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Benders-based approach for an integrated Lot-Sizing and Scheduling problem

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Abstract. The main concern of the current paper is to present mathematical model and a decision method for production planning issues of a manufacturing organization. We aim at integrating the medium term and the short term as two levels of decision. These consist in periodical planning with determining the intended produced quantity and scheduling the functioning of machines. It is worth noting that in the literature there exist only few works on the issue of integration because of the shortage of numerical results. Thus, the integrated model presented here allows us to take into consideration the scheduling constraints in the Lot-sizing model. A recent algorithm, based on a heuristic approach to find a production planning with a feasible schedule for each period, has recently been published in which the two levels of decision were applied. In this paper, some of these ideas are developed in order to get an optimal solution. For this, an exact algorithm of Benders’ decomposition method is adopted to the integration problem. This has been proved efficient with reliance primarily on modeling view and the link between the two levels of decision and secondly on the numerical view.

Keywords: Integration of Lot - Sizing and Scheduling (LSS) decisions in a job-shop, Integer Programming, Benders decomposition method.

1. Introduction

The traditional hierarchical approach in production management has long been recognized and accepted in practice. It consists of multiple decision levels (usually three: strategic, tactical and operational) with different characteristics. In particular:

- The higher in the hierarchy the more strategic are the decisions,
- The higher in the hierarchy the more aggregate are the models and the longer the time,
- The decision at some level becomes a constraint or an objective to be satisfied at lower levels,
- And each decision level has its own decision models and solving procedures.

Many research teams are interested in the study of the integration of LSS problem in its global aspect. Others were limited to the Lot-Sizing problem while a third group of researchers were interested solely in the Scheduling problem.

This works fits in the first category, whose two levels of decision are integrated- medium and short terms. It’s thus necessary to determine the volumes of production of various products, which correspond to the scheduling determination of different operations of production and their distributions on machines.

The present paper is organized as follows: the main concern of the first section is to present a brief literature review and to describe the prominent characteristics of an integrated issue of production LSS Problem. The second section centers on the problem modeling while the third one deals with the method of resolution. The focus of the last section is on the reporting of the numerical results as well as their interpretations.
2. The integration of Lot-Sizing and Scheduling Problem

2.1 Short literature review

Because of the complexity and the different particularities to which companies are subjected, we find a significant number of alternatives for the initial problem of LSS. They differ in that it depends on the consideration of one or more products, one or more machines, and on whether a set-up time is taken into account during the passage from one product to another. For more detailed presentation of various problems and models, [9] and [10] are illustrative. On the light of the studied problems, different approaches (exact or heuristic) were proposed. The following table gives an exhaustive list of there contributions:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Exacts methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSS problem</td>
<td>[2]</td>
</tr>
<tr>
<td>LSS problem with time and cost preparation</td>
<td>[8]</td>
</tr>
<tr>
<td>Problem</td>
<td>Heuristics</td>
</tr>
<tr>
<td>LSS problem</td>
<td>[3]</td>
</tr>
<tr>
<td>LSS problem with time and cost preparation</td>
<td>[5]</td>
</tr>
</tbody>
</table>

2.2 Characteristics of an integrated problem of production lot sizing and scheduling

As cited below, the issue of decision making in a context where information is exchanged between the tactical and the operational levels is reflected on the characteristics of problems governing each level. So, when dealing with the subject matter, it is worth defining the notion of the 2 decisions levels. To begin with the tactical level, it consists in determining for each product and each considered period the quantity to be produced in an attempt to minimize the total cost of production, surplus and lacks. The constraints that are included in the formulation are those of machines capacity, of non-simultaneity and of the classical inventory balance equation. Besides, when it comes to the operational level, it should be noted that it consists in setting up concrete scheduling taking into account the quantities determined by the resolution of the lot sizing problem. The main issue is to verify the effect of the following constraints: The operation of the same product needs to respect a certain order of transition of products on machines. The respect of the disjunction between two operations that machines cannot carry out simultaneously.

In the present case, the issue of scheduling and more precisely the issue of job-shop is put into light. In the literature, few works on the integration issue were conducted. It is worth noting that the current paper is mainly based on [3] and [4]. For the integration of LSS problem, the adopted strategy is said to be ‘period by period’, where an operation of production associated to a certain period t must start and finish during the same period t. The following figure illustrates an instance of LSS problem:

![An instance of LSS problem.](image)
3. Problem modeling

There are various ways to model the integrated Lot-Sizing and Scheduling problems. So, this section deals with the description of the formulation of the problem.

