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PARTICLES AND FIELDS

Analytical Formulation for ϕ^4 Field Potential Dynamics

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Abstract An analytical model for adding a space-dependent potential to the ϕ^4 field equation of motion is presented, by constructing a collective coordinate system for the solitary solutions of this model. The interaction of ϕ^4 solitons with a delta function potential barrier and also delta function potential well is investigated. Most of the characters of interaction are derived analytically while they are calculated by other models numerically. We will find that the behaviour of a solitary solution is like a point particle which is moved under the influence of a complicated effective potential. The effective potential is a function of the field initial conditions and also of parameters of the added potential.

Keywords ϕ^4 field theory · Soliton · Metric of space–time

1 Introduction

The dynamical evolution of a field theory in the presence of an external potential is an important phenomenon from the mathematical point of view, also because of its applications. External potentials generally come from medium disorders and impurities. An external potential can be added to the equation of motion as perturbative terms [1, 2]. These effects also can be taken into account by forcing some parameters of the equation of motion to be functions of space or time [3, 4]. The external potential can also be added to the field equation of motion through the metric of background space–time [5–7]. This method can be used for objects whose equation of motion results from a Lorentz invariant action, such as sine-Gordon model, ϕ^4 theory, CP^N model, Skyrme model, Faddeev–Hopf equation, chiral quark soliton model, Gross–Neveu model, nonlinear Klein–Gordon models and so on.

As is well-known, when waves scatter on a potential, they can be partly reflected and partly transmitted. For the fields with solitonic solutions, the situation is more complicated. Solitons cannot split and thus must either bounce, pass through or become trapped inside the potential. This behaviour is very sensitive to the value of all the parameters of the model as well as to the initial conditions of the scattering. Most of the research is based on numerical studies because such these systems are generally non-integrable. So it is interesting if we can construct suitable models with analytic solutions to test the validity of such phenomenon and also predict their behaviour.

Scattering of ϕ^4 solitons on barriers and wells has been widely investigated, and several models have been presented for this phenomenon. In this paper, an analytical model for the interaction of ϕ^4 field with an external potential is presented. A model for the ϕ^4 field in a space-dependent potential is presented in Section 2. The analytic model is introduced and will be solved in Section 3. In Section 4, the scattering of a ϕ^4 soliton with a potential barrier is discussed, and the

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results are compared with the other models. The results for the soliton-well system are presented in Section 5. Our conclusions and general remarks are presented in Section 6.

2 ϕ^4 Field and a Space-Dependent Potential

Consider a scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{1}$$

and the following potential

$$U(\phi) = \lambda(x)\left(\phi^2 - 1\right)^2 \tag{2}$$

The equation of motion for the field becomes

$$\partial_\mu\partial^\mu\phi + 4\lambda(x)\phi\left(\phi^2 - 1\right) = 0 \tag{3}$$

The effects of the potential are added to the equation of motion by using a suitable definition for $\lambda(x)$, like $\lambda(x) = 1 + V(x)$. For a constant value of parameter $(\lambda(x) = 1)$, (2) has a solitary solution as follows [8]:

$$\phi(x, t) = \tanh\left(\sqrt{2}\frac{x - X(t)}{\sqrt{1 - v^2}}\right) \tag{4}$$

where $X(t) = x_0 - vt$. x_0 and v are solitary wave initial position and its initial velocity, respectively. This equation is used as an initial condition for solving (3) with a space-dependent $\lambda(x)$ when the potential $V(x)$ is small.

