $$u(T) = (t - 1)\, a(n)$$

(15) $$u\,(S \cup \{1\}) = s\, a(n)$$

First, we are going to prove that:

$$u(S \cup T) \geq w(S) + u(T) \text{ then } w(S) \leq u(S \cup \{1\})$$

Then:

If $u(S \cup T) \geq w(S) + u(T)$ then by (12), (13), (14) we have:

$$(s + t - 1)\, a(n) \geq$$

$$\sum_{k=0}^{s} k\binom{s}{k} c^{k}(1-c)^{s-k} a(n-s+k)+(t-1) a(n)$$

Simplifying $(t - 1)\ a(n)$ we have:

$$s\,a(n) \geq \sum_{k=0}^{s} k \binom{s}{k} c^k (1-c)^{s-k} a(n-s+k)$$

By (15), $u(S \cup \{1\}) = s\ a(n)$, we have:

$$w(S) \leq u(S \cup \{1\})$$

This proves the first part.
Now we are going to prove that: $w(S) \leq u(S \cup \{1\})$ then $u(S \cup T) \geq w(S) + u(T)$.
$w(S) \leq s\ a(n)$ then:

$$u(S \cup T) - w(S) - u(T) \geq u(S \cup T) - w\ (S) - s\ a(n) =$$

$$(s + t - 1)\ a(n) - (t - 1)\ a(n) - s\ a(n) = 0$$

It follows from (12), (14). Then we have:
$u(S \cup T) \geq w(S) + u(T)$
It completes the proof.

Remark 3:

1. This lemma indicates that the players in a noninformed coalition $S \subseteq U$ have incentives to join a informed coalition $T \subseteq N$, if the utility they obtain is less than they would obtain buying the information. We do not need a restriction for T because by assumption 2.3, for the computation of the characteristic function $v(T)$ we assumed that the noninformed agents outside the coalition take the right decision. Thus it is always better joining them to the coalition.

2. If $|S| = 1$ then $a(n) \geq 0$, and it always hold by 2.6. It indicates that a sole uninformed player always have incentives to buy the information.

Now we will analyze what happens if $s > 1$. We analyze the restrictions given in Lemma 2 depending on the number of agents in the market.

For each set S we have an inequation, thus we have $\binom{n}{s}$ inequations, but as all the sets having the same cardinality s give the same inequation, we have only n relevant equations.

If $u(S \cup T) \geq w(S) + u(T)$, then, using (12), (13), (14) we have:

$$a(n) \geq \frac{\sum_{j=0}^{s-1} j \binom{s}{j} c^j (1-c)^{s-j} a(n-s+j)}{s(1-c^s)} \tag{16}$$

The first term in the numerator in (16) is 0. Thus we will consider $j \geq 1$.

We will use:

(17)
$$\frac{j}{s}\binom{s}{j} = \binom{s-1}{j-1}$$

and

(18)
$$(1-c^s) = (1-c)\sum_{j=0}^{s-1} c^j$$

Replacing (17) and (18) in (16) we have:

(19)
$$a(n) \geq \frac{\sum_{j=1}^{s-1}\binom{s-1}{j-1} c^j (1-c)^{s-j-1} a(n-s+j)}{\sum_{j=0}^{s-1} c^j}$$

For each s with $2 \leq s \leq n$ we have an inequation. Then we have:

$$\begin{cases} a(n) \geq \dfrac{c}{1+c}\, a(n-1) \\ a(n) \geq \dfrac{\sum_{j=1}^{2}\binom{2}{j-1} c^j (1-c)^{2-j}\, a(n-3+j)}{\sum_{j=0}^{2} c^j} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a(n) \geq \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1} c^j (1-c)^{k-j-1} a(n-k+j)}{\sum_{j=0}^{k-1} c^j} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a(n) \geq \dfrac{\sum_{j=1}^{n-1}\binom{n-1}{j-1} c^j (1-c)^{n-j-1} a(j)}{\sum_{j=0}^{n-1} c^j} \end{cases}$$

We also have the general assumption 2.6.
$0 \leq a(n) \leq a(n-1) \leq \ldots \leq a(2) \leq 1$.

