Elamine Talbi, Mohamed; Benayat, Djilali
The homotopy exact sequence of a pair of graphs
Universidade Estadual de Maringá
Maringá, Brasil

Available in: http://www.redalyc.org/articulo.oa?id=303228848017
The homotopy exact sequence of a pair of graphs

Mohamed Elamine Talbi and Djilali Benayat

Department of Mathemathiques, Universite Saad Dahlab, Bida, Algeria. 2Ecole Normale Superieure Vieux-Kouba, Alger, Algeria. *Author for correspondence, E-mail: talbi_mea@yahoo.fr

ABSTRACT. Higher homotopy of graphs has been defined in several articles. However, the existence of a long exact sequence associated to a pair \((G, A)\) has not been touched at. We treat it here. Applied to the discrete spheres, this lead to interesting open questions.

Keywords: graph, homotopy, exact sequence.

A sequência exata de homotopia de um par de gráficos

RESUMO. A homotopia superior de gráficos foi definida em vários artigos. Todavia, a existência de uma longa sequência exacta associada a um par \((G, A)\) de gráfico não foi abordada. Vamos tratar disto aqui. Aplicada às esferas discretas, isto levanta interessantes questões abertas.

Palavras-chave: gráficos, homotopia, sequência exata.

Introduction

Homotopy groups of graphs have been defined in Benayat and Kadri (1997) and Babson et al. (2006). One of the main property of any consistent homotopy theory is the existence of a long exact sequence associated to any based pair \((G,H,v_0)\) of graphs. In this paper, we define a relative homotopy theory and prove such a sequence exists. We give some immediate applications and formulate two important open questions related to the homotopy of the discrete spheres.

Definitions and notations

As usual, an undirected graph is a pair \(G = (V,E)\) where \(V = V(G)\) and \(E = E(G)\) are, respectively, the sets of vertices and edges of \(G\). We consider only simple (i.e. without multiple edges) and connected graphs. The neighborhood (or the neighbors) of a vertex \(v \in V\) is defined by the neighborhoods \(N(a) = \{b \in V : \{a,b\} \in E\}\). We use the same letter \(N\) for all graphs; this will cause no confusion. We will adopt the convention that any vertex is a neighbor of itself: \(\forall a \in G\) then \(a \in N(a)\). Recall that a graph can be defined by giving the neighbors of its vertices. We write \(N^k(G)\) (or just \(N^k\)) for all the neighborhoods of the graph \(G\). A pair \((G,A)\) of graphs is just a graph \(G\) and a subgraph \(A\). A morphism \(f : G \rightarrow G'\) is an application \(f : V(G) \rightarrow V(G')\) such that \(f(N(a)) \subset N(f(a))\) for any \(a \in G\). Graphs and morphisms define a category \(\mathcal{C}\). A morphism of pairs of graphs \(f : (G,A) \rightarrow (G',A')\) is a morphism \(f : G \rightarrow G'\) such that \(f : A \rightarrow A'\) is also a morphism. A based pair (of graphs) is a pair \((G,A)\) and a distinguished vertex \(x_0 \in A\). A morphism of based pairs sends base vertices to base vertices.

The set \(\mathcal{G} (G,G')\) of morphisms can be endowed with a natural graph structure. If \(f, g : G \rightarrow G'\) are morphisms, they are contiguous if \(\forall a \in V(G)\), \(\exists \sigma \in N^* (G')\), depending upon \(a\), such that \(f(N(a)) \cup g(N(a)) \subset \sigma\). The graph structure on \(\mathcal{G} (G,G')\) is defined by the neighborhoods \(N(f) = \{g \in \mathcal{G} (G,G') : f \text{ and } g \text{ are contiguous}\}\).

The infinite path (graph) is \(\square\) with the neighborhoods \(N(m) = \{m-1,m,m+1\}, \forall m \in \mathbb{N}\). We will also consider the subgraphs \(\square_a\) and, if \(p < q\), \([p,q] = \{p,p+1,...,q\}\) of \(\square\). The particular graph \([0,m]\) will be written \(I_m\).

The graphs \(\square\), \(\square_a\) and \(I_m\) are based by \(0\). A sequence in \(G\) is a morphism \(s : N \rightarrow G\); it is convergent if \(\exists m \in \mathbb{N}\) such that \(s(n) \in N(s(m)), \forall n \geq m\). Notice that, if we put \(x_n = s(n)\), then \(x_n \in N(x_{n-1})\) for \(n \geq 1\). As in topology, we have the following result.