The potential also can be added to the Lagrangian of the system, through the metric of background space–time. So the metric includes characteristics of the medium. The general form of the action in an arbitrary metric is:

$$I = \int \mathcal{L}(\phi, \partial_\mu\phi)\sqrt{-g}d^n x\mathrm{d}t \tag{5}$$

where g is the determinant of the metric $g^{\mu\nu}(x)$. The energy density of the "field + potential" can be found by varying "both" the field and the metric [6]. For the Lagrangian of the form (1), the equation of motion becomes [6, 7]

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\phi\partial^\mu\sqrt{-g}\right) + \frac{\partial U(\phi)}{\partial\phi} = 0 \tag{6}$$

One can add a space-dependent potential to the Lagrangian of the system by introducing a suitable nontrivial metric for the back ground space–time, without missing the topological boundary conditions [6, 7]. In the other words, the metric carries the introduced potential. The suitable metric in the presence of a weak potential $v(x)$ is [5–7]:

$$g_{\mu\nu}(x) \cong \begin{pmatrix} 1 + V(x) & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

The equation of motion (6) (describes by Lagrangian (1)) in the background space–time (7) is

$$(1 + V(x))\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{1}{2\,|1 + V(x)|}\frac{\partial V(x)}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial U(\phi)}{\partial\phi} = 0 \tag{8}$$

By inserting the solution (4) in the Lagrangian (1) and using the metric (7), with adiabatic approximation [1, 2], we have

$$\mathcal{L} = \sqrt{1 + v(x)}\left((1 + v(x))\,\dot{X}^2 - 2\right)\mathrm{sech}^4\left(\sqrt{2}\,(x - X(t))\right) \tag{9}$$

For a weak potential $v(x)$, (9) becomes

$$\mathcal{L} \approx \left(\left(1 + \frac{3}{2}v(x)\right)\dot{X}^2 - 2\left(1 + \frac{1}{2}v(x)\right)\right)\mathrm{sech}^4 \times \left(\sqrt{2}\,(x - X(t))\right) \tag{10}$$

3 Collective Coordinate Variable

The center of a soliton can be considered as a particle, if we look at this variable as a collective coordinate. The collective coordinate could be related to the potential by using the above model. This model is able to give us an analytic solution for most of the features of the soliton-potential system. For the Lagrangian (10) X(t) remains as a collective coordinate if we integrate (10) over the variable x

$$L = \int \mathcal{L}\mathrm{d}x = \frac{2\sqrt{2}}{3}\dot{X}^2 - \frac{4\sqrt{2}}{3} + \left(\frac{3}{2}\dot{X}^2 - 1\right) \times \int \mathrm{sech}^4\left(\sqrt{2}\,(x - X(t))\right)v(x)\mathrm{d}x \tag{11}$$

If we take the potential $V(x) = \epsilon\delta(x)$, then (11) becomes

$$L = \frac{2\sqrt{2}}{3}\dot{X}^2 - \frac{4\sqrt{2}}{3} + \left(\frac{3}{2}\dot{X}^2 - 1\right)\epsilon\,\mathrm{sech}^4\left(\sqrt{2}X(t)\right) \tag{12}$$

The equation of motion for the variable $X(t)$ is derived from (12)

$$\left(\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X\right)\right)\ddot{X}-\sqrt{2}\epsilon\tanh(\sqrt{2}X)\text{sech}^4 \times\left(\sqrt{2}X\right)\left(3\dot{X}^2+2\right)=0 \tag{13}$$

The above equation shows that the peak of the soliton moves under the influence of a complicated force which is a function of external potential $V(x)$, soliton position and its velocity. If $\epsilon > 0$, we have a barrier and $\epsilon < 0$ creates a potential well. By substituting $\ddot{X} = \dot{X}\frac{d\dot{X}}{dX}$ into (13) and integrating of this equation, we have

$$\frac{3\dot{X}^2+2}{3\dot{X}_0^2+2}=\frac{\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X_0\right)}{\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X\right)} \tag{14}$$

where X_0 and $\dot{X}_0$ are soliton initial position and its initial velocity, respectively. This equation gives soliton collective velocity $\dot{X}(t)$ as a function of its collective position $X(t)$. The energy of the soliton in the presence of the potential $V(x) = \epsilon\delta(x)$ becomes

$$E=\left(\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X\right)\right)\dot{X}^2+\frac{4\sqrt{2}}{3} +\epsilon\text{sech}^4\left(\sqrt{2}X\right) \tag{15}$$

