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A stochastic model of endogenous growth: the mexican case, 1930-2002

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Abstract

In this research, we develope a stochastic model of endogenous growth. We assume that the exchange rate is driven by a mixed diffusion-jump process, and the tax rate on wealth is governed by a geometric Brownian motion. We also suppose that contingent claims for hedging against future exchange-rate depreciation are not available. Finally, we use the proposed model to carry out a Monte Carlo simulation experiment that explains the observed mean growth rate of output for the Mexican case between 1930 and 2002.

JEL Classification: O41, F41, H31
Keywords: Endogenous growth, stochastic modelling.

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Introduction

The impact of uncertainty on economic growth has been of great interest to policy-makers for a long time. The purpose of this paper is to develop a stochastic model of endogenous growth in which agents have expectations of depreciation of the exchange rate driven by a mixed diffusion-jump process. Although uncertainty is a key element in studying growth, there are few studies considering a stochastic setting. In this regard we may mention, for instance, Canton (2001) and Gokan (2002).

It is important to point out that mixed diffusion-jump processes provide heavy tails and skeweness in the distribution of the exchange rate to rationalize dynamics that cannot be generated by using only the Brownian motion. This fact is not just a sophistication to be included in growth models, but an important issue with significant qualitative implications in the determinants of the growth rate of output.

In our approach, revenue raised by taxes, including seignorage, is wasted in unproductive government purchases. The model assumes that contingent-claims markets to hedge against future exchange-rate depreciation are unavailable. It is also assumed that an un-certain tax rate on wealth is governed by a geometric Brownian motion. Assuming risk averse agents, we examine the growth rate of consumption and output in the presence of taxes on wealth and consumption. The production function has constant return to capital (Rebelo, 1991), we combine this technology with the optimizing behavior of consumers and firms to obtain the stochastic per capita growth rate of consumption, capital and output.

Our modelling has several distinctive features in studying growth determinants and their dynamic implications: it takes into account all risk factors in the exchange-rate dynamics, providing a more realistic stochastic environment; it derives tractable closed-form solutions, making easier the understanding of the key issues of growth; it examines the effects of various forms of distortionary taxes on growth; and it explains the observed output, in the Mexican case between 1930 and 2002, by using Monte Carlo simulation methods.

The paper is organized as follows. In the next section, we work out a stochastic endogenous model where agents have expectations of depreciation of the exchange rate guided by a mixed diffusion-jump process. In this economy, agents pay a tax on wealth at an uncertain rate driven by a geometric Brownian motion. In

[1] A study where exchange-rate derivatives are available can be found in Venegas-Martínez (2005). Other studies fos the Mexican case within a stochastic: framework can be found in Venegas-Martínez (2004) and (2004b).

section 2, we solve the consumer's choice problem. In section 3, we undertake policy experiments. In section 4, we study the dynamic behavior of wealth. In section 5, we examine consumption dynamics and address a number of fiscal policy issues. In section 6, we specify the technology to derive the per capita growth rates of consumption, capital and output. In section 7, we carry out a Monte Carlo simulation experiment that replicates the mean and variance of the observed annual growth rate of output in Mexico between 1930 and 2002. Finally, we draw conclusions, acknowledge limitations, and make suggestions for further research.

1. The setting of the model

Let us consider a small open economy with a single infinite-lived household in a world with a single perishable consumption good. We assume that the good is freely traded, and its domestic price level, Pt, is determined by the purchasing power parity condition, namely

$$P_t = P_t^* e_t, \tag{1}$$

where Pt^* is the foreign-currency price of the good in the rest of world, and e_t is the nominal exchange rate. We will assume, for the sake of simplicity, that P_t^* is equal to 1.

We suppose that the number of extreme movements in the exchange rate, *i.e.*, jumps in the exchange rate, per unit of time, follows a Poisson process Q_t with intensity h, so $IP^{(N)}\{dQ_t = 1\} = hdt$ and $IP^{(N)}\{dQ_t = 0\} = 1 - hdt + o(dt)$. Thus,