Proposition 1. The application \(\mathcal{I} \rightarrow G'\) is a morphism \(\Leftrightarrow\) for any convergent sequence \(s\) in \(G\), then \(f \circ s\) is convergent in \(G'\).

Proof. The necessary part is obvious. We prove the converse by contradiction. Assuming \(f\) is not a morphism, then there exists \(x_n \in G\) such that \(f(N(x_n)) \not\subset N(y_n)\) where \(y_n = f(x_n)\). Let \(x_n \in N(x_{n-1})\) and \(y_{n+1} = f(x_n) \not\in N(y_n)\); consider the sequence \(s(2n) = x_{n}\) and...
Higher homotopy of graphs

If $I^n_n = I_n \times \cdots \times I_n$ its boundary is:

$$\partial I^n_n = \left\{ (p_1, \ldots, p_m) \in I^n : \text{ at least one of the } p_i s \text{ is 0 or } m \right\}$$

Let $(G,x_0)$ be a based graph. An n-spheroid of $(G,x_0)$ is a morphism $\gamma : I^n_n \to G$ such that $\partial I^n_n = \{x_0\}$ and we put $\Omega^n(G,x_0)$ for the set of all of them. We can assume that any finite number of n-spheroids are defined on the same $I^n_n$. by extending them outside their domain by the constant value $x_0$. If $\gamma_i : I^n_n \to G$, $i = 1, 2,$ are two n-spheroids, they are homotopic, written $\gamma_1 \sim \gamma_2$, if there is a morphism $H : I^n_n \times I^n_n \to G$ such that $H(s,0) = \gamma_1(s)$, $H(s,p) = \gamma_2(s)$, $s \in I^n_n$, and $H(\partial I^n_n,t) = \{x_0\}$, $t \in I^n_n$. Homotopy is an equivalence relation on $\Omega^n(G,x_0)$. The $n^{th}$ homotopy group of the based graph $(G,x_0)$ is defined as $\pi_n(G,x_0) = \Omega^n(G,x_0)/\sim$. It is known that $\pi_n(G,x_0)$ is a group for $n \geq 1$ which is abelian for $n \geq 2$. The following results have been proved in Benayat and Kadi (1997) and Babson et al. (2006).

**Proposition 2.** 1) We have the isomorphism: $\pi_n(G,x_0) = \pi_1(\Omega^n(G,x_0),c_0)$, $n \geq 2$, where $c_0$ is the constant spheroid at $x_0$; 2) $\pi_1(G,x_0)$ is abelian for $n \geq 2$.

**Definition 1.** A graph $G$ is simply connected if $\pi_1(G,x_0) = \{0\}$ and n-connected ($n \geq 2$) if $\pi_1(G,x_0) = \{0\}$, $1 \leq j \leq n - 1$ and $\pi_1(G,x_0) \neq \{0\}$, for some base vertex $x_0$.

Relative homotopy

Let $(G,A,x_0)$ be a based pair of graphs. For $n \geq 1$, we put:

$I^{n+1}_w = I_n \times \cdots \times I_n \times \{0\}$

and

$$\partial I^{n+1}_w = \left\{ (q_1, \ldots, q_n) \in I^n : \text{ (one of the } q_j s \text{ is 0 or } m \text{ for } 1 \leq j \leq n - 1 \text{ or } (q_n = m)) \right\}$$

We have $\partial I^{n+1}_w = I^n \cup \partial I^n$.

**Definition 2.** An n-spheroid of $(G,A,x_0)$ is a morphism of triples $\gamma : (I^n, I^{n+1}, \partial I^n) \to (G,A,x_0)$.

**Remark 1.** We have $\gamma(I^n) \subseteq A$, $\gamma(I^{n+1}) \subseteq \Omega^n(A,x_0)$ and $\gamma(\partial I^n) = \{x_0\}$. When $A = \{x_0\}$ we get back $\Omega^n(G,x_0)$.

We write $\Omega^n(G,A,x_0)$ for the set of n-spheroids of $(G,A,x_0)$. Let us recall that any finite number of n-spheroids can and will be defined on the same $I^n_n$ which will be written $I^n$.

**Definition 3.** Let $\gamma_i : (I^n, I^{n+1}, \partial I^n) \to (G,A,x_0)$, $i = 1, 2$, be two n-spheroids. They are homotopic if there is a morphism $H : I^n \times I^n \to G$ such that $H(0,0) = \gamma_1$, $H(0,p) = \gamma_2$ and $H(\partial I^n, t) \in \Omega^n(G,A,x_0)$, $t \in I^n$.