By substituting $\dot{X}$ from (14) in (15), one can show that the energy is a function of initial conditions X_0 and $\dot{X}_0$. Therefore, the energy of the system is conserved. Some features of the soliton-potential dynamics can be investigated using (14) and (15) analytically which are discussed in next sections. We will compare the results of our analytic model with the results of direct simulation of (3) with $\lambda(x) = 1 + \epsilon\delta(x)$. It is clear that the results are not completely equal to each other because they come from different models. But we can compare the results by adjusting the potential parameter ϵ between two models. Therefore, we will find an effective potential parameter in the analytic model to fit our analytic results on a set of simulations using (3).

4 Potential Barrier

Suppose that a potential barrier with the height ϵ is located at the origin. When the soliton is far from the center of the potential ($X \to \infty$), (15) reduces to $E = \frac{2\sqrt{2}}{3}\dot{X}_0^2 + \frac{4\sqrt{2}}{3}$. It is the energy of a particle with the rest mass $\frac{4\sqrt{2}}{3}$ and a velocity of $\dot{X}_0$. A soliton with a low velocity reflects back from the barrier and a high-energy soliton climbs over the barrier and passes over it. So we have a critical value for the soliton velocity which separates these two situations. The energy of a soliton at the origin is $E(X=0) = \left(\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\right)\dot{X}_0^2 + \frac{4\sqrt{2}}{3} + \epsilon$. The minimum energy for a soliton in this position is $E = \frac{4\sqrt{2}}{3} + \epsilon$. On the other hand, a soliton which comes from infinity with initial velocity v_c has the energy of $E\,(X=\infty) = \frac{2\sqrt{2}}{3}v_c^2 + \frac{4\sqrt{2}}{3}$. Therefore, the minimum velocity for the soliton to pass over the barrier is $v_c = \sqrt{\frac{3}{2\sqrt{2}}\epsilon}$. The same result is derived by substituting $\dot{X} = 0$, $\dot{X}_0 = v_c$, $X_0 = \infty$ and $X = 0$ in (15).

We can adjust the potential parameter ϵ between the analytic model and the results of direct numerical solution of the (3) using their predictions for the critical velocity. Figure 1 presents simulation results of the critical velocity with (3) and fitted curve $v_c = \sqrt{\frac{3}{2\sqrt{2}}\epsilon_{\text{effective}}}$ where $\epsilon_{\text{effective}} = \alpha\epsilon + \beta$. v_c of analytic model can be fitted on the simulation results with $\alpha = 0.523 \pm 0.0123$ and $\beta = 0.0234 \pm 0.004$. Simulations have been done using Runge–Kutta method for time derivatives and finite difference method for space derivatives. Space grids have been chosen $\Delta x = 0.001, 0.005$, and time cells are $\Delta t = \frac{\Delta x}{4}$ in the simulations. Delta function has been simulated using the function $\sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}$ with several values for α.

If the soliton is located at some position like X_0 (which is not necessary infinity), the critical velocity will not be $\sqrt{\frac{3\sqrt{2}}{4}\epsilon}$. Soliton passes over the barrier if

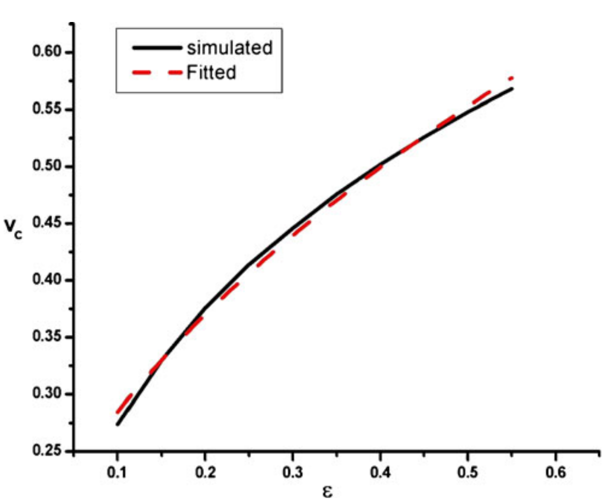

Fig. 1 Critical velocity as a function of ϵ with results of simulation using (3) (*solid line*) and analytic model (*dash line*)