$$E^{(N)}[dQ_t] = Var\,(N)\,[dQ_t] = hdt \tag{2}$$

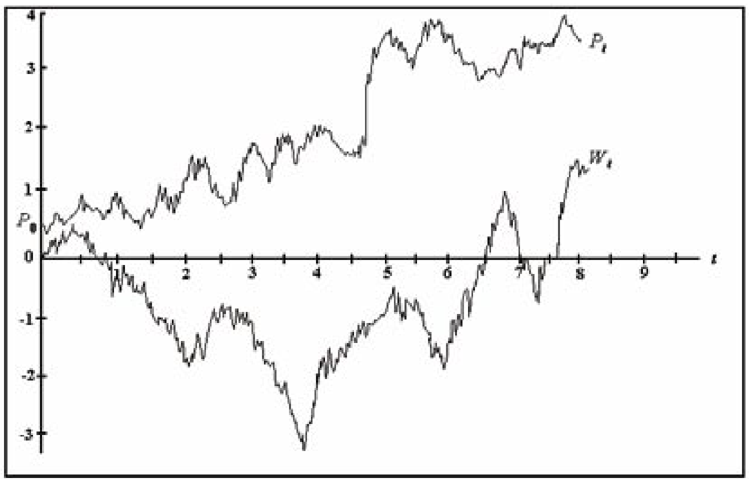
Let us consider a Brownian motion, dV_t, that is $E[dV_t] = 0$ and $Var[dV_t] = dt$. We assume that the consumer perceives that the expected inflation rate, dP_t/P_t, and consequently the expected rate of depreciation, de_t/e_t, follows a geometric Brownian motion with Poisson jumps in accordance with

, (3)

where $\in$ is the mean expected rate of depreciation conditional on no jumps, σ_P is the instantaneous volatility of the expected price level, and μ is the mean expected size of an exchange-rate jump. Process V_t is supposed to be independent of Q_t. In what follows, $\in$, σ_P, h and μ are all supposed to be positive constants. Figure 1 shows a Brownian motion and a mixed diffusion-jump process.

The agent holds real cash balances, $m_t = M_t/P_t$, where M_t is the nominal stock of money. The stochastic rate of return of holding real cash balances, dR_m, is given by the percentage change in the price of money in terms of goods, dm_t/m_t. By applying

Figure 1
A Brownian motion and a mixed diffusion-jump process



Itô's lemma for diffusion-jump processes to the inverse of the price level, with (3) as the underlying process, it can be shown

(4)

The agent also holds capital, k_t, that pays a risk-free real interest rate r, which is constant for all terms, satisfying

$$dk_t = rk_t dt,\ k_0 \text{ given} \tag{5}$$

The representative agent takes r as given.

Let us consider now a Brownian motion dW_t, that is $E[dW_t] = 0$ and $Var[dW_t] = dt$. We assume that the representative consumer perceives that his/her wealth is taxed at an uncertain rate, τ_t, in accordance with the following stochastic differential equation:

(6)

where

, (7)

and

$$\rho \in (-1, 1). \quad (8)$$

Here τ is the mean expected growth rate of the taxes on wealth, σ_τ is the volatility of the tax rate on wealth, and $\rho \in (-1, 1)$ is the correlation between changes in inflation and changes in wealth taxes. Notice that an increase in the rate of depreciation will produce a higher depreciation in real cash balances. This, in turn, will reduce real assets, which could lead to the fiscal authority to modify tax rates. Processes Q_t, V_t, and W_t are supposed to be independent.

Consider a cash-in-advance constraint of the Clower-Lucas-Feenstra form:

, (9)

where c_t is consumption, and $\alpha > 0$ is the time that money must be held to finance consumption. Condition (9) is critical in linking exchange-rate dynamics with consumption. Observe that

In the sequel, we will assume that the error $o(\alpha)$ is negligible.

2. The consumer's decision problem

In this section, we characterize the household's optimal decisions on consumption and portfolio shares through the Hamilton-Jacobi-Bellman condition of the continuous-time stochastic dynamic programming.

The stochastic consumer's wealth accumulation in terms of the portfolio shares,

$$w_t = m_t / a_t,$$

$$1 - w_t = k_t/a_t,$$

and consumption, c_t, is given by the following system:

,

, (10)

where $dR_k = dk_t/k_t$, and τ is a resident-based *ad valorem* tax rate on consumption. By substituting (4), (5) and (9) into the first equation of (10), we get

, (11)

where

The von Neumann-Morgenstern utility at time t, V_t, of the competitive consumer is assumed to have the time-separable form:

, ˆ (12)

Notice that the agent's subjective discount rate has been set equal to the constant real international rate of interest, r, to avoid unnecessary technical difficulties. We consider the logarithmic utility function, $u(c_t) = \log(c_t)$, in order to derive closed-form solutions and make the analysis easy to manage.