Then we have the following system:

$$\begin{cases} a(n) \geq \dfrac{c}{1+c}\, a(n-1) \\ a(n) \geq \dfrac{\sum_{j=1}^{2}\binom{2}{j-1} c^{j}(1-c)^{2-j} a(n-3+j)}{\sum_{j=0}^{2} c^{j}} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a(n) \geq \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1} c^{j}(1-c)^{k-j-1} a(n-k+j)}{\sum_{j=0}^{k-1} c^{j}} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a(n) \geq \dfrac{\sum_{j=1}^{n-1}\binom{n-1}{j-1} c^{j}(1-c)^{n-j-1} a(j)}{\sum_{j=0}^{n-1} c^{j}} \\ 0 \leq a(n) \leq a(n-1) \leq \ldots \leq a(2) \leq 1 \end{cases}$$

We should solve this system finding the variation of the variables $a(n)$, $a(n-1)$, ..., $a(2)$:

We will use the Fourier method (Bertsimas, D. and J. N. Tsitsiklis (1997)) for inequations. This method consists of eliminating in each step a variable obtaining a equivalent system with the remaining variables.

Thus we have:

Proposition: The solution to the system

$$\begin{cases} a(n) \geq \dfrac{c}{1+c}\, a(n-1) \\ \vdots \\ a(n) \geq \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1} c^{j}(1-c)^{k-j-1} a(n-k+j)}{\sum_{j=0}^{k-1} c^{j}} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a(n) \geq \dfrac{\sum_{j=1}^{n-1}\binom{n-1}{j-1} c^{j}(1-c)^{n-j-1} a(j)}{\sum_{j=0}^{n-1} c^{j}} \\ 0 \leq a(n) \leq a(n-1) \leq \ldots \leq a(3) \leq a(2) \leq 1 \end{cases}$$

is:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(n) \leq a(n-1) \leq \cdots \leq a(3) \leq a(2) \leq 1$$

The proof is done by induction and it appears in the Appendix.

Remark 4: This proposition shows that the restriction of Lemma 2 can be simplified, giving only a new inequality

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(n) \tag{20}$$

The other inequalities ($a(i) \leq a(i-1)$), are given in 2.3.

Restriction 20 fulfills that when the number of players grows it gives a lower value. Thus we have more freedom for choosing $a(n)$.

Moreover, when $c \to 0$, or $c \to 1$ (with n fixed), we have:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \to 0$$

Thus in these extreme cases we only have the basic restriction $a(n) \geq 0$.

3. Solution of the Cooperative Game

The solution is the Shapley Value. We compute it for the given characteristic function v. Given a game (N, v), the Shapley Value is defined by the following vector $\varphi(v) = (\varphi_1(v), \ldots, \varphi_{n+1}(v))$ where:

$$\varphi_i(v) = \sum_{S \subseteq N-\{i\}} \frac{s!(n-s)!}{(n+1)!}[v(S \cup \{i\}) - v(S)]$$

with $|S| = s$ and $|N| = n + 1$.

In the following theorem we give the formulation for the Valor de Shapley using the characteristic function definition.

Theorem 1: The Shapley for the users is given by:

$$\varphi_i(v) = \frac{1}{2}a[n] + \sum_{s=0}^{n-1} \frac{n-s}{n(n+1)}(w(S \cup \{i\}) - w(S))$$

The Shapley Value for the innovator is given by:

$$\varphi_1(v) = \sum_{s=0}^{n} \frac{1}{n+1}(u(S \cup \{1\}) - w(S))$$

Proof:

1st Case: Let's be $i \neq 1$, it means i is an user. Let's compute the Shapley Value for i.