3.1 Problem statement

Consider the problem in which we want to plan the production of \(i\) products in a job-shop environment, and on a planning horizon of \(T\) periods. Each planning period \(t\) has a capacity of \(d_t\) time units. Let \(d_{it}\) be the demand for each product \(i\) and period \(t\). Let \(X_{it}\) be the production quantity to be processed before the end of period \(t\), and \(I_{it}\) be the inventory level of products \(i\) at the end of period \(t\). Where \(I_{it}\) is replaced by \(I_{it}^+ - I_{it}^-\). Here \(I_{it}^+\) is the inventory (surplus) at the end of period \(t\), and \(I_{it}^-\) represents the quantity of the demand which is not satisfied at the end of period \(t\) (backlogged).

The goal is to minimize a given cost function, depending on the production quantity \(X_{it}\), and the inventory \(I_{it}\) and the quantity of the demand which is not satisfied. Furthermore, \(c_{it}^+\) is a per unit holding cost, \(c_{it}^-\) a per unit backlogging cost, and \(q_{it}\) a per unit production cost.

Hence, we introduce the following notation. So, let:

Variables:

\[ X_{it} \quad \text{Quantity of item } i \text{ available at the end of period } t \]
\[ S_{imt} \quad \text{Start time of operation } o_{imt} \]
\[ y_{ijmt} \quad \text{Boolean variable equal to 1 if operation } o_{imt} \text{ needs to precede operation } o_{jmnt} \text{ and 0, otherwise.} \]
\[ z_{immt} \quad \text{Boolean variable equal to 1 if operation } o_{imt} \text{ needs to precede operation } o_{im't} \text{ and 0, otherwise.} \]

Parameters:

\[ M \quad \text{Set of machines } = \{1, 2, 3, \ldots, m\} \]
\[ P \quad \text{Set of all jobs to process } = \{1, 2, 3, \ldots, i\} \]
\[ p_{im} \quad \text{Processing time of operation } o_{imt} \text{ per unit of production } i \]
\[ I_0 \quad \text{Initial inventory.} \]
\[ o_{imt} \quad \text{Operation corresponding to the passage of the product } i \text{ on the machine } m \text{ during the period } t \]
\[ d_{it} \quad \text{Demand of item } i \text{ at the end of period } t \]
\[ d_t \quad \text{Length of period (available capacity)} \]

3.2 Mathematical model

The MIP is simply formulated as:

\[
\begin{align*}
\text{Min} & \sum_{i \in P} \sum_{t \in T} c_{it}^+ I_{it}^+ + c_{it}^- I_{it}^- + q_{it} X_{it} , \\
\text{subject to} & \quad l_{it}^+ - l_{it}^- = l_{it-1}^+ - l_{it-1}^- + X_{it} - d_{it} , \quad \forall i \in P, t \in T .
\end{align*}
\]
The interpretation of these constraint sets is as follows:

The objective function constraints (1) is to minimize a given cost function, depending on the production quantity, the inventory and the quantity of the demand which is not satisfied.

Constraints (2) present the classical inventory balance equations.

The second set (3) gives the relations between operations to be performed on a same machine.

The constraints (4) force the jobs to be finished at their given due date.

Constraints (5) and (6) represented the precedence relations.

The set of the two following constraints (7) and (8) gives the disjunctive constraints.

Constraints (9) guarantees that \( S_{imt} \) the start time of operation \( s_{imt} \) is performed between periods \( (t-1) \) and \( t \), and does not start before.

Finally, constraints (10) state that the decision variables are on the one hand Boolean and on the other hand nonnegative.

So, we can easily see that this model present two types of variables: the binary and continuous variables. This mixed nature of the variables encouraged us to choose the Benders decomposition method for solving the problem cited above.

We now explain the main ideas of Benders decomposition method.

4. Method of resolution: Benders decomposition

The present section is concerned with the introducing of the proposed method that solves the problem at hand. In the next sub–section, we describe the method with reference to literature and in the second we implement the Benders decomposition to the model described in the third section.
4.1 Main ideas of Benders decomposition

As their name suggests, the decomposition methods have a role to decompose, that is to say separate, one initial problem into two or more sub-problems. It differs in most decomposition methods literature a Master Problem, and one or more Sub-Problems.

The resolution of each sub-problem provides a set of information that is then used by the master problem to bring proceedings to a global optimum.

