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An Example of a Heavy Tailed Distribution

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Abstract

We study some properties of the distribution function of a random variable of the form $X = C^D$, where C and D are independent random variables. We assume that C is absolutely continuous and limited to a finite interval, such that its probability density function has definite limits at the endpoints of the interval and D is exponentially distributed. We show that the tail function $\bar{F}(.) := 1 - F(\cdot)$ is of regular variation and that the distribution function F is asymptotically equivalent to a log-gamma distribution. Then F can be considered as a heavy tailed distribution. It is also shown that it is contained is an special subclass of the subexponential distributions.

Keywords: Regular variation, subexponential distributions, heavy tailed distributions, probability of ruin, decreasing hazard rate function.

AMSC(2000): Primary: 60E05.

1 Introduction

We consider the distribution function of a random variable of the form $X = C^D$ with C and D independent random variables, C absolutely continuous and limited to a finite interval and D exponentially distributed. The article is organized as follows. In section 2 it is shown that the random variable $X = C^D$ posseses finite moments up to order k and that the distribution function F of X belongs to the class of Pareto-type distributions and then, to the class of subexponential distributions, which is an important class ([3] and [4]). In section 3 some additional properties are developed, related to the probability of ruin in the collective risk theory, when the claim amount follows the F distribution. In section 4 some conclusion are presented.

2 Definitions and Properties

Let D be an exponential random variable with parameter $\ln(q^{-1})$, $q \in (0, 1)$, i.e. $P(D > t) = \exp(-t \ln(q^{-1})) = q^t$, $t \geq 0$. And let C be an absolutely continuous random variable, independent of D, with values in the interval $[1, q^{-1/k}]$, where $k = 1, 2, \ldots$, with cumulative distribution function (cdf) $G(x) = P(C \leq x)$ such that $G'(x) = g(x)$ is continuous and satisfies the following condition: there exist constants $\gamma > 0$ and $\lambda > 2$ such that

$$g(x) \sim \gamma(q^{-1/k} - x)^{\lambda-1},\ x \uparrow q^{-1/k}, \tag{1}$$

where $f(x) \sim g(x),\ x \to x_0$, means $\lim_{x \to x_0} f(x)/g(x) = 1$. Then define the random variable $X := C^D$ and denote by F its cdf, i.e., $F(x) = P(X \leq$

x), $x \geq 1$ and by $\bar{F}(x) = 1 - F(x)$ its tail function. The next proposition shows that the random variable X has finite moments up to order k.

Proposition 1. *Let* $r = 0, 1, 2, \ldots$

(i) *If* $r < k$ *then* $1 \leq E(X^r) \leq 1/(1 - r/k)$.

(ii) *If* $r > k$ *then* $E(X^r) = +\infty$.

(iii) *If* $r = k$ *then* $E(X^r) < \infty$.

Proof. To prove *i)* see the following: $E(X^r) = E(E(C^{rD}|C))$ and for $1 < s < q^{-1/k}$

$$\begin{aligned} E(C^{rD}|C = s) &= E(s^{rD}|C = s) = E(s^{rD}) \\ &= E(e^{r\ln(s)D}) = \frac{\ln(q^{-1})}{\ln(q^{-1}) - \ln s^r} \\ &\leq \frac{\ln(q^{-1})}{\ln(q^{-1}) - r\ln(q^{-1/k})} = \frac{1}{1 - \frac{r}{k}}, \end{aligned}$$

then $1 \leq E(X^r) \leq 1/(1 - r/k)$. Notice that independence of C and D allows to eliminate the conditional $C = s$.

ii) If $r > k$ and $q^{-1/r} < s < q^{-1/k}$ then $E(C^{rD}|C = s) = +\infty$ and so $E(X^r) = +\infty$.

iii) If $r = k$ a direct application of L'Hopital rule and condition (1) proves that the function $\frac{\ln(q^{-1})g(s)}{\ln(q^{-1}) - \ln s^k}$ has an absolutely convergent integral in $[1, q^{-1/k}]$. □