By the Shapley Value definition and splitting the sums in informed and noninformed coalitions, we have:

$$\varphi_i(v) = \sum_{S \subseteq N - \{i\}} \frac{s!(n-s)!}{(n+1)!}[v(S \cup \{i\}) - v(S)] =$$

$$\sum_{\substack{S \subseteq N - \{i\} \\ 1 \in S}} \frac{s!(n-s)!}{(n+1)!}[u(S \cup \{i\}) - u(S)] + \sum_{\substack{S \subseteq N - \{i\} \\ 1 \notin S}} \frac{s!(n-s)!}{(n+1)!}[w(S \cup \{i\}) - w(S)]$$

Using Observation 2 in the last equality and as a consequence of 2.13. and 2.14:

2.13. If $S \subseteq N - \{i\}$, with $1 \in S$, we count how many subsets we have of the type $S - \{1\} \subseteq N - \{1, i\}$, (the innovator is a fixed player in all the coalitions S we could form). They are $\binom{n-1}{s-1}$. Then $\binom{n-1}{s-1}\frac{s!(n-s)!}{(n+1)!} = \frac{s}{n(n+1)}$, with $s = 1, ..., n$.

2.14. If $S \subseteq N - \{i\}$, with $1 \notin S$, then there exists $\binom{n-1}{s}$ subsets with $S \subseteq N - \{i, 1\}$, so that $\binom{n-1}{s}\frac{s!(n-s)!}{(n+1)!} = \frac{n-s}{n(n+1)}$, with $s = 0, ..., n-1$.

Thus we have:

$$\varphi_i(v) = \sum_{\substack{s=1 \\ 1 \in S}}^{n} \frac{s}{n(n+1)}[u(S \cup \{i\}) - u(S)] + \sum_{\substack{s=0 \\ 1 \notin S}}^{n-1} \frac{n-s}{n(n+1)}[w(S \cup \{i\}) - w(S)]$$

As a consequence of the definition 3 of the given characteristic function we have:

$$\varphi_i(v) = \sum_{s=1}^{n} \frac{s}{n(n+1)}(u(S \cup \{i\}) - u(S)) + \sum_{s=0}^{n-1} \frac{n-s}{n(n+1)}(w(S \cup \{i\}) - w(S))$$

By the definition of u we have:
$u(S \cup \{i\}) = s\, a(n)$
$u(S) = (s-1)a(n)$ then $u(S \cup \{i\}) - u(S) = a(n)$

and $\sum_{s=1}^{n} \frac{s}{n(n+1)} = 1/2$

then $\sum_{s=1}^{n} \frac{s}{n(n+1)} (u(S \cup \{i\}) - u(S)) = \frac{1}{2} a(n)$.

So that: $\varphi_i(v) = \frac{1}{2} a(n) + \sum_{s=0}^{n-1} \frac{n-s}{n(n+1)} (w(S \cup \{i\}) - w(S))$.

2nd Case: Let's be $i = 1$, it means that i is the innovator. Let's compute the Shapley Value for i.

By the definition of the Shapley Value and the definition of v we have:

$$\varphi_1(v) = \sum_{S \subseteq N - \{i\}} \frac{s!(n-s)!}{(n+1)!} [v(S \cup \{1\}) - v(S)] = \sum_{S \subseteq N - \{i\}} \frac{s!(n-s)!}{(n+1)!} [u(S \cup \{1\}) - w(S)].$$

Using Observation 2 in the last equality and 2.15:

2.15. As $S \subseteq N - \{1\} = U$, then there exist $\binom{n}{s}$ subsets $S \subseteq N - \{1\}$, with $s = 0$, 1, ..., n so that: $\binom{n}{s} \frac{s!(n-s-1)!}{n!} = \frac{1}{(n-s)}$.

Then we have:

$$\varphi_1(v) = \sum_{s=0}^{n} \frac{1}{n+1} (u(S \cup \{1\}) - w(S)).$$

We will prove that under certain conditions, φ results a payoff distribution for the game (N, v). Let's consider the following definition:

Definition 4: *An imputation or payoff distribution for the game* (N, v) *is a vector* $x = (x_1, \ldots, x_{n+1})$ *satisfying:*

1. $\sum_{i \in N} x_i = v(N)$
2. $x_i \geq v(\{i\})$ for each $i \in N$.