In this case, the Hamilton-Jacobi-Bellman equation for the stochastic optimal control problem of maximizing the agent's life-time expected utility subject to the intertemporal budget constraint is:

(13)

where

is the agent's indirect utility function (or welfare function) and $I_a(a_t, \tau_t, t)$ is the co-state variable.

Given the exponential time discounting in (14), we specify $I(a_t, \tau_t, t)$ in a time-separable form as

$$I\,(a_t, \tau_t, t) \equiv F\,(a_t, \tau_t)e^{-rt} \tag{14}$$

where

, (15)

where δ_0, δ_1 and $H(\tau_t;\, \delta_2, \delta_3)$ are to be determined from equation (15). Coefficients δ_2 and δ_3 must satisfy

$$H(\tau_0) = 0 \text{ and } H'(\tau_0) = 0 \tag{16}$$

Substituting (14) into (13), we have

(17)

The first-order conditions of the intertemporal optimization of the risk averse representative agent lead to a time-invariant $w_t \equiv w$, and

(18)

Figure 2 shows optimal w^* as a function μ y h. We choose now $H(\tau_t)$ as a solution of

(19)

Coefficients δ_0 and δ_1 are determined from (15) after substituting optimal w^*. Thus, $\delta_1 = r^{-1}$, so the coefficient of $\log(a_t)$ in (17) becomes zero, and

(20)

Logarithmic utility implies that w depends only upon the parameters determining the stochastic characteristics of the economy, and hence w is constant.

In other words, the consumer's attitude toward currency risk is independent of his/her wealth, *i.e.*, the resulting level of wealth at any instant has no relevance for portfolio decisions. Moreover, due to the logarithmic utility, the correlation coefficient, ρ, plays no role in the consumer's optimal portfolio, only matters the trend and volatility componentes of the stochastic processes driving the dynamics of the exchange rate and the tax policy. Finally, it is important to point out that equation (18) is cubic, therefore it has at least one real root.

Notice also that from $\delta_1 = r^{-1}$, it can be shown that the solution of (19) is

(21)

where

,

and

Coefficients δ_2 and δ_3 are determined in such a way that $H(\tau_0) = 0$ and $H'(\tau_0) = 0$.

Equation (18) is cubic with one negative and two positive roots. This can be seen by intersecting the straight line defined by the right-hand side of (18) with the graph defined by the left-hand side of (18). In such a case, there is only one intersection defining a unique, perfectly viable, steady-state share of wealth set apart for consumption such that $w^* \in (0, 1)$.

3. Policy experiments and comparative statics

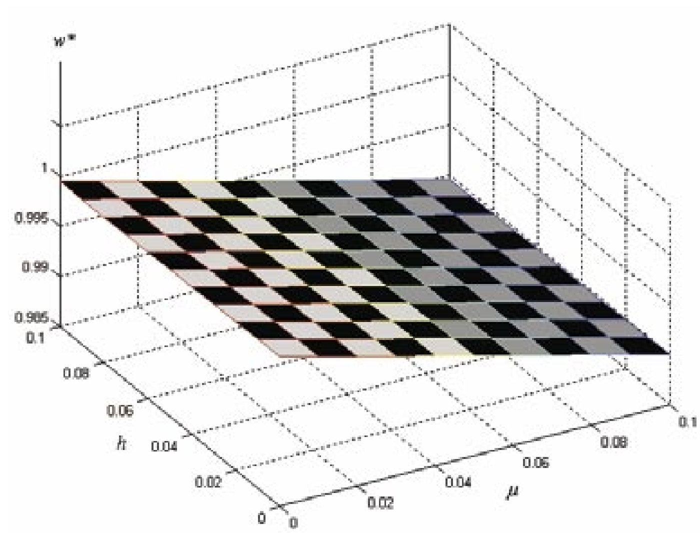
We are now in a position to derive the first result: a once-and-for-all increase in the rate of depreciation, which results in an increase in the future opportunity cost of purchasing goods, leads to a permanent decrease in the proportion of wealth devoted to future consumption. To see this, we may differentiate (18) to find that

(22)

where

(23)

Figure 2
Optimal *w as a function μ y h**



A second result is the response of the equilibrium share of real monetary balances, *w**,to once-and-for-all changes in the intensity parameter, *h*. A once-and-for-all increase in the expected number of exchange-rate jumps per unit of time causes an increase in the future opportunity cost of purchasing goods. This, in turn, permanently decreases the proportion of wealth set aside for future consumption. Indeed, after differentiating (18), we get

(24)

A similar effect is obtained for a once-and-for-all change in the mean expected size of an exchange-rate jump:

(25)

Finally, an increase in the *ad valorem* tax on consumption, will produce a permanent reduction in the proportion of wealth devoted to future consumption.

