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CARACTERIZACIÓN DE LA TRANSICIÓN AL CAOS EN ECONOMÍA

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Resumen

Básicamente, cualquier proceso que evoluciona con el tiempo es un sistema dinámico. Los sistemas dinámicos aparecen en todas las ramas de la ciencia y, virtualmente, en todos los aspectos de la vida. La Economía es un ejemplo de un sistema dinámico: las variaciones de precios en la Bolsa de Valores son un ejemplo simple de la evolución temporal de dicho sistema. El principal objetivo del estudio y análisis de un sistema dinámico es la posibilidad de predecir el resultado final de un proceso.

Algunos sistemas dinámicos son predecibles y otros no lo son. Existen sistemas dinámicos muy simples que dependen de una sola variable y muestran un comportamiento sumamente no predecible, debido a la presencia del "caos", esto es, poseen una dependencia sensible a los valores iniciales.

El objetivo principal de este trabajo es investigar cuáles son los factores que producen caminos alternativos para pasar del orden al caos en problemas económicos.

***Palabras clave*:** caos, atractores y repulsores, bifurcación, atractor extraño, contornos fractales.

CHARACTERIZATION OF THE ONSET TO CHAOS IN ECONOMY

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Abstract

Basically, any process evolving with time is a dynamical system. Dynamical systems appear at every branch of Science and virtually at every aspect of life. Economy is an example of a dynamical system: the prices variations at the Stock Exchange is a simple illustration of the temporal evolution of this system. The main objective of the study and analysis of a dynamical system is the possibility of predicting the final result of a process.

Some dynamical systems are predictable and some are not. There are very simple dynamical systems depending only on one variable that show a highly non predictable behavior, due to the presence of "chaos", that means they possess a sensitive dependence on the initial values.

The main aim of this paper is to investigate which are the factors that produce alternative roads to pass from order to chaos in economic problems.

***Keywords*:** chaos, attractors and repellers, bifurcation, strange attractor, fractal basin boundaries.

1. DYNAMICAL SYSTEMS

Basically, any process evolving with time is a dynamical system. Dynamical systems appear at every branch of Science and virtually at every aspect of life. Economy is an example of a dynamical system: the prices variations at the Stock Exchange is a simple illustration of the temporal evolution of this system.

The main objective of the study and analysis of a dynamical system is the possibility to predict the final result of a process. A natural question arising in this field is the following:

(1) If we know with every detail the past evolution of a temporal process, can we predict what will happen in the future?

Mathematically, the question is posed in this way:

(1) Can we derive the asymptotic behavior of the dynamical system if we know exactly its past evolution?

The answer to this fundamental question is sometimes affirmative and sometimes negative. This means that some dynamical systems are predictable -- i.e., we can be sure that the sun will rise every morning -- and some are not predictable -- i.e., we cannot be completely sure when will it rain.

What is that makes the difference between predictable and non predictable dynamical systems? It is not the number of parameters intervening in the process, as may be believed by the analysis of the previous examples: dynamical systems with a great number of variables, like the economy of a country, are certainly not predictable. But on the contrary, there are very simple dynamical systems depending only on one variable that show a highly non predictable and essentially stochastic behavior.

The blame -- if any -- for this high lack of predictability is to be put on the mathematical notion of **"chaos"**, which surprisingly appears more frequently than we expect, even in the simplest dynamical system.

If the mathematical description of the dynamical system is given by one or more differential equations, we say it is a **"continuous dynamical system"**. If, instead, its behavior is described by one or more equations in differences, like the one we shall expose in the following example, it is a **"discrete dynamical system".** We know real dynamical systems are continuously evolving with time, but in many situations the dynamical system state in a certain point at a certain instant of time depends on the state of the system at a previous instant: in most cases it is more convenient to choose a discrete dynamical model.