Homotopy of relative spheroids is an equivalence relation on $\Omega^n(G,A,x_0)$; the set of equivalence classes is written $\pi_n(G,A,x_0)$. The n-spheroid $\gamma \in \Omega^n(G,A,x_0)$ is nullhomotopic if it is homotopic to a $\gamma' \in \Omega^n(G,A,x_0)$ such that $\gamma'(0) = A$.

**Law of composition in $\pi_1(G,A,x_0)$, $n \geq 2$**

Let $\gamma_i : (I^n_n, I^{n+1}_n, \partial I^{n+1}_n) \to (G,A,x_0)$, $i = 1, 2$, be representatives of two homotopy classes $\tilde{\gamma}_i$, $i = 1, 2$, in $\pi_1(G,A,x_0)$. Let $I^n$ be the domain shown underneath where the second square is actually a translated copy of $I^n$. The base $I^{n+1}$ is just the juxtaposition of $I^n$ and a translation of itself.
We define $\gamma_i \ast \gamma_j$ as the morphism $\gamma : J^n \to (G, A, x_0)$ which is $\gamma_i$ on $I^n$ and $\gamma_j$ on the translated of $I^j$. It is easy to show that the homotopy class $\bar{\gamma}$ of $\gamma$ depends only upon the homotopy classes $\bar{\gamma}_i$,\ $i.e.$ two vertices $\bar{\gamma}$ depends only upon the homotopy classes $\bar{\gamma}_i$.

**Proposition 3.** The $\Pi_n(G, A, x_0)$ are sets for $n = 1$, groups for $n \geq 2$ which are abelian for $n \geq 3$.

Proof. The associativity of the law is easy but lengthy; we omit it. The class of the constant spheroid at $x_0$ is clearly the identity element. Let us show that every class has an inverse. We assume $n \geq 2$. Let $\bar{\gamma} : (I^n, I^{n-1}, S^n_{\infty}) \to (G, A, x_0)$ and define $\bar{\gamma} : I^n \to G$ by $\bar{\gamma}(q_1, \ldots, q_n) = \gamma(m - q_1, q_2, \ldots, q_n)$ for $(q_1, \ldots, q_n) \in I^n$. The composed morphism $\gamma \ast \bar{\gamma}$ is defined on $K = I_m \times I_n \times \cdots \times I_n$. We have $(\gamma \ast \bar{\gamma})(K^{n-1}) \subset A$ and $(\gamma \ast \bar{\gamma})(\delta K) = \{x_0\}$. So $\gamma \ast \bar{\gamma}$.

is an $n$-spheroid of $(G, A, x_0)$. Let us show that it is nullhomotopic. So let us consider $H : K \times I_n \to G$ defined by:

$$H(q_1, s, t) = \begin{cases} \gamma(q_1, s) & \text{if } q_1 \leq t \\ \gamma(t, t) & \text{if } t \leq q_1 \leq m \\ \gamma(m - q_1, s) & \text{if } 2m - t \leq q_1 \leq 2m \end{cases}$$

where $s = (q_2, \ldots, q_n)$. The application $H$ is a morphism since all parts of its definition glue together. Moreover, for $t = 0$, we have $H(q_1, s, 0) = \gamma(0, q_2, \ldots, q_n) = \{x_0\}$ and, for $t = m$, we have where $q = (q_1, q_2, \ldots, q_n)$. For all $t \in I_n$, the values of $H(t, q_1, s, t)$ are given in term of values of $\gamma$ and, consequently, define an $n$-spheroid of $(G, A, x_0)$. So is homotopic to the constant $n$-spheroid.

We show that $\Pi_n(G, A, x_0)$ is abelian for $n \geq 3$. Let $\gamma_1 : I^m \to G$ and $\gamma_2 : I^m \to G$ two elements of $\Omega^m(X, A, x_0)$. The following displacements of the domains $I^m$ and $J^m$ in the hyperplane $q_n = 0$ and around the axis $(0, \ldots, 0) \times I$ define a homotopy between $\gamma_1 \ast \gamma_2$ and $\gamma_2 \ast \gamma_1$. Points of the empty space are sent to $x_0$. This has been possible only because of the extra degree of freedom allowing rotation around the axis $0 \times I$.