Now we will prove that $\varphi(v)$, is an imputation for the game (N, v).

Theorem 2: Under the hypothesis of Lemma 1 and Lemma 2, we obtain that $\varphi(v) = (\varphi_1(v), \ldots, \varphi_{n+1}(v))$ is an imputation for the game (N, v).

Proof.

Taking into account the definition of imputation, we should prove that:

1) $\sum_{i \in N} \varphi_i(v) = v(N)$

2) $\varphi_i(v) \geq v(\{i\})$ for each $i \in N$.

Let's prove 1). Splitting the sums in the Shapley Value for the innovator and using Theorem 1 we have:

$$\sum_{i \in N} \varphi_i(v) = \varphi_1(v) + \sum_{i \in N-\{1\}} \varphi_i(v)$$

In the last sum, as all the $\varphi_i(v)$ have the same value, we obtain:

$$\sum_{i \in N} \varphi_i(v) = \varphi_1(v) + n\varphi_i(v) =$$

(21)

$$\sum_{|S|=0}^{n} \frac{1}{n+1}(u(S \cup \{1\}) - w(S)) + n\left[\frac{1}{2}a(n) + \sum_{\substack{s=0 \\ |S|=s}}^{n-1} \frac{n-s}{n(n+1)}(w(S \cup \{i\}) - w(S))\right]$$

Splitting the sums in (21), we have:

(22)

$$\frac{1}{2} n\, a(n) + \frac{1}{n+1}\sum_{s=0}^{n} u(S \cup \{1\}) -$$

$$\frac{1}{n+1}\sum_{s=0}^{n} w(S) + \frac{n}{n+1}\left[\sum_{s=0}^{n-1}\frac{n-s}{n}(w(S \cup \{i\}) - w(S))\right]$$

Developing the last two sums we have that:

(23)

$$-\frac{1}{n+1}\sum_{s=0}^{n} w(S) + \frac{n}{n+1}\left[\sum_{s=0}^{n-1}\frac{n-s}{n}(w(S \cup \{i\}) - w(S))\right] =$$

$$-\frac{1}{n+1}\sum_{s=0}^{n} w(S) + \frac{1}{n+1}\left[\sum_{s=0}^{n-1}(n-s)(w(S \cup \{i\}) - w(S))\right]$$

By 2.5, $w(S \cup \{i\}) = w(S \cup \{j\})$ for all i and j in U, from (23) simplifying we have:

$$-\frac{1}{n+1}\sum_{s=0}^{n} w(S) + \frac{1}{n+1}\left[\sum_{s=0}^{n-1}(n-s)(w(S \cup \{i\}) - w(S))\right] = 0.$$

Replacing in (23)

$$\sum_{i \in N} \varphi_i(v) = \frac{1}{2} n\, a(n) + \frac{1}{n+1} \sum_{s=0}^{n} u(S \cup \{1\})$$

By the definition of u the above expression becomes:

$$\frac{1}{2} n\, a(n) + \frac{1}{n+1} \sum_{s=0}^{n} s\, a(n).$$

Using $\sum_{s=0}^{n} s = \frac{n(n+1)}{2}$ in the last sum we have:

$$\sum_{i \in N} \varphi_i(v) = \frac{1}{2} n\, a(n) + \frac{1}{n+1} a(n) \sum_{s=0}^{n} s = n\, a(n)$$

As $na(n) = v(N)$ we have:

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

Lets prove 2). $x_i \geq v(\{i\})$ for each $i \in N$.
1st Case: Lets consider the Shapley Value for the innovator, *i.e.* $i = 1$:

$$\varphi_1(v) = \sum_{s=0}^{n} \frac{1}{n+1} (u(S \cup \{1\}) - w(S)) \geq \sum_{s=0}^{n} \frac{1}{n+1} (w(S) + w(\{1\}) - w(s)) = 0.$$