Example: Let us assume that today I deposit $ 1,000 at a bank that pays an interest rate of 10% per year. At the end of the first year, I shall have

$C(1) = 1{,}000 + 0.1\ (1{,}000) = 1.1\ (1{,}000) = 1{,}100.$

Similarly, at the end of the second year, my capital will amount to

$C(2) = 1.1\ (1.1)\ 1{,}000 = (1.1)^2\ 1{,}000.$

Obviously, if we call $C(0)$ the initial capital, the discrete dynamical system will satisfy the following equation:

$$C(n+1) = C(n) + 0.1\ C(n) \qquad \text{where } n = 0,1,2,\ldots$$

whose solution is

$$C(k) = (1.1)^k\ C(0) \qquad \text{for } k = 1,2,3,\ldots$$

In this case, we say it is a **"discrete first order dynamical system"**, because every iteration depends only on the previous one. More generally, if a dynamical system is described by a set of equations such as

$$A(n+1) = f\,(A(n)) \qquad \text{for } n = 0,1,2,\ldots \tag{1.1}$$

where f is any function, we say it is of first order. Evidently, a dynamical system can be of higher order, for example: $A(n+2) = 3\,A(n+1) + A(n)$ is a second order dynamical system. To solve this dynamical system, it is necessary to know not only the value of $A(0)$ but also the value of $A(1)$, because if not one could not find the value of $A(2)$. These values of $A(0)$ and $A(1)$ are called **"initial values"**. It is easy to generalize this result:

To solve a dynamical system of order m it is necessary to fix m initial values.

If the function f is linear and its graph $(x,f(x))$ passes through the coordinate origin, the dynamical system is called **"linear",** i.e. $A(n+1) = k\,A(n)$, where k is a constant. If $f(x)$ is linear but its graph does not cross the origin, the dynamical system is called **"affine"**, i.e. $A(n+1) = m\,A(n) + b$, where m and b are constants.

Finally, if $f(x)$ is not linear, i.e. $A(n+1) = A^2(n) + A^3(n-1)$, the dynamical system is called **"nonlinear".** Evidently, linear systems are much easier to handle, but nonlinear dynamical systems are more successful in modeling the real world phenomena.

2. FIXED POINTS. ATTRACTORS AND REPELLERS

Given a first order dynamical system

(2.1) $A(n+1) = f(A(n))$ with $A(0) = a$

we say that the point a is an **“equilibrium point”** or **“fixed point”** for this system if $A(k) = a$ for every value of k. That is, $A(k) = a$ is a constant solution for the dynamical system. Constant solutions are important in the sense that they inform us about the future behavior of the dynamical system. It is very easy to prove that the point a is a fixed point for (2.1) if and only if $a = f(a)$. Indeed, if we state a is a fixed point, being $A(0) = a$, it results

$$A(1) = f(A(0)) = f(a) = a$$

$$A(2) = f(A(1)) = f(f(A(0)) = f(f(a)) = f(a) = a$$

....

and conversely, if all these equations are satisfied, we are sure a is, by definition, a fixed point for (2.1).

Example 1: Let us have the following first order dynamical system

$A(n+1) = -0.8\, A(n) + 3.6.$

The fixed point a of this system must satisfy the first degree algebraic equation

$a = -0.8\, a + 3.6$ from where $a = 2$.

Example 2: Let us have the nonlinear dynamical system

$A(n+1) = 1.5\, A(n) - 0.5\, A^2(n).$

To find the fixed points of this system, we must solve the second degree equation

$a = 1.5\, a - 0.5\, a^2$,

whose solutions are $a = 0$ and $a = 1$.

Now, the question is: how do we analyze the stability of a fixed point? The answer is intuitively evident in the sense that the point a will be a **“stable fixed point”** or **“attractor”** if, as time passes by, the values of the iterates $A(k)$ for very big values of k are very near the point a. Mathematically, we say that a is an attractor if there exists a number ε such that

$$\lim_{n\to\infty} A(n) = a$$

when $|a_0 - a| < \varepsilon$.

On the opposite, a will be called **"unstable"** or **"repeller"** if there exists a number ε such that

$$| A(n) - a | > \varepsilon$$

when $0 < |a_0 - a| < \varepsilon$. for some values of n but not necessarily all values.

It is necessary to mention that there exist also fixed points that are neither stable nor unstable, i.e., the fixed point a may be half stable, attracting solutions coming from its right and repelling solutions coming from its left, or vice versa.