We will now obtain an asymptotic equivalent expresion for the tail function of X, $\bar{F}(t) = P(X > t)$, $t \geq 1$, by applying Laplace's method for asymptotic expansions of definite integrals of the form:

$$I(t) = \int_a^b e^{-tp(x)} h(x) dx \tag{2}$$

given in the following lemma:

Lemma 2. *([2], chap. 5) Consider an integral of the form (2) and assume*

(i) *The minimum value of* $p(x)$ *is located at* $x = a$.

(ii) $p'(x)$ *and* $h(x)$ *are continuous functions in a vicinity of* $x = a$ *(except, possibly, at* $x = a$*).*

(iii) $p(x) \sim p(a) + \beta(x - a)^{\mu}$, $t \downarrow a$, $\beta > 0, \mu > 0$.

(iv) $h(x) \sim \gamma(x - a)^{\lambda - 1}$, $x \downarrow a$, $\lambda > 0, \gamma > 0$.

(v) $I(t)$ is absolutely convergent for t large enough.

Then

$$I(t) \sim \frac{\gamma}{\mu}\,\Gamma\left(\frac{\lambda}{\mu}\right)\, e^{-tp(a)}(\beta t)^{-\lambda/\mu},\ t \to \infty\ , \tag{3}$$

where $\Gamma(\cdot)$ denotes the gamma function.

Proposition 3.

(i) $\bar{F}(t) = E\left(q^{\frac{\ln t}{\ln C}}\right),\ t \geq 1\ .$

(ii) $\bar{F}(t) \sim c\,t^{-k}(\ln t)^{-\lambda},\ t \to \infty$, *with* $c = \gamma\,\Gamma(\lambda)/\beta^{\lambda}$ *and* $\beta = k^2 q^{1/k}/\ln(q^{-1})$.

Proof. First, note that

$$\begin{aligned}
\bar{F}(t) &= \int_1^{q^{-1/k}} P(C^D > t | C = s) g(s) ds \\
&= \int_1^{q^{-1/k}} P\left(D > \frac{\ln t}{\ln s}\right) g(s) ds \\
&= \int_1^{q^{-1/k}} q^{\frac{\ln t}{\ln s}} g(s) ds = E\left(q^{\frac{\ln t}{\ln C}}\right)\ .
\end{aligned}$$

Now, let us denote by $\bar{H}(t)$ the tail function of $Y = \ln X$. The expectation $\bar{H}(t) = \bar{F}(e^t) = E(q^{\frac{t}{\ln C}}),\ t > 0$ can be represented as a definite integral of the form (2) by writing

$$\begin{aligned}
\bar{H}(t) &= \int_1^{q^{-1/k}} \exp\left(-t\frac{\ln(q^{-1})}{\ln x}\right) g(x) dx \\
&= \int_{-q^{-1/k}}^{-1} \exp\left(-t\frac{\ln(q^{-1})}{\ln(-x)}\right) g(-x) dx.
\end{aligned}$$

Conditions (i) to (v) can readily be verified by definining $a = -q^{-1/k}$, $b = -1$, $p(x) = \ln(q^{-1})/\ln(-x)$ and $h(x) = g(-x)$. We have $p'(x), h(x)$ continuous functions in $[a, b]$, the minimum of $p(x)$ is located at $a = -q^{-1/k}$, and

$$p(x) \sim k + \frac{k^2 q^{1/k}}{\ln(q^{-1})}(x + q^{-1/k}),\ x \downarrow -q^{-1/k}\ .$$

The assumption about $g(x)$ in (1) is equivalent to condition (iv) and the integral $I(t) = \bar{H}(t)$ is easily seen to be absolutely convergent for every $t > 0$. Replacing $\beta = k^2 q^{1/k}/\ln(q^{-1})$ and $\mu = 1$ in Lemma (2), we arrive at the following asymptotic equivalence:

$$\bar{H}(t) \sim c\,e^{-kt}\,t^{-\lambda},\ t \to \infty\ ,$$

from which the result follows with $\bar{F}(t) = \bar{H}(\ln(t))$. $\square$

We use Proposition (3) to prove that $\bar{F}$ is a function of regular variation. The class of distribution functions of regular variation is included in the class of subexponential distributions, both defined ahead. As the latter class is usually identified as the class of "heavy tailed distributions", this result justifies the denomination of F as "heavy tailed" distribution.

