Arteaga, Carlos; Alves, Alexandre
A note on the connectedness locus of the families of polynomials $P_c(z) = z^n - cz^n-j$
Anais da Academia Brasileira de Ciências, vol. 84, núm. 1, 2012, pp. 5-8
Academia Brasileira de Ciências
Rio de Janeiro, Brasil

Available in: http://www.redalyc.org/articulo.oa?id=32721622002
A note on the connectedness locus of the families of polynomials $P_c(z) = z^n - cz^n - j$

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Manuscript received on November 22, 2010; accepted for publication on June 10, 2011

ABSTRACT

Let $j$ be a positive integer. For each integer $n > j$ we consider the connectedness locus $M_n$ of the family of polynomials $P_c(z) = z^n - cz^n - j$, where $c$ is a complex parameter. We prove that $\lim_{n \to \infty} M_n = D$ in the Hausdorff topology, where $D$ is the unitary closed disk $\{ c; |c| \leq 1 \}$.

Key words: Julia set, connectedness locus, hyperbolic components, principal components.

1 INTRODUCTION

In (Milnor 2009), J. Milnor considers the complex 1-dimensional slice $S_1$ of the cubic polynomials that have a superattracting fixed point. He gives a detailed pictured of $S_1$ in dynamical terms. In (Roesch 2007), Roesch generalizes these results for families of polynomials of degree $n \geq 3$ having a critical fixed point of maximal multiplicity. This set of polynomials is described -modulo affine conjugacy- by the polynomials $P_c(z) = z^n - cz^{n-1}$. Roesch proved that the global pictures of the connectedness locus of this family of polynomials is a closed topological disk together with “limbs” sprouting off it at the cusps of Mandelbrot copies. In this note, we consider a positive integer $j$, and for each integer $n > j$, we consider the family of polynomials $P_c(z) = z^n - cz^n - j$, where $c$ is a complex parameter. By definition, the connectedness locus $M_n$ of this family of polynomials consists of all parameters $c$ such that the Julia set of $P_c(z)$ is connected or equivalently if the orbit of every critical point of $P_c(z)$ is bounded (see Carleson and Gamelin 1992). Since for all parameter $c; z = 0$ is a superattracting fixed point of $P_c(z)$, we deduce that $M_n$ consists of all parameter $c$ such that the orbit of every non-zero critical point of $P_c(z)$ is bounded. We also consider the space of non-empty compacts subsets of the plane equiiped with the Hausdorff distance (see Douady 1994). We obtain the following result about the size of $M_n$.

AMS Classification: Primary 37F45; Secondary 30C10.
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An Acad Bras Cienc (2012) 84 (1)
THEOREM A. $\mathcal{M}_n$ is a non-empty compact subset of the plane and

$$\lim_{n \to \infty} (\mathcal{M}_n) = D,$$

in the Hausdorff topology, where $D$ is the unitary closed disk $\{c; |c| \leq 1\}$.

2 PROOF OF THEOREM A

The proof of the Theorem is based in the following results.

**LEMMA 2.1.** For $n > 3j$, the closed unitary disk $D$ is contained in $\mathcal{M}_n$.

**PROOF.** Let $c \in D$ and let $k = \left(\frac{n-j}{n} \right)^{\frac{n-1}{n}} \left(\frac{j}{n-j} \right)^j$. Since $n > 3j$, we have that $\frac{j}{n-j} < \frac{1}{2}$, so $k < 1$. Let $z_c$ be a non-zero critical point of $P_c(z)$. Then, $z_c = \frac{n-j}{n} c$, and this implies that

$$P_c(z_c) = z_c^n - cz_c^{n-j} = z_c^n - \left(\frac{n}{n-j}\right)z_c^{n-j} = -\left(\frac{j}{n-j}\right)z_c.$$

This and the fact that

$$|z_c|^{n-1} = \left(\frac{n-j}{n}\right)^{\frac{n-1}{n}} |c|^{\frac{n-1}{n}},$$

imply that

$$|P_c(z_c)| = \frac{j}{n-j} \left(\frac{n-j}{n}\right)^{\frac{n-1}{n}} |z_c| = k|c|^{\frac{n-1}{n}} |z_c|.$$

Hence, since $|c| < 1$, $P_c(z_c)| \leq k |z_c|$.