Obviously, it is in general impossible to solve analytically a nonlinear dynamical system. We may calculate the successive iterates $A(0)$, $A(1)$, ... , $A(n)$ for n sufficiently big, but we are not able to find an analytic expression of $A(n)$ in terms of n. Notwithstanding, if there are fixed points in the dynamical system, we have a very simple criterion for determining when the fixed point is stable or not. In fact, a will be an attractor if $| f'(a)| < 1$ and a repeller if $| f'(a) | > 1$. This is because the first derivative at the fixed point $f'(a)$ is the best linear approximation to the curve $f(x)$ in a sufficiently small neighborhood of the point a, and this linear approximation is all what we need to analyze the stability of the fixed point. If, finally, $f'(a) = \pm 1$, the first derivative gives not enough information, it is necessary to calculate also the second and the third derivatives of $f(x)$ to determine the stability of the fixed point.

In Example 1, being $f(x) = - 0.8\ x + 3.6$; $f'(x) = - 0.8$ and $f(2) = - 0.8$, with absolute value $0.8 < 1$, the fixed point $a = 2$ is an attractor, while in Example 2, being $f(x) = 1.5\ x - 0.5\ x^2$, the derivative is $f'(x) = 1.5 - 2\ x$. So $f'(0) = 1.5 > 1$ and the fixed point is a repeller; instead $f'(1) = - 0.5$ with absolute value $0,5 < 1$ and the fixed point is an attractor.

Instead of calculating derivatives, it is much easier to analyze the stability through the use of a **"graphical iteration method"** that consists in the following procedure:

Let us have the dynamical system

$$A(n + 1) = f(A(n)) \qquad \text{with } A(0) = x_0.$$

The sequence of iterates is

$$A(1) = f(x_0);\ A(2) = f(A(1)) = f(f(x_0));\ \ldots;$$
$$A(n+1) = f(A(n)) = f(f(\ldots(f(x_0)) = f^n(x_0)$$

where f^n indicates the nth iterate of the function f.

If we represent in the same coordinate system the graph of $y = f(x)$ and the straight line $y = x$, obviously the point of intersection of both graphs is a fixed point a. If we begin with a certain x_0 and mark $f(x_0) = A(1)$ on the y-axis, tracing an horizontal line until it cuts the line $y = x$, we get $A(2)$ on the x-axis and $f(A(2)) = A(3)$ on the y-axis and so on. If, like in Fig. 1 that corresponds to Example 1, the behavior is stable because the sequence of iterates tends to the fixed point $a = 2$, we say that this point is an attractor. Instead, if like in Fig. 2 that corresponds to Example 2, the sequence of iterates tends to go away from the fixed point $a = 0$, we conclude that this point is repelling or unstable. Clearly, the other fixed point $x = 1$ is attracting or stable.

3. BIFURCATIONS

Mostly, we do consider our universe as continuous, in the sense that a small variation in the *"entry"* produces a small variation at the *"exit"*. But this is not always the rule! Precisely, **"bifurcation theory"** is the study of the point at which the qualitative behavior of a dynamical system changes.

A physical example in which there is a bifurcation is the melting point of ice. As everybody knows, the qualitative behavior of water when the temperature is above freezing is quite different from the behavior when the temperature is below freezing. The value of the temperature of water at which there is a breaking point is called the **"bifurcation value"**.

Another physical system where we find bifurcations is the *"turbulence"*. In the 19th century, the English physicist, Osborne Reynolds (1842-1912), experimented with pipes of different diameters and discovered a number -- called today **"Reynolds number"**-- that indicates the engineer at which instant of time the system will become turbulent. This number depends on the pipe width, the fluid viscosity and the flow velocity. It is one of the ends of a spectrum that covers from regular flow to vortices, periodic fluctuations and chaos. A curious characteristic of this spectrum is its *"self-similarity"*, repeating itself in different scales.