A positive function f defined on $(0, \infty)$ is called a function of regular variation at infinity if it exists a real number δ such that

$$\lim_{t\to\infty} f(tx)/f(t) = x^{\delta} \ , \tag{4}$$

for any $x > 0$. The number δ is called the index of regular variation, and we write $f \in \mathcal{R}_\delta$. A function $f(.)$ of regular variation then satisfy $f(x) \sim x^\delta l(x), \ x \to \infty$ where $l(.)$ is a function of regular variation with index $\delta = 0$, or a funcion of slow variation. A simple calculation shows that $c(\ln t)^{-\lambda}$ is a function of slow variation, and we obtain from Proposition (3):

Corollary 4. $\bar{F} \in \mathcal{R}_{-k}$.

In general, distributions F such that $\bar{F} \in \mathcal{R}_{-\rho}$ with $\rho > 0$ are called Pareto-type distributions. Some examples of distributions in this class are the Pareto, Burr, Generalized Pareto, Frechet, Log-hyperbolic, Log-logistic and Log-gamma ([1],ch.2). In particular, the tail function $\bar{V}(.)$ of the log-gamma distribution with parameters $k > 0$ and $\lambda < 1$, is defined as:

$$\bar{V}(t) = \frac{1}{k^{\lambda-1}\Gamma(1-\lambda)} \int_t^\infty u^{-k-1}(\ln u)^{-\lambda} du, \ t \geq 0.$$

We now use the following result, Karamata's theorem, for proving the equivalence of the tail functions $\bar{V(t)}$ and $\bar{F(t)}$.

Theorem 5. *([4], th. A.3.6(b)) Let $l(.) \in \mathcal{R}_0$ be locally bounded in $[x_0, \infty)$ for some $x_0 \geq 0$. Then*

(i) for $\alpha > -1$, $\int_{x_0}^x t^\alpha l(t)dt \sim (\alpha+1)^{-1}x^{\alpha+1}l(x), \ x \to \infty$.

(ii) for $\alpha < -1$, $\int_x^\infty t^\alpha l(t)dt \sim -(\alpha+1)^{-1}x^{\alpha+1}l(x), \ x \to \infty$.

On applying Karamata's theorem we obtain

$$\bar{V}(t) \sim \frac{1}{k^\lambda \, \Gamma(1-\lambda)} t^{-k}(\ln t)^{-\lambda}, \ t \to \infty \ .$$

Comparing the last expression with Proposition $(3.ii)$:

$$\bar{F}(t) \sim \frac{\gamma\Gamma(\lambda)}{\beta^\lambda} t^{-k}(\ln t)^{-\lambda}, \ t \to \infty \ ,$$

we can conclude that the distribution function F of the random variable $X = C^D$ is asymptotically equivalent to the distribution function V of a log-gamma distribution with parameters $k,\ \lambda$, in the sense that

$$\lim_{t\to\infty} \bar{F}(t)/\bar{V}(t) = \text{constant} .$$

Another class of interest is the class of subexponential distributions. F belongs to the class $\mathcal{S}$ of subexponential distributions if

$$\lim_{x\to\infty} \bar{F}^{2*}(x)/\bar{F}(x) = 2 ,$$

where F^{2*} is the convolution product $F^{2*}(x) = \int_0^x F(x-y)F(dy)$, and $\bar{F}^{2*} := 1 - F^{2*}$.