Benders decomposition [1] and [4] starts from a mixed integer programming of the following form:

\[
\begin{align*}
(P) : \quad & \min \left\{ c^T x + \{d, y\} \right\} \\
& \text{s.t.:} \quad A x + B y \geq e \\
& \quad x \in X, y \geq 0 \\
& \text{with:} \quad c \in \mathbb{R}^n, d \in \mathbb{R}^m, e \in \mathbb{R}^p \\
& \quad A \in \mathbb{R}^{p,n}, B \in \mathbb{R}^{p,m} \\
& \quad X \subset \mathbb{R}^n
\end{align*}
\]

Reformulation in the form of Benders

The method first separates the continuous variables from the other ones, and the continuous part is dualized for practical reasons. We obtain

\[
\begin{align*}
(P) : \quad & \min \left\{ c^T x \right\} + f(x) \\
& \text{s.t.:} \quad x \in X
\end{align*}
\]

With:

\[
\begin{align*}
f(x) = & \max \left\{ e - Ax, u \right\} \\
& \text{s.t.:} \quad B^T u \leq d \\
& \quad u \geq 0
\end{align*}
\]

Using Minkowski-well theorem: So, a non-empty polyhedron \( U = \left\{ u \in \mathbb{R}^n / A^T u \leq c, u \geq 0 \right\} \) can be written as \( U = K + C \), where

\[
K = \left\{ u / u = \sum_{i=1}^{p} \lambda_i u_i^i, \lambda_i \geq 0, \sum_{i=1}^{p} \lambda_i = 1 \right\}, \quad \text{with} (u_i)_{i=1,\ldots,p} \text{ the extreme points of } U
\]

\[
C = \left\{ u / u = \sum_{i=p+1}^{p+q} \lambda_i u_i^i, \lambda_i \geq 0 \right\}, \quad \text{with} (u_i)_{i=p+1,\ldots,p+q} \text{ the extreme rays of } U
\]

and the relations between feasible, unbounded solutions, extreme points and rays, we can reformulate \((P)\) as follows:

\[
\begin{align*}
(P) : \quad & \min \left\{ c^T x \right\} + \max_{i \in \{1,\ldots,p\}} \left\{ e - Ax, u^i \right\} \\
& \text{s.t.:} \quad x \in X \\
& \quad \left\{ e - Ax, u^i \right\} \leq 0, i = p+1,\ldots,p+q
\end{align*}
\]

Or

\[ (P): \min \{c, x\} + x_0 \]
\[ \text{s.t.:} \quad x \in X \]
\[ \begin{align*}
& e - A x, u \leq 0, i = p + 1, \ldots, p + q \\
& x_0 \geq e - A x, u^i, i = 1, \ldots, p
\end{align*} \]

The method

This new formulation has usually less variables than the original one, but more constraints. Benders decomposition is a relaxation where these constraints are generated iteratively.

Given the Master Problem \((MP)\):

\[ (MP): \min \{c, x\} + x_0 \]
\[ \text{s.t.:} \quad x \in X \]
\[ \begin{align*}
& e - A x, u \leq 0, i = p + 1, \ldots, p + \mu \\
& x_0 \geq e - A x, u^i, i = 1, \ldots, \eta
\end{align*} \]

with: \( \mu \leq q, \eta \leq p \)

One computes at each iteration an optimal solution \(\overline{x}\) of \(MP\). Then the Sub Problem \(SP\) can be solved, giving an optimal solution \(\overline{x}_0\).

\[ (SP): \max \{e - Ax, u\} \]
\[ \text{s.t.} \quad B^t u \leq d \]
\[ u \geq 0 \]

If \(\overline{x}_0\) is unbounded (resp. bounded) then an extreme ray (resp. point) is found and the corresponding cut is added to the \(MP\).

An optimal solution of the Problem \(P\) is found when the cut is not violated by the current solution \(\overline{x}, \overline{x}_0\).

We now apply this decomposition to the integration of LSS problem using the mathematical model presented in the previous section and we explain the improvements made to the classical method.

4.2 Implementation

The Benders’ decomposition method is appropriate for the resolution of mathematical problems in mixed variables, the structure of which is enabling to fix a few variables. This procedure leads to an iterative algorithm of generating constraints which after successive iteration give either an optimal solution or else the conclusion that the problem cannot have a feasible solution.