The last inequality follows from Lemma 2.
Then by the definition of v : $v(\{1\}) = 0$

$$\varphi_1(v) \geq v(\{1\})$$

2nd Case: Lets consider the Shapley Value for the users, *i.e.* $i = 2, ..., n + 1$

$$\varphi_i(v) = \frac{1}{2} a(n) + \sum_{s=0}^{n-1} \frac{n-s}{n(n+1)} (w(S \cup \{i\}) - w(S))$$

Using Lemma 1 we have that:

$$\varphi_i(v) \geq \frac{1}{2} a(n) + \frac{1}{n(n+1)} \sum_{s=0}^{n-1} (n-s)(w(S) + w(\{i\}) - w(S))$$

As $w(\{i\}) = ca(n)$ we have:

$$\frac{1}{2} a(n) + \frac{1}{n(n+1)} \sum_{s=0}^{n-1} (n-s)\, c\, a(n)$$

Using that $\sum_{s=0}^{n-1}(n-s)=\frac{n(n+1)}{2}$ in the last sum

$$\varphi_i(v) \geq \frac{1}{2}a(n)+\frac{1}{n(n+1)}\frac{n(n+1)}{2}c\,a(n) = \frac{1}{2}a(n)(1+c) \tag{24}$$

As $0 \leq c \leq 1$ then $\frac{1}{2}(1+c) \geq c$ replacing in (24)

$$\frac{1}{2}a(n)(1+c) \geq c\,a(n)$$

Then by the definition of v:
$\varphi_i(v) \geq v(\{i\})$.

4. Comparison of the Shapley Value with the payoff of the unique equilibrium in the noncooperative game

The cooperative game studied in this paper was analyzed by Quintas (1995) form a noncooperative point of view and then it was observed that the innovator obtained a neat profile by selling the information to the n firms. However the situation was not so appealing for the buyers. The expected utility each one finally obtained after buying the information was that one he would have obtained if he was the only uniformed agent. Nevertheless they couldn't ignore the existence of the information and they should buy it.

The main result of the noncooperative study mentioned above states as follows:

Theorem 3: Under condition (2), the price p that the innovator can ask to the n users such that all them acquire the information, is determined by the unique Nash equilibrium of the noncooperative game. This price is: $p = (1-c)a(n) - \varepsilon$, with $\varepsilon \geq 0$ arbitrarily small, and the payoff $n - upla$ is:

$$((1-c)na(n) + n\varepsilon,\ ca(n) + \varepsilon,\ \ldots,\ ca(n) + \varepsilon)$$

Lets compare the expected utility obtained in the unique equilibrium of the main theorem in the paper of Quintas (1995) with the results presented in our article.

Theorem 4: Let $N = \{1, 2, \ldots, n+1\}$, with $n \geq 2$, where $I = \{1\}$ (the innovator) and $U = \{2, \ldots, n+1\}$ (the users) and $0 \leq c \leq 1$, then:

1. The innovator prefers the noncooperative model.

2. The users prefer the cooperative model.

Proof.

Lets prove 1:

As $\varphi(v) = (\varphi_1(v), ..., \varphi_{n+1}(v))$ is an imputation for the game (N, v), it satisfies:

I. $\sum_{i \in N} \varphi_i = v(N) = n\,a[n]$

II. $\varphi_i \geq v(\{i\})$ for each $i \in N$. Then

If $i \in U : \varphi_i \geq v(\{i\}) = c\ a[n]$

Then $n\ a[n] = \sum_{i \in N} \varphi_i = \varphi_1 + \sum_{i \in N/\{1\}} \varphi_i \geq \varphi_1 + n\ c\ a[n]$.

and thus we have $\varphi_1 \leq n\ (1 - c)\ a[n]$.

By Theorem 3, the expected payoff for the innovator is: $n(1 - c)\ a[n]$, then we have proved that:

The Shapley Value gives a lower utility for the innovator than the utility obtained in the unique equilibrium in the noncooperative model.