the soliton energy is greater than the energy of a static soliton at top of the barrier. So a soliton in the initial position X_0 with initial velocity $\dot{X}_0$ has the critical initial velocity if its velocity becomes zero at top of the barrier $X = 0$. Consider a soliton with initial conditions of X_0 and $\dot{X}_0$. If we set $X = 0$ and $\dot{X} = 0$ in (14), then $v_c = \dot{X}_0$. Therefore, we have

$$v_c = \sqrt{\frac{\epsilon\left(1 - \mathrm{sech}^4\left(\sqrt{2}X_0\right)\right)}{\frac{2\sqrt{2}}{3} + \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}x_0\right)}} \tag{16}$$

Figure 2 shows critical velocity as a function of initial position. Simulated results using (3) are in a very good agreement with (16). Equation 16 in Fig. 2 has been plotted with $\epsilon_{\text{effective}} = 0.53\epsilon$ as we have found later.

If soliton initial velocity is less than the v_c, then there exists a return point in which the velocity of the soliton is zero. For this situation, we have

$$\left(\frac{3}{2}\dot{X}_0^2 + 1\right)\left(\frac{2\sqrt{2}}{3} + \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}X_0\right)\right) - \frac{2\sqrt{2}}{3} = \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}X_{\text{stop}}\right) \tag{17}$$

Therefore, this model predicts that $\mathrm{sech}^4\left(\sqrt{2}X_{\text{stop}}\right)$ is proportional to $\mathrm{sech}^4\left(\sqrt{2}X_0\right)$. Also (17) shows that $\mathrm{sech}^4\left(\sqrt{2}X_{\text{stop}}\right)$ is proportional to $\frac{1}{\epsilon}$.

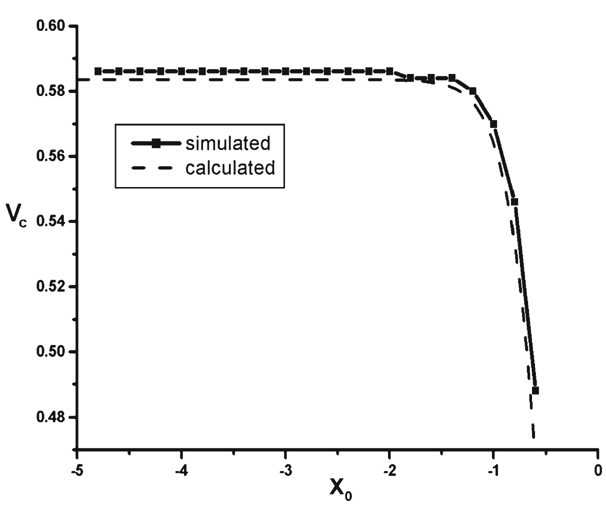

Fig. 2 Critical velocity as a function of initial position X_0 with results of simulation using (3) (*solid line*) and analytic model (*dash line*)

A soliton with initial conditions X_0 and $\dot{X}_0$ will go to infinity after the interaction with a potential barrier. The final velocity of the soliton at infinity after the interaction is

$$\dot{X} = \sqrt{\dot{X}_0^2 + \frac{3\epsilon}{2\sqrt{2}}\left(\frac{3\dot{X}_0^2}{2} + 1\right)\mathrm{sech}^4\left(\sqrt{2}X_0\right)} \tag{18}$$

which is greater than the initial velocity $\dot{X}_0$.

Equations 14 and 15 show that the soliton finds its initial velocity after the interaction when it reaches its initial position. This means that the interaction is completely elastic. Numerical simulations are in agreement with above analytical results [7–9].