**Meaning of $\Pi_0[G, x_0]$**

Elements of $\Pi_0[G, x_0]$ are homotopy classes of morphisms $\gamma : \{0+ \ast \gamma : \{0\}, 0) \to (G, A)$; two such morphisms $\gamma$ and $\rho$, i.e. two vertices $\gamma(0)$ and $\rho(0)$ of $G$, are homotopic if there is a path in $G$ joining them. So $\Pi_0[G, x_0]$ is in bijection with the (path) connected components of $G$.

**The homotopy exact sequence**

Let $(G, A, x_0)$ be a based pair of graphs, $i : (A, x_0) \subset (G, x_0)$ and $j : (G, x_0) \to (G, A, x_0)$ be the obvious inclusions which are morphisms. By functoriality of the $\Pi_n$ we get homomorphisms $i_n : \Pi_n[A, x_0] \to \Pi_n[G, x_0]$ and $j_n : \Pi_n[G, x_0] \to \Pi_n[G, A, x_0]$ for $n \geq 2$; $i_n$ is also a homomorphism when $n = 1$. We define a boundary operator $\partial : \Pi_n[G, A, x_0] \to \Pi_{n-1}[G, x_0]$ as follows: $\partial[\gamma] = \partial_0[\gamma]$ where $\gamma : \{0\} \to (G, A, x_0)$ is an element of $\alpha^{n-1}(G, A, x_0)$. We get a long homotopy sequence:

$$\cdots \to \Pi_n[A, x_0] \xrightarrow{i_n^{-1}} \Pi_n[G, x_0] \xrightarrow{j_n} \Pi_n[G, A, x_0] \xrightarrow{\partial} \Pi_{n-1}[G, x_0] \xrightarrow{i_{n-1}^{-1}} \cdots$$

$$\cdots \to \Pi_n[A, x_0] \xrightarrow{i_n^{-1}} \Pi_n[G, x_0] \xrightarrow{j_n^{-1}} \Pi_n[G, A, x_0] \xrightarrow{\partial} \Pi_{n-1}[G, x_0] \xrightarrow{i_{n-1}^{-1}} \cdots$$

**Theorem 1.** The long homotopy sequence of a based triple $(G, A, x_0)$ is exact in degrees $n \geq 1$.

Proof. 1) $j \circ i_n = 0$ : let $\gamma : I^m \to A$ represents a class in $\Pi_n[A, x_0]$. Then $j \circ i_n[\gamma] : (I^m, I^{m-1}, 0) \to (G, A, x_0)$ represents the null class in $\Pi_n[G, A, x_0]$ since its image is already in $A$.

2) $\partial \circ j_n = 0$ : let $\gamma : I^m \to G$ defining a class in $\Pi_n[G, x_0]$. Then $\gamma \ast j_n : I^m \to \{x_0\}$ is a constant map whose class is 0.
3) $i \circ \tilde{\vartheta} = 0$: let $\gamma : I^* \to G$ represents a class in $\Pi_{n}(G,A,x_0)$. Then $\gamma = i \circ \vartheta = : I^* \to \Delta$ represents $i \circ \tilde{\vartheta}$. The map $H = \gamma : I^* \times I^* \to I^*$ is a homotopy between $H(a,0) = \gamma (a)$ and $H(a,m) = x_0$.

4) $\operatorname{Ker}(j) \subset \operatorname{Im}(i)$: let $\gamma : I^* \to G$ represents a class in $\Pi_{n}(G,A,x_0)$ homotopic, as an element of $\Omega^n(G,A,x_0)$, to a morphism $\delta : I^* \to A$; thus we have $[\gamma] = [i \circ \delta] = [\delta]$

5) $\operatorname{Ker}(i) = \operatorname{Im}(\delta)$: we consider an element $\gamma : I^* \to A$ of $\Omega^n(G,A,x_0)$ whose image $\gamma : I^n \to G$ of $\Omega^n(G,A,x_0)$ is nullhomotopic and let $K : I^* \times I^* \to G$ be such a homotopy. So $K(\alpha,p) = x_0$, $\forall \alpha \in I^*$. The restriction of $K$ to $I^* = 1$ is precisely $\gamma$ and the rest of the border of $I^* \times I^*$ is sent to $x_0$. We have gotten an element in $\Omega^n(G,A,x_0)$ whose image is $\gamma$.