(26)

4. Wealth dynamics

We now derive the stochastic process that generates wealth when the optimal rule is applied. After substituting the optimal share w^* into (11), we get

(27)

where

, (28)

and

$$\varepsilon \sim N(0, 1) \quad (29)$$

We also have that

(30)

and

(31)

It can be shown that the solution of the stochastic differential equation in (27), conditional on a_0, is

(32)

where

(33)

$$\phi_t = L(w^*)Q_t, \quad (34)$$

and

$$Q_t \sim P\,(ht) \qquad (35)$$

As usual $P(\cdot)$ denotes a Poisson distribution. The stationary components of the parameters of the above distributions are:

and

Notice also that

(36)

and

(37)

Moreover, it readily follows that

(38)

and

(39)

Finally, according to (32), the last two equations determine the mean and variance of the growth rate of real assets.

5. Consumption dynamics

In virtue of (9) and (32), the stochastic process for consumption can be written as

(40)

This indicates that, in the absence of contingent-claims markets, the exchange-rate depreciation risk has an effect on wealth via the uncertainty in ξt, that is, uncertainty changes the opportunity set faced by the consumer. On the other hand, the depreciation risk also affects the composition of portfolio shares via its effects on w^*. Thus, a policy change will be accompanied by both wealth and substitution effects. From (40), we can compute the probability that, in a given time interval, certain levels of consumption occur. It is also important to note, regarding (40) and (12), that the assumption that the agent's time-preference rate is equal to the world's interest rate does not ensure a steady-state level of consumption. However, we do have a steady-state share of wealth set aside for consumption. We may conclude that uncertainty is the clue to rationalize richer consumption dynamics that could not be obtained from deterministic models. Finally, in virtue of (40), equations (38) and (39) determine the mean and variance of the growth rate of consumption. Figure 3 shows consumption as a function of a_0 y τ_0.

6. Technology specification

We suppose that technology is of the form $y_t = Ak_t$ where A >0. That is, the marginal product of capital is constant and equal to A. We assume that capital does not depreciate. The condition for profit maximization require $r = A$. Since

$$1 - w^* = k_t / a_t$$

we have

$$y_t = A\,(1 - w^*)\,a_t$$

In virtue of (32), we obtain

$$y_t = A\,(1 - w^*)\,a_0 e^{\xi t}$$

Notice first that the production-consumption ratio remains constant according to

(41)

and due to the cash-in-advance constraint, the money-consumption ratio, m_t/

$c_t = \alpha$, remains also constant. On the other hand, from (40) and (41), we have that

and since $m_t = w^* a_t$, we obtain

Therefore,

(42)

where, in [t, $t + dt$],

and

$$dQ_t \sim P(hdt)$$

Hence, if $dt = T - t$, in virtue of (36), the expected growth rate of output, in [t, T], satisfies

(43)

Thus, $\psi_{t,T}$ depends upon the parameters determining risk factors (uncertain fiscal and monetary policies), which shows significant qualitative differences with respect to the deterministic framework. Also, from (37), the variance of the growth rate of output, in [t, T], is given by

(44)

Finally, from (42), the expected growth rates of consumption and real cash balances, as well as their variances, are also determined by (43) and (44), respectively.