Lets prove 2:

Lets consider the Shapley Value for the users, *i.e.* $i = 2, ..., n + 1$ and using Lemma 1, we have:

$$\varphi_i(v) = \frac{1}{2}a(n) + \sum_{|S|=0}^{n-1} \frac{n-s}{n(n+1)}(w(S \cup \{i\}) - w(S)) \geq$$
$$\frac{1}{2}a(n) + \frac{1}{2}c\,a(n) = \frac{1}{2}(1+c)\,a(n).$$

As $0 \leq c \leq 1$ then:

$$\frac{1}{2}\ (1 + c)\ a(n) \geq c\ a(n).$$

By theorem 3 the expected payoff of the users is: $c\ a(n)$. Then we then proved that The Shapley Value gives a better utility for the users than the utility obtained in the unique equilibrium in the noncooperative model.

5. Conclusions

We observe that the Shapley Value gives a better utility for the possible users that what they obtained in the unique equilibrium in the noncooperative model. It means that they avoid being exploited by the innovator. An opposite situation is observed from the innovator point of view because its utility is lower than in the noncooperative model.

Using a characteristic function that takes into account condition 2.2 (saying that the players that acquire the information always make use of it), avoids monopolies formation. It models real situations where there exists antimonopoly laws. The resulting games can be nonsuperaditive. The superaditivity assumptions is usual in cooperative studies, however we show that under a weaker form of superaditivity it is still possible to use the Shapley Value. It is used in Lemma 1 and Lemma 2, and it reinforces the relevance of Theorem 2. The study on the conditions for the weak superaditivity helps understanding the implications of those conditions on the resulting market.

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5.1. APPENDIX

Proof of Lemma2:

In order to solve the system we use the Fourier method. It consists of eliminating one variable in each step, obtaining a new system equivalent to the previous one but with one variable less.

1st step: We eliminate $a(n)$.

1. We reorder the system as follows:

$$\begin{cases} a(n)-\dfrac{c}{1+c}a(n-1)\geq 0 \\ a(n)-\dfrac{\sum_{j=1}^{2}\binom{2}{j-1}c^{j}(1-c)^{2-j}a(n-3+j)}{\sum_{j=0}^{2}c^{j}}\geq 0 \\ \vdots \\ a(n)-\dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1}c^{j}(1-c)^{k-j-1}a(n-k+j)}{\sum_{j=0}^{k-1}c^{j}}\geq 0 \\ \vdots \\ a(n)-\dfrac{\sum_{j=2}^{n-1}\binom{n-1}{j-1}c^{j}(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^{j}}\geq\dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^{j}} \\ 0\leq a(n)\leq a(n-1)\leq\ldots\leq a(2)\leq 1 \end{cases}$$

2. We determine the variation of $a(n)$.

$$\max\begin{cases} 0,\dfrac{c}{1+c}a(n-1),\dfrac{\sum_{j=1}^{2}\binom{2}{j-1}c^{j}(1-c)^{2-j}a(n-3+j)}{\sum_{j=0}^{2}c^{j}},\ldots, \\ \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1}c^{j}(1-c)^{k-j-1}a(n-k+j)}{\sum_{j=0}^{k-1}c^{j}},\ldots, \\ \dfrac{\sum_{j=2}^{n-1}\binom{n-1}{j-1}c^{j}(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^{j}}+\dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^{j}} \end{cases}$$

$\leq a(n)$

$\leq \min\{1, a(2), ..., a(n-1)\}$

As min $\{1, a(2), ..., a(n-1)\} = a(n-1)$ we have:

$$\max\left\{\begin{array}{l} 0, \dfrac{c}{1+c}a(n-1), \dfrac{\sum_{j=1}^{2}\binom{2}{j-1}c^{j}(1-c)^{2-j}a(n-3+j)}{\sum_{j=0}^{2}c^{j}}, \ldots, \\ \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1}c^{j}(1-c)^{k-j-1}a(n-k+j)}{\sum_{j=0}^{k-1}c^{j}}, \ldots, \\ \dfrac{\sum_{j=2}^{n-1}\binom{n-1}{j-1}c^{j}(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^{j}} + \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^{j}} \end{array}\right\} \leq a(n) \leq a(n-1)$$