6) We prove, now, exactness at $\Pi_{n}(G,A,x_0)$. Let $\gamma : (I^*,I^*,\delta^*) \to (G,A,x_0)$ an element of $\Omega^n(G,A,x_0)$ such that $\gamma = \gamma_{\gamma} : (I^*,0) \to (A,x_0)$ is homotopic, in $A$ to the constant loop $x_0$ and let $H : I^* \times I^* \to A$ such a homotopy where, for convenience, we take $I^* = \{-p,-p+1,...,0\}$. So we have:

$$H(q_1,...,q_n,0) = \gamma_0(q_1,...,q_n),$$

$$H(q_1,...,q_n,s) \in A, \forall s \in I^*;$$

$$H(q_1,...,q_n,0-p) = x_0, \forall (q_1,...,q_n) \in I^n.$$

We have to show that $\gamma$ is homotopic in $G$ relatively to $A$ to an $(n+1)$-spheroid $\omega \in \Omega^n(G,x_0)$. Let us consider the following extension $\tilde{\gamma}$ of $\gamma$ to $J = I^{n+1} \cup I^* \times I^*$:

$$\tilde{\gamma}(q_1,...,q_{n+1}) = \gamma(q_1,...,q_n)$$ if $(q_1,...,q_n) \in I^{n+1};$

$$\tilde{\gamma}(q_1,...,q_n,0) = \gamma(q_1,...,q_n,0)$$ if $(q_1,...,q_{n},t) \in I^n \times I^*.$

It is clear that $\gamma$ and $\tilde{\gamma}$ are homotopic in $\Omega^n(G,A,x_0)$. Let us construct a relative homotopy, that is in $(G,A)$, between $\tilde{\gamma}$ and an $(n+1)$-spheroid $\omega \in \Omega^n(G,x_0)$.

For $t \in [0,p]$ we define:

$$K(q_1,...,q_{n+1},t) = \gamma(q_1,...,q_n),$$

$$K(q_1,...,q_n,s,t) =$$

$$\begin{cases} H(q_1,...,q_n,-t+s+p) & p \leq s \leq t-p \\ H(q_1,...,q_n,0) & t-p \leq s \leq 0 \end{cases}$$

For each value of $t$, the application $K : J \times I^* \to G$ is an element of $\Omega^n(G,A,x_0)$. For $t=0$, we start with $\gamma_0$ and we end up, for $t=p$, with $K(\cdot,p)$ which is in $\Omega^n(G,x_0)$ since all the boundary of $J$ is sent to $(x_0)$.

**Corollary 1.**

a) If $A$ is contractible, then $\Pi_{n}(G,A,x_0) \to \Pi_{n}(G,A,x_0)$ is an isomorphism for $n \geq 2$.

b) If $G$ is contractible, then $\Pi_{n}(G,A,x_0) \to \Pi_{n}(G,A,x_0)$ is an isomorphism for $n \geq 2$.

Proof. These are direct consequences of the exact sequence.

**Proposition 4.** The image of the homomorphism $\Pi_{n}(G,A,x_0) \to \Pi_{n}(G,A,x_0)$ is included in the center of $\Pi_{n}(G,A,x_0)$.

Proof. Let $\gamma : I^* \times I^* \to G$ be a 2-spheroid of $(G,x_0)$ representing a class in $\Pi_{n}(G,A,x_0)$ and $\lambda : I^* \times I^* \to G$ representing a class in $\Pi_{n}(G,A,x_0)$. We have the following representation for $\int_{\gamma} \gamma + \lambda$:

Using the same trick we used before, we can move the domain of $\gamma$ everywhere in $\square \times \{0\}$ since $\partial(I^* \times I^*)$ is sent to $\{x_0\}$ by $\gamma$. 

---

This is not the case for $\delta$ which is constrained by $\delta(I_n \times [0]) \subseteq A$. So we move the domain of $\gamma$ round the domain of $\delta$. All these moves are globally a homotopy between $j_*\gamma \ast \lambda$ and $\lambda \ast f_*\gamma$.

**Example 1.** The path graph $I_n = [0, m]$ is contractible; a homotopy of the identity map with a constant is $H : I_n \times I_n \to I_n$ given by:

$$H(n, t) = \begin{cases} 0 & \text{if } n < t \\ n - t & \text{if } n \geq t \end{cases}$$

**Proposition 5.** Let $G = \sqcup_{k \in N} A_k$ where, for any $k, A_k$ is contractible and $A_k \subseteq A_{k+1}$. Then $\Pi_n[G, x_0] = 0, \forall n \geq 1$.

Proof. Let $\gamma : I^n \to G$ be an n-spheroïd at $x_0$. Then, there is $k \in N$, such that $\gamma(I^n) \subseteq A_k$. The latter being contractible, $\gamma$ is homotopic to a constant.