To describe the behavior of a dynamical system, we use a **"bifurcation diagram"**. How do we construct it? Let us assume we have a dynamical system involving a parameter *b*. We find all fixed points *a* as a function of *b* and plot these functions on the (*b*-*a*) plane. We look for ranges of *b* for which each of these fixed points is attracting and draw vertical arrows towards them. Similarly, for repelling fixed points, we draw arrows away from them. Finally, we also draw arrows either up or down for values of *b* for which there are no fixed points.

Example 1. Deer population growth

Consider a population where it is allowed to hunt *b* units of deer per season. The dynamical system that models growth is the following

$A(n + 1) = 1.8\, A(n) - 0.8\, A^2(n) - b.$

The fixed points are the solutions of equation

$a = 1.8\, a - 0.8\, a^2 - b$

or else

$0.8\, a^2 - 0.8\, a + b = 0$ from where $a^2 - a + 1.25\, b = 0$,

whose solutions are: $a = \dfrac{1 \pm \sqrt{1-5b}}{2}$.

There are three cases:

1) $1 - 5b < 0$, that is, $b > 0.2$: in such a case, there are <u>no fixed points</u>;

2) $1 - 5b > 0$, that is, $b < 0.2$: in this case we have <u>two fixed points</u> and it is easy to prove that $a_1 = \dfrac{1+\sqrt{1-5b}}{2}$ is attracting or stable if $-1.05 < b < 0.2$ and $a_2 = \dfrac{1-\sqrt{1-5b}}{2}$ is repelling or unstable if $b < 0.2$;

3) $b = 0.2$ and there is only one double root $a = 0.5$.

The value $b = 0.2$ is the bifurcation value and the bifurcation diagram is the one depicted in Fig. 3.

Example 2. In some cases, like the following dynamical system

$A(n + 1) = - A(n) + 4$,

whose fixed point is the solution of equation: $a = - a + 4$, that is, $a = 2$, if we let the initial condition be $A(0) = 6$, we get a closed figure when we follow the vertical and horizontal procedure for determining graphically the stability of the fixed point $a = 2$ (see Fig. 4). This means that the solution forms a **"2-cycle"** oscillating between $x = -2$ and $x = 6$.

In this case, the fixed point is said to be **"neutral"**.

4. CHAOS AND STRANGE ATTRACTORS

Intuitively, a dynamical system exhibits **"chaos"** if it has a sensitive dependence on the initial values. More precisely, a dynamical system has *sensitive dependence on the initial values* if, whenever you take two initial values, a_0 and b_0, which are close together. Then the two corresponding sequences of successive iterates $A(k)$ and $B(k)$ eventually get further apart.

To detect chaos, we need two new concepts:

1. The set of all attracting points in a dynamical system is called an **"attractor".**

2. A dynamical system is said to be **"transitive"** if, when the initial value a_0 is close to some point in an attractor S, the sequence of iterates $A(k)$ gets *"close"* to every point in S.

Suppose now a dynamical system that is:

I) transitive on its attractor S;

II) has sensitive dependence of initial values and

III) has repelling cycles that are close to the attractor S.

Then this dynamical system exhibits **"chaos"** and the attractor is called a **"strange attractor"**. In other words, a dynamical system exhibits chaos if in one sense there is *unpredictability* (the sensible dependence on initial values makes impossible to state precise predictions), but in another sense, there is *predictability* (the property of transitivity assures we will be at a point on the strange attractor, but we do not know when!). As curious as it may sound, we have also *order out of chaos*, in the sense that we have lots of cycles (nice solutions that repeat every 2^n time periods).

In Fig. 5 we show a beautiful example of a strange attractor: it is the so called *Lorenz attractor,* discovered by the meteorologist Edward N. Lorenz in 1962, at the Massachusetts Institute of Technology, long before this concept was introduced in the scientific world.

In this figure, the solution of the system of three nonlinear differential equations starts from the origin (0,0,0) at time t = 0, then makes one loop to the right, then a few loops to the left, then to the right and so on in irregular manner. If one would take, instead of (0,0,0), a nearby initial condition, the new solution would soon deviate from the old one and the numbers of loops to the left and to the right would no longer be the same: this is a proof of the sensitive dependence with respect to initial conditions.