Thus the system is equivalent to:

$$\begin{cases} \max\left\{\begin{array}{l} 0, \dfrac{c}{1+c}a(n-1), \dfrac{\sum_{j=1}^{2}\binom{2}{j-1}c^{j}(1-c)^{2-j}a(n-3+j)}{\sum_{j=0}^{2}c^{j}}, \ldots, \\ \dfrac{\sum_{j=1}^{k-1}\binom{k-1}{j-1}c^{j}(1-c)^{k-j-1}a(n-k+j)}{\sum_{j=0}^{k-1}c^{j}}, \ldots, \\ \dfrac{\sum_{j=2}^{n-1}\binom{n-1}{j-1}c^{j}(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^{j}} + \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^{j}} \end{array}\right\} \leq a(n) \leq a(n-1) \\ 0 \leq a(n-1) \leq \ldots \leq a(2) \leq 1 \end{cases}$$

3. As $0 < c < 1$, the variation of $a(n)$, is bounded. Comparing the extremes of the variation interval we obtain a new equivalent system:

$$\begin{cases} a(n-1) - \dfrac{c(1-c)}{1+c(1-c)}a(n-2) \geq 0 \\ \vdots \\ a(n-1) - \dfrac{\sum_{j=1}^{k-2}\binom{k-1}{j-1}c^{j}(1-c)^{k-j-1}a(n-k+j)}{\sum_{j=0}^{k-1}c^{j} - (k-1)c^{k-1}} \geq 0 \\ \vdots \\ a(n-1) - \dfrac{\sum_{j=2}^{n-2}\binom{n-1}{j-1}c^{j}(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^{j} - (n-1)c^{n-1}} \geq \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^{j} - (n-1)c^{n-1}} \\ 0 \leq a(n-1) \leq \ldots \leq a(2) \leq 1 \end{cases}$$

2nd Step: We eliminate $a(n-1)$ *in* a similar way obtaining:

$$\begin{cases} a(n-2) - \dfrac{c(1-c)^2}{1+c(1-c)^2}\, a(n-3) \geq 0 \\ \vdots \\ a(n-2) - \dfrac{\sum_{j=1}^{k-3}\binom{k-1}{j-1} c^j (1-c)^{k-j-1} a(n-k+j)}{\sum_{j=0}^{k-1} c^j - (k-1)c^{k-1} - (k-2)c^{k-2}} \geq 0 \\ \vdots \\ a(n-2) - \dfrac{\sum_{j=2}^{n-3}\binom{n-1}{j-1} c^j (1-c)^{n-j-1} a(j)}{\sum_{j=0}^{n-1} c^j - (n-1)c^{n-1} - (n-2)c^{n-2}} \geq \\ \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1} c^j - (n-1)c^{n-1} - (n-2)c^{n-2}} \\ 0 \leq a(n-2) \leq \ldots \leq a(2) \leq 1 \end{cases}$$

After k steps we have a system with the following variables, $a(n-k-1)$, $a(n-k-2)$, ..., $a(2)$ and we proceed to eliminate the variable $a(n-k-1)$.

This system has as a first inequality:

$$a(n-k-1) - \frac{c(1-c)^k}{\sum_{j=0}^{k-1} c^j - \sum_{j=2}^{k-1}\binom{k-1}{j-1} c^j (1-c)^{k-j-1}}\, a(n-k-2) \geq 0$$

1. We will rewrite the denominator as follows:

$$\sum_{j=0}^{k-1} c^j - \sum_{j=2}^{k-1}\binom{k-1}{j-1} c^j (1-c)^{k-j-1} = 1 + c(1-c)^k$$

Thus we have the following system:

$$\begin{cases} a(n-k)-\dfrac{c(1-c)^k}{1+c(1-c)^k}\,a(n-k-1)\geq 0 \\ \vdots \\ a(n-k)-\dfrac{\sum_{j=2}^{n-k-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}a(j)}{\sum_{j=0}^{n-1}c^j-\sum_{j=n-k}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}}\geq \\ \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^j-\sum_{j=n-k}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}} \\ 0\leq a(n-k)\leq a(n-k-1)\leq\ldots\leq a(2)\leq 1 \end{cases}$$