7. Simulation exercise

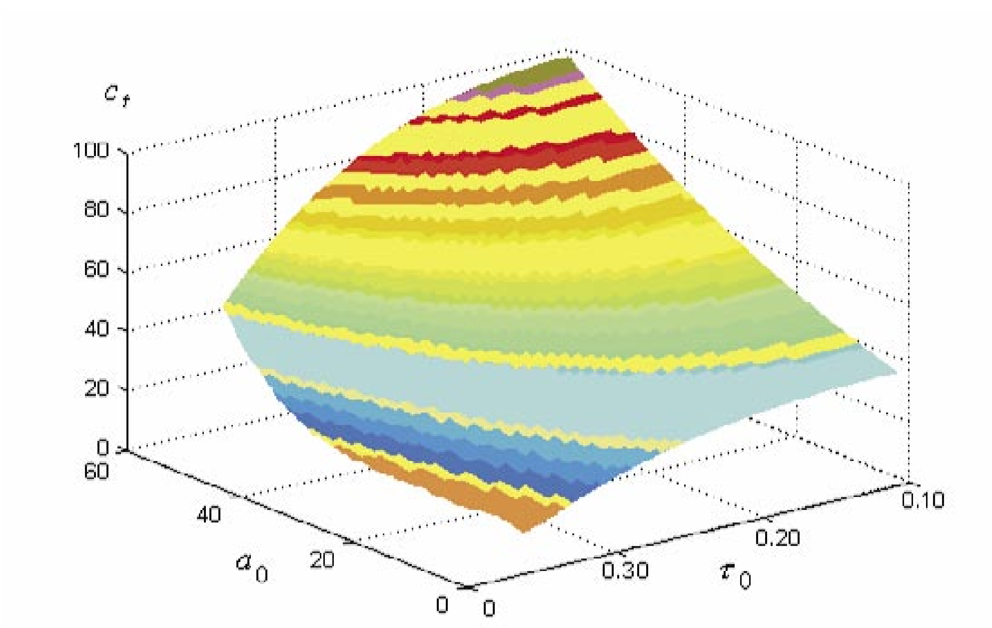
The following experiment is intended to replicate, via Monte Carlo methods, the mean and variance of the observed annual growth rate of output, $E[dy_t/y_t]=E[d\xi_t]$ and $Var[dy_t/y_t] = Var[d\xi_t]$, by using equations (43) and (44), in Mexico between 1930 and 2002. In Figure 4, we show the observed annual growth rate of output from 1930 to 2002.[2]

Table 1 presents a vector of diffusion and jump parameter values, $(\in, \sigma_P^{-1}, h, \mu)$, that replicate the mean and variance of the annual growth rate. In order to choose such a vector, we tried about 800 different feasible combinations of parameter values. For simulation purposes, we have used a standard discrete-time version of (40) with an appropriate unit of time, see, for instance, Ripley (1985). Results are based on 10,000 iterations.

Conclusions

Most of the existing models of endogenous growth ignore uncertainty, providing elaborate

Figure 3
Consumption as a function of a_0 y τ_0,
(a_0 in 10^{11} pesos of 1993)



justification why uncertainty does not need to be considered. We have shown that risk factors may lead to significant qualitative changes in the determinants of growth in contrast with the deterministic setting. The consideration of uncertainty in the expected dynamics of both the exchange rate and the tax policy have led to more complex transitional dynamics, but results were certainly richer.

Our stochastic framework, in which a Brownian motion and a Poisson process drive the expectations of exchange-rate jumps, and a geometric Brownian motion guides a tax rate on wealth, has provided new elements to carry out simulation experiments and empirical research. In particular, our stochastic model was capable of explaining average growth for the Mexican case of 1930-2002.

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Figure 4
Observed growth rate from 1930 to 2002 (pesos of 1993)

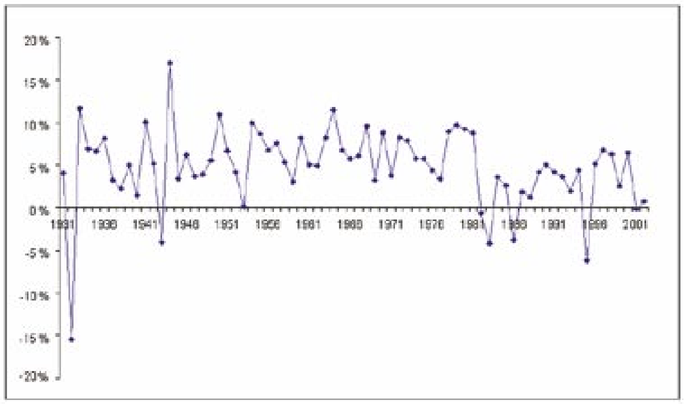

Table 1
Optimal consumption shares, parameters, and estimates

w^*	0.430004
$\in$	0.300000
σ	0.009999
h	0.100000
μ	0.300000
$F(w^*)$	0.011026
$G(w^*)$	0.000019
$L(w^*)$	0.004539
α	0.980000
r	0.085000
$\bar{\tau}$	0.060000
σ_τ	0.180000
a_0	1.849000×10^{12} (pesos of 1993)

estimated growth rate mean = 0.0470	observed growth rate mean = 0.0473
estimated growth rate variance = 0.0223	observed growth rate variance = 0.0222