Strange attractors are relatively abstract mathematical entities but computers give them some life and draw pictures of them. Finally, it is interesting to mention that all strange attractors that have been found up to now have fractal dimensions that are not necessarily integer numbers. The magnitude of the fractal dimension is, intuitively, a measure of the *"roughness"* of the configuration, either a line or a surface.

5. FRACTAL BASIN BOUNDARIES

Sensitive dependence on initial conditions is one of the main properties of chaos generation. Notwithstanding, there exists a different kind of sensitivity, namely the so called **"final state sensitivity".** This phenomenon may arise whenever there are several coexisting attractors and not only one. These several attractors may be strange attractors or simply attractive fixed points of a dynamical system. Therefore, there must be a *"boundary"* of the corresponding basins of attractions. Such boundaries are often fractals.

Physically, an initial point can only be specified numerically up to some precision *b*. If all orbits that started within the distance *b* from the initial point converge to the *same* attractor, then it is possible to predict the final state. However, if some of these orbits converge to one attractor and the rest of them to another, no longer can we predict the final state corresponding to the initial point. Obviously, this difficulty grows when the fractal dimension of the basin boundaries gets larger.

Summarizing

Fractal basin boundaries with a large fractal dimension are an impediment for the predictability in nonlinear dynamical systems with several attractors.

A beautiful computer model of such a behavior is thoroughly analyzed by Peitgen et al., p. 757.

6. ROUTES TO CHAOS. ARE THEY UNIVERSAL?

The concept of "*chaos*" is very ancient: it comes from the Pelasgian people, who lived at the Peloponnesian Peninsula before the Greek culture. In his religious ideology, afterwards adopted by the Greeks -- as is mentioned by Robert R. Graves (1895-1985) -- the meaning of chaos was the "*abysmal void*", previous to the birth of the world (**cosmos***: order).* It was considered like a goddess -- Eurynome, all the names -- and conceived like the main element or shapeless mass, including the future constituent elements of the world (air, water, earth and fire) confusedly mixed. Until the '80 decade, the word "*chaos*" indicated a state of disorder, of deterioration and even of death. Since the eighties, the paradigm has notably changed. Scientists have recognized that Nature may use chaos in a constructive way. Through the amplification of small fluctuations, it facilitates natural systems the access to *creativity*. Biological evolution needs genetic variability; the concept of chaos supplies a way of structuring random changes, making it possible that variability be under evolutive control.

The same process of intellectual progress is based upon the injection of new ideas and new ways of connecting the old ones. Under what is known as innate creativity, there could be an underlying chaotic process that amplifies selectively small fluctuations and molds them in coherent and macroscopic mental stages that are experimented as *thoughts*. In some cases, thoughts may be decisions or what is felt like an exercise of will. From this point of view, chaos supplies a mechanism that allows free will in a world governed by deterministic laws.

Precisely in these last years, together with researches about certain nonlinear phenomena that pass from order to chaos in different scientific fields, there have been a lot of investigations on subjects like Medicine, Neuronal Nets (research on the functioning of the human brain), Evolution and History, Enterprises Management, Factories Organization, Development and Planning of Cities, Criminality and Society, Urban Morphology, etc.

How do we arrive practically to "*chaos*"? Given a physical system there are, obviously, many alternative roads to pass from order to chaos, but some of them show certain characteristics that are common to another processes that have nothing to do with them! That is, these roads to chaos are completely independent of the concrete system in observation. For this reason, they are called **"universal"**, adopting a terminology introduced by the physicists Leo Kadanoff, Kenneth Wilson and Michael Fischer in the seventies, analyzing the properties of phase changes. In this context, a description has been developed, playing a very important role in the analysis of complex systems. It is the description of the invariance in the presence of "*scale changes*" in natural structures. This property has been designed as "*self-similarity*" and is an essential feature of "*fractals*".

But if we ask ourselves: *Which are the routes to chaos*? the answer is still open! There exist some universal scenarios of the roads to chaos, such as **"period doubling bifurcations"**.