After $(n-3)$ steps we have:

$$\begin{cases} a(3)-\dfrac{c(1-c)^{n-3}}{1+c(1-c)^{n-3}}\,a(2)\geq 0 \\ a(3)-\dfrac{(n-1)c^2(1-c)^{n-3}a(2)}{\sum_{j=0}^{n-1}c^j-\sum_{j=3}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}}\geq \\ \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^j-\sum_{j=3}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}} \\ 0\leq a(3)\leq a(2)\leq 1 \end{cases}$$

Then in the step $(n-2)$ we eliminate $a(3)$.

$$\begin{cases} a(2)\geq\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \\ 0\leq a(2)\leq 1 \end{cases}$$

It is:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\leq a(2)\leq 1$$

Now working backwards. In order to determinate the variation de $a(3)$, we replace this in the previous systems:

$$\begin{cases} a(3) - \dfrac{c(1-c)^{n-3}}{1+c(1-c)^{n-3}}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right] \geq 0 \\ a(3) - \dfrac{(n-1)c^2(1-c)^{n-3}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right]}{\sum_{j=0}^{n-1} c^j - \sum_{j=3}^{n-1}\binom{n-1}{j-1} c^j (1-c)^{n-j-1}} \geq \\ \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1} c^j - \sum_{j=3}^{n-1}\binom{n-1}{j-1} c^j (1-c)^{n-j-1}} \\ 0 \leq a(3) \leq a(2) \leq 1 \end{cases}$$

Operating in the inequations of the system we obtain:

1. $$a(3) \geq \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$$

Thus we obtain:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(3) \leq a(2) \leq 1$$

In order to determine the variation of the $a(n–k)$ variable with $2 \leq n–k \leq n$, we must replace the variables $a(n-k-1)$, ..., $a(3)$, $a(2)$, with its bounds in the $k-th$ system.

By induction hypothesis we assume that:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(n-k-1) \leq \mathrm{K} \leq a(3) \leq a(2) \leq 1$$

Thus in the $k-th$ system we have:

$$\begin{cases} a(n-k) - \dfrac{c(1-c)^k}{1+c(1-c)^k}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right] \geq 0 \\ \vdots \\ a(n-k) - \dfrac{\sum_{j=1}^{s-k-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right]}{\sum_{j=0}^{s-1}c^j - \sum_{j=s-k}^{s-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}} \geq 0 \\ \vdots \\ a(n-k) - \dfrac{\sum_{j=2}^{n-k-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right]}{\sum_{j=0}^{n-1}c^j - \sum_{j=n-k}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}} \geq \\ \dfrac{c(1-c)^{n-2}}{\sum_{j=0}^{n-1}c^j - \sum_{j=n-k}^{n-1}\binom{n-1}{j-1}c^j(1-c)^{n-j-1}} \\ 0 \leq a(n-k) \leq a(n-k-1) \leq \ldots \leq a(2) \leq 1 \end{cases}$$

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(n-k)$$

Operating as before, we obtain:

1. $a(n-k) \geq \dfrac{\sum_{j=1}^{s-k-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}}{\sum_{j=0}^{s-1}c^j - \sum_{j=s-k}^{s-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}}\left[\dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right]$

We also have that:

$$\frac{\sum_{j=1}^{s-k-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}}{\sum_{j=0}^{s-1}c^j - \sum_{j=s-k}^{s-1}\binom{s-1}{j-1}c^j(1-c)^{s-j-1}} \leq 1$$

Thus we obtain:

$$a(n-k) \geq \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \geq$$

$$\frac{\sum_{j=1}^{s-k-1}\binom{s-1}{j-1}c^{j}(1-c)^{s-j-1}}{\sum_{j=0}^{s-1}c^{j}-\sum_{j=s-k}^{s-1}\binom{s-1}{j-1}c^{j}(1-c)^{s-j-1}}\left[\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right]$$

And we finally obtained:

$$\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a(n-k) \leq a(n-k-1) \leq \ldots \leq a(3) \leq a(2) \leq 1$$