By induction, suppose that $|P_c^q(z_c)| \leq k^q |z_c|$. Then,

$$|P_c^{q+1}(z_c)| = |P_c^q(z_c)|^{n-j} |(P_c^q(z_c))^{j-1} - c| = |P_c^q(z_c)|^{n-j} |(P_c^q(z_c))^{j-1} - \frac{n}{n-j} z_c^j|$$

$$= |P_c^q(z_c)|^{n-j} |z_c^j| \left| \left(\frac{P_c^q(z_c)}{z_c} \right)^{j-1} - \frac{n}{n-j} \right| \leq k^{q(n-j)} |z_c|^n \left(\frac{k^j + \frac{n}{n-j}}{n-j} \right)$$

$$\leq k^{q(n-j)-1} \left(\frac{k + \frac{n}{n-j}}{n-j} \right) k^{q+1} |z_c|,$$

where the last inequality is true because $|z_c| < 1$ and $k < 1$.

On the other hand, since $n > 3j$, $\frac{n}{n-j} < \frac{3}{2}$ and $q(n-j) - 1 > 1$. Thus,

$$k^{q(n-j)-1} \left(\frac{k + \frac{n}{n-j}}{n-j} \right) < k \left(\frac{k + \frac{3}{2}}{2} \right) < \frac{1}{2} \left(\frac{1}{2} + \frac{3}{2} \right) = 1.$$

Combined with the estimate above, this gives $|P_c^{q+1}(z_c)| \leq k^{q+1} |z_c|$. Hence, $|P_c^q(z_c)| \leq k^q |z_c|$ for all positive integer $q$. Since $k < 1$, we deduce that the orbit $\{P_c^q(z_c)\}$ is bounded and Lemma 2.1 is proved.

**LEMMA 2.2.** If $n > j$, then $\mathcal{M}_n$ is a subset of the disk $\left\{c; |c| \leq \left(\frac{n-j}{j}\right)^{\frac{j}{n-1}} \left(\frac{n}{n-j}\right)^2 \right\}$.
THE CONNECTEDNESS LOCUS OF POLYNOMIALS $P_c(z) = z^n - cz^{n-j}$

PROOF. Let $|c| > \left(\frac{n-j}{j}\right)^{\frac{j}{n-j}} \left(\frac{n}{n-j}\right)^{\frac{j}{n}}$. By definition of $\mathcal{M}_n$, we have that, in order to prove Lemma 2.2, it is sufficient to prove that, for each non-zero critical point $z_c$ of $P_c(z) = z^n - cz^{n-j}$, the orbit $\{P_c^q(z_c)\}$ is not bounded.

Let $k = \frac{j}{n-j} |z_c|^{n-1}$. We claim that $k > \left(\frac{n}{n-j}\right)^{\frac{n-1}{j}}$ and hence $k > 1$.

In fact, since $z_c^j = \frac{n-j}{n} c$,

$$k = \frac{j}{n-j} \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}} |c|^{\frac{n-1}{j}} > \left(\frac{j}{n-j}\right)^{\frac{n-1}{j}} \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}} \left(\frac{n}{n-j}\right)^{\frac{n-1}{j}}$$

and the claim is proved.

Now, we have that

$$|P_c(z_c)| = |z_c^n - cz_c^{n-j}| = |z_c^n - \frac{n}{n-j} z_c^j| = \frac{j}{n-j} |z_c|^n = k |z_c|$$

By induction, suppose that $|P_c^q(z_c)| \geq k^q |z_c|$. Then,

$$|P_c^{q+1}(z_c)| = |P_c^q(z_c)|^{n-j} |P_c(z_c)| - c| = |P_c^q(z_c)|^{n-j} |z_c| \left| \frac{P_c^q(z_c)}{z_c} \right| - \frac{n}{n-j}$$

$$\geq k^{q(n-j)} |z_c|^{n-j} k^{q(j)} \left(\frac{n-j}{n} \right)^{\frac{n-1}{j}} \left(\frac{n}{n-j}\right)^{\frac{n-1}{j}} \left(\frac{n-j}{n} \right)^{q(j-1)} |z_c|$$

$$\geq \frac{n}{j} \left(\frac{n-j}{n} \right)^{q(j-1)} k^{q+1} |z_c| \geq \frac{n}{j} \left(\frac{n}{n-j}\right)^{q(n-1)-1} - 1 \right) k^{q+1} |z_c|.$$

where the last inequality follows from the Claim above.