This method consists in firstly decomposing the problem into two other problems (a primal master problem and a sub-problem) and secondly reducing the original problem to one or more sub-problems much easier to solve. Applied to the integration problem, each iteration of this procedure consists of two parts:

- Solving an optimization problem defined only by variables of scheduling, called Master Problem.
- Solving a Sub-Problem leading to constraints generation of the Lot- Sizing problem.

4.2.1 Master Problem \([MP]\)

It consists in determining scheduling of pairs of disjuncted operations. We have introduced an auxiliary variable \(y_0\) to define the Relaxed MP (RMP) which is formulated as follows:
\[ \min_{y \in Y, y_0 \in R} y_0 \quad (31) \]

\[ y_{ijmt} + y_{jimt} = 1, \quad \forall i, j \in P; i \neq j, m \in M, t \in T. \quad (12) \]

\[ y_{ijmt} \in \{0,1\}, \quad \forall i \neq j \in P, t \in T, m \in M. \quad (43) \]

### 4.2.2 The Sub Problem [SP]

It consists in determining the optimal plan of production (volumes of production) and it is obtained by the resolution of the RMP through fixing the \( y \) variables to some \( \bar{y} \) ones [SP] is stated as:

\[ \text{Min} \sum_{i \in P} \sum_{t \in T} c_{it}^+ l_{it}^+ + c_{it}^- l_{it}^- + q_{it} x_{it}. \quad (54) \]

\[ l_{it}^+ - l_{it}^- = l_{it}^+ - l_{it-1}^- + x_{it} - d_{it}, \quad \forall i \in P, t \in T. \quad (15) \]

\[ \sum_{i \in P} p_{im} x_{it} \leq d_t, \quad \forall m \in M, t \in T. \quad (16) \]

\[ s_{int} + p_{im} x_{it} \leq \sum_{t=1}^{t} d_{t'}, \quad \forall i \in P, t \neq t' \in T, m \in M. \quad (17) \]

\[ s_{int} + p_{im} x_{it} \leq s_{jmt} + (1 - z_{immt}) d_{t}, \quad \forall m \neq m' \in M, i \in P, t \in T. \quad (18) \]

\[ z_{jimt} + z_{immt} = 1, \quad \forall m \neq m' \in M, i \in P, t \in T. \quad (19) \]

\[ s_{int} + p_{im} x_{it} \leq s_{jmt} + M \bar{y}_{jimt}, \quad \forall i, j \in P; i \neq j, m \in M, t \in T. \quad (20) \]

\[ s_{int} \geq \sum_{t=1}^{t-1} d_{t'}, \quad \forall i \in P, t \neq t' \in T, m \in M. \quad (21) \]

\[ x_{it}, l_{it}^+, l_{it}^- \geq 0, \quad \forall i \in P, t \in T. \quad (22) \]

The Benders decomposition algorithm iterates between solving the SP and the RMP. The latter yields the \( \bar{y} \) variables for SP that generates new extreme solution as used to construct Benders cuts which are to be included in RMP

\[ y_o \geq \nu(y) + M \sum_{j \in P; m \in M} \left( y_{jmt} - \bar{y}_{jimt} \right) \bar{u}_{jmt}^* \quad (\ast). \]
The Benders decomposition method is carried out according to two versions: the traditional Benders in accordance with the literature and the Benders with the introduction of constraints, which are added in the initial RMP, serving to eliminate the cycles $\sum_{(\text{int}, \text{int}) \in \mathcal{E}} y_{\text{int}} \leq |\mathcal{I}| - 1$ (**).

The following figure illustrates the structure of the method of Benders decomposition.

![Fig.2. Structure of the Benders decomposition method.](image)

5. Computational results

This section presents the computational experiments obtained by the Benders decomposition to illustrate the effectiveness of the proposed method described in this paper. Since we are not aware of existing examples in the literature, we have selected static instances of the famous job-shop problem with 6 jobs and 6 machines duplicated in three periods, given respectively in [3] and [2].

The selected method is one of the default search goal of the ILOG CPLEX, especially ILOG Scheduler library. We have coded the method in C++.

To evaluate the efficiency of this framework we compare the results obtained by the proposed method with reference to two versions of the Benders decomposition.