5 Soliton-Well System

The soliton-well system is a very interesting problem. Suppose a particle moves toward a frictionless potential well. It falls in the well with increasing velocity and reaches the bottom of the well with its maximum speed. After that, it will climb the well with decreasing velocity and finally pass through the well. Its final velocity after the interaction is equal to its initial speed. But there are some differences between point particle and a soliton in a potential well. Our analytic model explains several features of soliton-well system correctly.

Changing ϵ to $-\epsilon$ in (12) changes potential barrier to potential well. The solution for the system is

$$\frac{3\dot{X}^2 + 2}{3\dot{X}_0^2 + 2} = \frac{\frac{2\sqrt{2}}{3} - \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}\left(X_0\right)\right)}{\frac{2\sqrt{2}}{3} - \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}\left(X\right)\right)} \tag{19}$$

There is not a critical velocity for a soliton-well system, but we can define an escape velocity. A soliton with initial position X_0 reaches the infinity with a zero final velocity if its initial velocity is

$$\dot{X}_{\text{escape}} = \sqrt{\frac{\epsilon\,\mathrm{sech}^4\left(\sqrt{2}X_0\right)}{\frac{2\sqrt{2}}{3} - \frac{3\epsilon}{2}\mathrm{sech}^4\left(\sqrt{2}X_0\right)}} \tag{20}$$

In other words, a soliton which is located at the initial position X_0 can escape to infinity if its initial velocity $\dot{X}_0$ is greater than the escape velocity $\dot{X}_{\text{escape}}$. From (20), analytic model predicts that $\frac{1}{\dot{X}^2_{\text{escape}}}$ is proportional to $\frac{1}{\epsilon}$ and also $\frac{1}{\dot{X}^2_{\text{escape}}}$ is proportional to $\cosh^4(\sqrt{2}X_0)$. Figure 3 presents $\cosh^4(\sqrt{2}X_0)$ as a function of $\frac{1}{\dot{X}^2_{\text{escape}}}$ using direct simulation of (3). Potential depth has been

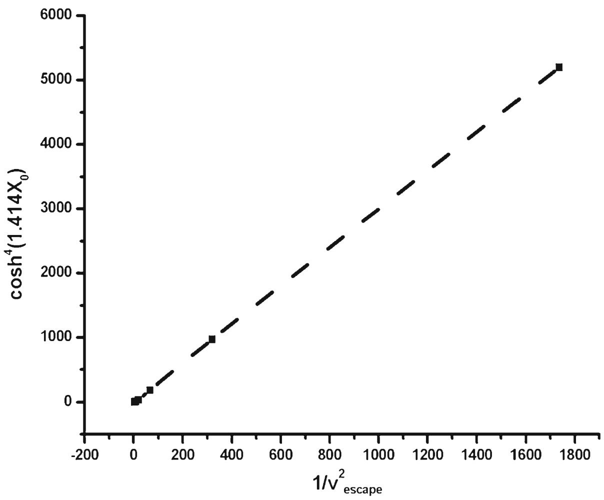

Fig. 3 $\cosh^4(\sqrt{2}X_0)$ as a function of $\frac{1}{\dot{X}^2_{\text{escape}}}$ with $\epsilon = -0.6$

chosen $\epsilon = -0.6$. This figure clearly shows linear relation between $\cosh^4(\sqrt{2}X_0)$ and $\frac{1}{\dot{X}^2_{\text{escape}}}$ as predicted by the analytic model.