Since $Z^n = \lim\left[-k, k\right]^n$, we deduce:

**Corollary 2.** We have $\Pi_n [Z^n, 0] = 0, \forall n \geq 1$ and $m \geq 1$.

**The discrete $n$-sphere**

The circle with $m$ vertices is the quotient graph $S^1_m = [0, m]/\langle 0, m \rangle$. A visual representation of $S^1_4$ is:

![Image](image)

A graph $G = (V, E)$ is trivial if $\exists v \in V$ such that $N(v) = V$. In particular, they are contractible and have trivial homotopy. All the complete graphs are trivial. So the minimal non-trivial circle is $S^1_2$.

**Suspension of a graph**

Let $G = (V, E)$ be a graph; the suspension $SG$ of $G$ is the quotient of $G \times [0, 2]$ by the relation $G \times \{0\} = S$ and $G \times \{2\} = N$:

$$SG = G \times [0, 2]/\langle G \times \{0\}, G \times \{2\} = S, G \times \{1\} = N \rangle$$

A useful remark is that a suspension $SG$ is a union $S^+G \cup S^-G$ of two contractible subgraphs which are the neighborhoods of the 'opposite' poles $N$ and $S$, and whose intersection is $G$.

We define the discrete $n$-sphere as $S^n = S\left(S^{n-1}\right)$ starting with the circle $S^1 = S_1^1$.

**Proposition 6.** Let $f : G \to H$ be a morphism of graphs. Then we have a natural morphism $Sf : SG \to SH$.

Proof. We define $Sf$ as follows: $Sf(x, 1) = (f(x), 1), \forall x \in G$, and North and South poles goes to the same named vertices respectively. It is clearly a morphism of graphs.

**Proposition 7.** We have a canonical homomorphism of suspension:

$$S : \Pi_n [G, x_0] \to \Pi_n [SG, x_0], n \geq 1.$$}

Proof. The suspension homomorphism is the result of the following compositions:

$$\Pi_n [G] \xrightarrow{\varepsilon} \Pi_n [SG, G] \xrightarrow{\delta} \Pi_n [SG, S], \Pi_n [SG]$$

The first isomorphism comes from the HES (homotopy exact sequence) applied to the pair $(S^+G, G)$ and the contractibility of $S^+G$; a similar argument gives the second isomorphism. The middle homomorphism comes from functoriality applied to the inclusion of pairs $(S^+G, G) \subseteq (SG, S^+G)$.

In particular, we have homomorphisms of suspension $S_n : \Pi_n [S^+, 1] \to \Pi_n [S^{n+1}, 1], n \geq 1$. As mentioned before, we have, in several ways, $S^n = S^+ \cup S^-$ with $S^+ \cap S^- = S^{n-1}$. The number of vertices and edges of $S^n$ is $2(n + 1)$ and $2n(n + 1)$ respectively. Applying the HES (homotopy exact sequence) to the pair $(S^n, S^n)$ and using results from Benayat and Kadri (1997), we get the exact sequence:

$$\Pi_n [S^+, 1] = \{0\} \xrightarrow{i_*} \Pi_n [S^2, 1] \xrightarrow{i_*} \Pi_n [S^2, S^+, 1] \xrightarrow{\delta} \Pi_n [S^1, 1] \approx \mathbb{Z} \xrightarrow{0}$$
Conclusion

By analogy with topology, there is a strong indication that \( \pi_n[S^r,1] \neq \emptyset \). This leads to the following future work:

1) Use technology to prove the latter isomorphism.

2) We have defined (not yet published) a notion of discrete fibration of graphs and proved the existence of a long exact sequence. Use the topological Hopf fibration \( S^r \rightarrow S^r \rightarrow S^s \) to construct a combinatorial version of it. This would imply, using the mentioned exact sequence that \( \pi_n[S^r,1] = 0 \) and \( \pi_n[S^r,1] = \pi_n[S^r,1] \).

3) Do we have \( \pi_n[S^r,1] \neq \emptyset \) for \( n > 1 \).

This would show that the discrete spheres can play the role of \( n \)-dimensional holes in a graph and that the homotopy of graphs is able to detect them.

4) Ultimately, compute the \( \pi_n[S^r,1] \) for small values of \( m \) and \( n \), and compare them with the topological homotopy groups of spheres.

References


Received on January 19, 2012.
Accepted on January 28, 2013.

License information: This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.