The computational results obtained are displayed on these following tables for set of instances. The following table illustrates the Lower and the Upper Limits (LL, UL) as well as the gap for the Branch and Bound and the 2 versions of the Benders (Traditional Benders and Benders without cycle):
Table 2: The Lower and Upper Bound for the Branch and Bound and the 2 versions of Benders

<table>
<thead>
<tr>
<th>Instances</th>
<th>Branch and Bound</th>
<th>Traditional Benders</th>
<th>Benders without cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LL</td>
<td>UL</td>
<td>Gap</td>
</tr>
<tr>
<td>1</td>
<td>195,68</td>
<td>330,09</td>
<td>40.71%</td>
</tr>
<tr>
<td>2</td>
<td>196,05</td>
<td>493,87</td>
<td>60.30%</td>
</tr>
<tr>
<td>3</td>
<td>237,97</td>
<td>338,9</td>
<td>29.78%</td>
</tr>
<tr>
<td>4</td>
<td>308,15</td>
<td>540,01</td>
<td>42.93%</td>
</tr>
<tr>
<td>5</td>
<td>296,27</td>
<td>564.34</td>
<td>47.51%</td>
</tr>
<tr>
<td>6</td>
<td>118,09</td>
<td>302,97</td>
<td>61.02%</td>
</tr>
<tr>
<td>7</td>
<td>117,53</td>
<td>379,10</td>
<td>68.99%</td>
</tr>
</tbody>
</table>

- Always, the Lower Limits is negative for the 2 versions of Benders.
- For the 7 instances, the Lower Limits is always lower than the Upper Limits for the Branch and Bound of Cplex and also for the 2 versions of Benders.
- In comparison with the traditional Benders, the method of Benders without cycle yields the best Limits for the Upper and Lower Limits.
- The Lower limits of traditional Benders are the worst because of the great number of cycles to be eliminated.

So, on the basis of Lower and Upper Limits criteria, the Benders decomposition method with constraints that eliminate the cycles is more efficient than traditional Benders.

The most convincing result of the comparisons between the two versions consists of the time execution of released MP and SP.

For this we indicate in the next following table the results obtained by the 2 versions of the proposed method:

Table 3: Time of resolution in (s) and numbers of the generated cuts for the 2 versions of Benders

<table>
<thead>
<tr>
<th>Instances</th>
<th>Traditional Benders</th>
<th>Benders without cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MP</td>
<td>SP</td>
</tr>
<tr>
<td>1</td>
<td>10266.5</td>
<td>46</td>
</tr>
<tr>
<td>2</td>
<td>10117.10</td>
<td>46.78</td>
</tr>
<tr>
<td>3</td>
<td>10083</td>
<td>42.6</td>
</tr>
<tr>
<td>4</td>
<td>10428</td>
<td>44.87</td>
</tr>
<tr>
<td>5</td>
<td>10545</td>
<td>43.38</td>
</tr>
<tr>
<td>6</td>
<td>10520</td>
<td>45.58</td>
</tr>
<tr>
<td>7</td>
<td>10569.8</td>
<td>38.24</td>
</tr>
</tbody>
</table>

- The resolution time of MP and SP in traditional Benders is generally higher than the resolution time of released MP with the constraints that eliminate the cycle and SP. This result is explained by the greatest number of constraints of MP.
- The number of cuts generated by the traditional Benders decomposition is more important with comparison to Benders without cycle.
- The significant time necessary to the resolution of the MP in Benders without cycle cannot be exploited by the low number of generated cuts.

On the basis of the resolution time of MP and SP as well as the numbers of cuts generated criteria, the Benders decomposition method with constraints that eliminate the cycles is more efficient than traditional Benders.

4. Conclusion

In this paper, we were interested in the study of the Benders decomposition approach as well as the resolution method for the production of integrated LSS problem. Two versions of the Benders decomposition were proposed. The first confirms with the literature and is said to be the “classical Benders”. The second is Benders decomposition with the introduction of the constraints that eliminate the cycles into the RMP and is said to be the “Benders without cycles”.

The use of Benders’ method is original. Despite the variety of the integration LSS problem, we could not find classical instances in the literature. Therefore, we have chosen the same instances which are commonly utilized. With regard to the upper and lower bounds, the resolution time and the numbers of generated cuts, the results obtained show that the Benders decomposition method without cycle are more efficient than those in traditional Benders.

What is worth noting is the limitation of the Benders' method in that it is adopted to solve the integration problem with only small sized-problems. That is, the Benders' method does not provide as many feasible results as those provided by heuristic methods.

The results could have been more significant when large data and a concrete rather than a hypothetical company are adopted. These ideas could be taken into account in a future research.

References