Trajectory of a soliton during the interaction with the potential, $X(t)$ follows from (14) as

$$t=\int_{X(t=0)}^{X(t)}\left(\sqrt{\left(\dot{X}_0^2+\frac{2}{3}\right)\left(\frac{\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X_0\right)}{\frac{2\sqrt{2}}{3}+\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X\right)}\right)-\frac{2}{3}}\right)^{-1}\text{d}x \quad (21)$$

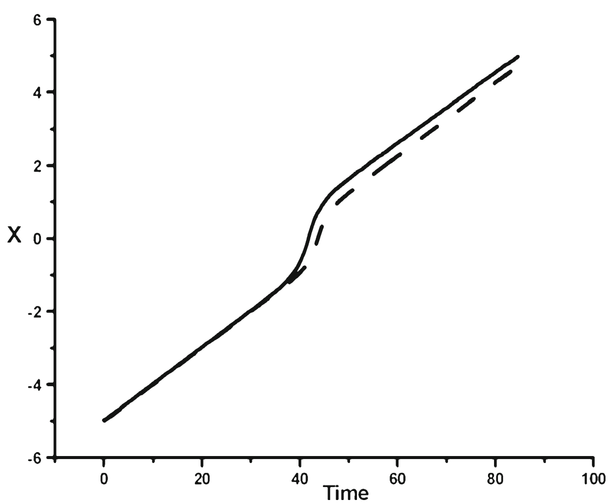

Fig. 4 Soliton trajectory during the interaction with potential well. Potential depth $\epsilon = -0.2$ has been chosen for direct simulation using (3) (*solid line*) and ($\epsilon_{\text{effective}} = -0.134$) using analytic model (*dash line*). Initial conditions are $X_0 = -5$ and $\dot{X}_0 = 0.1$

The above integral has been evaluated numerically by using Rubmerg's method and $X(t)$ was plotted versus t. This result was compared with direct simulation using (3). Figure 4 shows the result for a system with $X_0 = -5$, $\dot{X}_0 = -0.1$ and $\epsilon = -0.2$ ($\epsilon_{\text{effective}} = -0.134$). There is a little difference between the predicted final velocities from different models after interaction. The difference is reduced when the height of the potential(ϵ) reduces. The difference is due to the approximation which is used for deriving (10) from (9).

Consider a potential well with the depth of ϵ and a soliton at the initial position X_0 which moves toward the well with initial velocity $\dot{X}_0$ smaller than the $\dot{X}_{\text{escape}}$. The soliton interacts with the potential and reaches a maximum distance $X_{\max}$ from the center of the potential with a zero velocity and then come back toward the well. The soliton oscillates around the well with the amplitude $X_{\max}$. The required initial velocity to reach $X_{\max}$ is found from (19) as

$$\dot{X}_0 = \sqrt{\frac{\epsilon\left(\text{sech}^4\left(\sqrt{2}X_{\max}\right)-\text{sech}^4\left(\sqrt{2}X_0\right)\right)}{\frac{2\sqrt{2}}{3}-\frac{3\epsilon}{2}\text{sech}^4\left(\sqrt{2}X_{\max}\right)}} \quad (22)$$

If the initial velocity is lower than the escape velocity, the soliton oscillates around the well. The period of oscillation can be calculated numerically using (21). We have done simulations for investigating above equations. All of the simulations show very good agreement between direct results of (3) and results of our analytic model.

6 Conclusion and Remarks

A model for the ϕ^4 field potential interaction has been presented. Several features of soliton-potential characters were calculated using this model. Calculated characters have been compared with the results of direct simulation of soliton-potential system. In this paper, we have found a critical velocity for the soliton during the interaction with a potential barrier as a function of its initial conditions and the potential characters. The model predicts specific relations between some functions of initial conditions and other functions of final state of the field after the interaction. An escape velocity has been derived for the soliton-well system.

All of the simulations for the soliton-barrier and soliton-well systems show the validity of analytic results. In a previous study, we have applied the same analytic model for sine-Gordon solitons [7]. Therefore, we can conclude that the presented collective coordinate method can be used for most of the solitonic systems.

Although this model is able to explain most of the features of the system, but it is unable to explain fine structure of the islands of trapping in soliton-well system. This phenomenon is a very interesting features of soliton-potential systems. Perhaps one can find an acceptable explanation for this behaviour using a further improved version of the collective coordinate method.

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