7. PERIOD DOUBLING BIFURCATIONS IN ECONOMY

The classical deterministic model of economical growth, as is well known, depends on three elements:

1. An equation relating the net rate of births of the population with the money income
2. A production function describing the **"immediate product of labor"**
3. A distribution function defining the wages of labor.

The astonishing range of qualitative behaviors corresponding to the classical model and its onset to chaos may be analyzed when one specifies the production function. A reasonable production function is the one given by the following nonlinear equation

(7.1) $$f(P) = k\, P^b\, (1 - P)^d$$

in which the term AP^b represents the usual power production function and the term $(1\text{-}P)^d$ is a factor of productivity reduction due to a concentrated population surplus. Let us assume, for simplicity, that $b = d = 1$. Then, the production function is given by the following quadratic equation

(7.2) $$f(P) = k\, P\, (1 - P)$$

which is the well known **"logistic equation"**, discovered by Pierre F. Verhulst (1804-1849) studying the dynamics of population [6]. And what is most remarkable about this equation: it describes a nonlinear dynamical system of economical growth of complicated behavior! This means that we have to solve iteratively the simplest nonlinear map given by

(7.3) $$\boxed{P_{n+1} = k\ P_n\ (1 - P_n)}$$

The detailed dynamics of the logistic map described by equation (7.3) is easily followed on a computer. The experiment consists of studying the iterates P_n for successive values of k, taking as time unit a generation of 25 years. In Fig. 6 we notice a pitchfork bifurcation for the dynamical system $P(n + 2) = f(f(P(n)))$, where $f(x) = k\,x\,(1 - x);\ 0 \leq k \leq 3.45$. This 2-cycle corresponds to a period doubling bifurcation for the original system. Similarly, in Fig. 7 we have a pitchfork bifurcation for the dynamical system $P(n + 4) = f(f(f(f(P(n)))))$ that corresponds to a 4-cycle for the logistic equation; $0 \leq k \leq 3.54$. In Fig. 8 we see an 8-cycle for $3.55 \leq k \leq 3.57$. Finally, in Fig. 9 we have the whole bifurcation diagram from first period-doubling bifurcation at $k = 3.0$ to ergodic limit at $k = 4$. The value k_∞ is the period-doubling accumulation point.

Summarizing, for $k < k_\infty = 3,5699456\ldots$ the iterated values of the functions $f^{(n)}(P)$ are periodic. For $k = k_\infty$ the iterated values are *a*-periodic and converge to a **"strange attractor"** that is a two-scale Cantor set generated with a model of two different intervals $A_1 = 0,408$; $A_2 = A_1^{\,2}$ with equal probabilities $p_1 = p_2 = 0,5$. The fractal dimension D of this attractor model is given by equation (see Vera W. de Spinadel, p. 160)

$$A_1^{\,D} + A_2^{\,D} = 1 \tag{7.4}$$

Or else $A_1^{\,D} + (A_1^{\,D})^2 - 1 = 0$, which positive solution is

$$A_1^{\,D} = \frac{\sqrt{5} - 1}{2} = 0,618\ldots$$

This value is known in Physics as the **"Golden Mean"**, because the scientists use to work in the unitary interval, reducing all values mod 1. Notice that $0,618\ldots = 1/\phi$, where $\phi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots$ is mathematically known as the Golden Mean.

Then we have

$$D = \frac{\log 0,618}{\log 0,408} \cong 0,537\,,$$

that is really the maximum value of the logistic parabola (7.2).

In this example, the chaotic answer is generated by the nonlinearity of the production function. The time unity was considered as a generation of 25 years, because this was the period that the Classics thought appropriate for the study of long-run dynamics. However, it may be proved that there is also a chaotic behavior when there is a natural growth rate that would imply a duplication of population every two generations, figure that appears to be reasonable in the context of historical registers.

Finally, it is interesting to mention that the same type of phenomenon of onset to chaos in Economy appears also in many nonlinear continuous temporal models, that is models described by differential equations. Such equations, to exhibit chaos, have to be of third order, what implies much more sophisticated considerations and a greater dependence outline the numerical calculus (see Benoit Mandelbrot & Richard Hudson).

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