On the other hand, let $s = q(n-1) - 1$. Then, $s > 1$ and

$$\frac{n}{j} \left(\left(\frac{n}{n-j}\right)^{s-1} - 1 \right) = \frac{n}{j} \left(\frac{n}{n-j} - 1 \right) \left(\left(\frac{n}{n-j}\right)^{s-1} + \ldots + 1 \right)$$

$$= \frac{n}{n-j} \left(\left(\frac{n}{n-j}\right)^{s-1} + \ldots + 1 \right) > 1.$$

Combed with the estimates above, this gives $|P_c^{q+1}(z_c)| \geq k^{q+1} |z_c|$. Hence, $|P_c^q(z_c)| > k^q |z_c|$ for all positive integer $q$. Since $k > 1$, we conclude that, for each critical point $z_c$ of $P_c(z)$, the orbit $\{P_c^q(z_c)\}$ is not bounded, and Lemma 2.2 is proved.

Now, we prove Theorem A. By Lemma 2.2, $\mathcal{M}_n$ is bounded.

Let $J = \left(\frac{n-j}{j}\right)^{\frac{j}{n-j}} \left(\frac{n}{n-j}\right)^{\frac{j}{n-j}^2}$ and let $L$ be a positive integer such that $L^j - J > 1$. Suppose by contradiction that $\mathcal{M}_n$ is not closed. Then, there exists $d$ in the boundary $\partial \mathcal{M}_n$ of $\mathcal{M}_n$ such that the orbit $\{P_d^q(z_d)\}$ is not bounded for some non-zero critical point $z_d$ of $P_d(z)$. Hence, there exists a positive integer $q$ such that $|P_d^q(z_d)| \geq k^q |z_d|$. Since $k > 1$, we conclude that, for each critical point $z_d$ of $P_d(z)$, the orbit $\{P_d^q(z_d)\}$ is not bounded, and Lemma 2.2 is proved.
q such that \( |P^q_d(z_d)| > L \). Since \( z_d = \frac{n-j}{n} d \), we can choose a local branch of \( F(c) = \left( \frac{n-j}{n} c \right)^j \) in a neighborhood \( V \) of \( d \) such that \( |P^q_d(z_c)| > L \), for all \( c \in V \). Since \( d \in \partial M_n \), there exists \( c \in M_n \cap V \) such that \( |P^q_c(z_c)| > L \). By Lemma 2.2, \( |c| < j \). Let \( \omega = P^q_c(z_c) \). Then,

\[
|\omega|^j - |c| > L^j - J > 1,
\]

thus,

\[
|P_c(\omega)| = |\omega^{n-j}||\omega^j - c| > L.
\]

By induction, suppose that \( |P^m_c(\omega)| > L^m \). Then, \( |P^m_c(\omega)|^j - |c| > L^{mj} - J > L \). It follows that,

\[
|P^{m+1}_c(\omega)| = |P^m_c(\omega)||^{n-j} |(P^m_c(\omega))^j - c| > L^{m(n-j)} L > L^{m+1}.
\]

Hence, the orbit \( \{P^j_c(z_c)\} \) is not bounded. This is a contradiction because \( c \in \mathcal{M}_n \). Therefore, \( \mathcal{M}_n \) is closed, so it is compact. Now, Lemmas 2.1 and 2.2 and the fact that \( \lim_{n \to \infty} (\frac{n-j}{n})^{\frac{1}{n-j}} \left( \frac{n}{n-j} \right)^2 = 1 \) imply that \( \lim_{n \to \infty} \mathcal{M}_n = \mathbf{D} \) in the Hausdorff topology, and Theorem A is proved.

**RESUMO**

Seja \( j \) um inteiro positivo. Para cada inteiro \( n > j \), consideramos o locus conexo \( \mathcal{M}_n \) da família de polinômios \( P_c(z) = z^n - cz^{n-j} \), onde \( c \) é um parâmetro complexo. Provamos que \( \lim_{n \to \infty} \mathcal{M}_n = \mathbf{D} \) na topologia de Hausdorff; onde \( \mathbf{D} \) é o disco unitário \( \{c; |c| \leq 1\} \).

**Palavras-chave:** Conjunto de Julia, locus conexo, componentes hiperbólicas, componente principal.

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