It is a well known result that $\mathcal{R}_{-\rho} \subseteq \mathcal{S},\ \rho > 0$ ([6], pag.328, Exercise No 27, C. [8]). Then, by Corollary 4, the distribution F of C^D belongs to $\mathcal{S}$. The class $\mathcal{S}$ posseses several interesting properties, see, for instance [3] and [4].

3 Additional Properties

Other additional properties can be obtained by studying the corresponding hazard rate function i.e. the function $h(x) := -\bar{F}'(x)/\bar{F}(x),\ x \geq 0$. We will show that the hazard rate is such that $\lim_{x\to\infty} xh(x) = \text{constant} < \infty$. This result implies that F is in $\mathcal{S}$ but also in other sub-class which posseses important properties.

Proposition 6.

(i) $h(t) = \ln(q^{-1})E\left(\frac{1}{\ln C} q^{\frac{\ln t}{\ln C}}\right) \Big/ \left(t\, E\left(q^{\frac{\ln t}{\ln C}}\right)\right),\ t > 1$.

(ii) $h(t) \sim k/t,\ t \to \infty$.

Proof. (i) The functions $q^{\frac{\ln t}{\ln x}} g(x)$ and

$$(\partial/\partial t)(q^{\frac{\ln t}{\ln x}}\, g(x)) = -\ln(q^{-1})q^{\frac{\ln t}{\ln x}} g(x)/(t \ln x)$$

are both continuous in the region $1 < x \leq q^{-1/k},\ t > 1$ and can be extended as continuous functions in $1 \leq x \leq q^{-1/k},\ t > 1$. Then differentiation under the expectation sign in $E(q^{\frac{\ln t}{\ln x}})$ is permitted and:

$$-\bar{F}'(t) = -(d/dt)E\left(q^{\frac{\ln t}{\ln C}}\right) = \frac{\ln(q^{-1})E\left(\frac{1}{\ln C} q^{\frac{\ln t}{\ln C}}\right)}{t}$$

from where part *(i)* follows.

(ii) The expectation in the numerator of *(i)* can be put in the form of the integral (2), where the integrand is

$$\ln(q^{-1})q^{\frac{\ln t}{\ln x}}g(x)/(t\ln x).$$

Then applying the assumption about $g(x)$ in (1) we get

$$\frac{g(-x)}{\ln(-x)} \sim \frac{\gamma(x+q^{-1/k})^{\lambda-1}}{\ln q^{-1/k}},\ x \downarrow -q^{-1/k}$$

and then, using Laplace's method in Lemma (2)

$$E\left(\frac{1}{\ln C}q^{\frac{\ln t}{\ln C}}\right) \sim \frac{c}{\ln(q^{-1/k})}t^{-k}(\ln t)^{-\lambda},\ t\to\infty,$$

where $c = \gamma\Gamma(\lambda)/\beta^{\lambda}$. We can conclude from Proposition (3) and part *(i)* that

$$h(t) \sim \frac{\ln(q^{-1})}{t\ln(q^{-1/k})} = \frac{k}{t},\ t\to\infty.$$

□

Notice that from the last result it is also true that

$$\lim_{t\to\infty} t\,h(t) = k < \infty$$

This result is related to the class $\mathcal{S}^*$ introduced in [7], and defined as follows. F belongs to the class $\mathcal{S}^*$ if F has finite expectation μ and

$$\lim_{x\to\infty}\int_0^x \frac{\bar{F}(x-y)}{\bar{F}(x)}\bar{F}(y)dy = 2\mu$$

In [7] it is proved that if F is absolutely continuous, has finite mean μ and a hazard rate function $h(.)$ such that

$$\lim_{x\to\infty} x\,h(x) = \text{constant} < \infty$$

then $F \in \mathcal{S}^*$. ([7], th. 3.2 and 3.3). Using this result together with Proposition 1 and Lemma 6 we can conclude that if $k > 1$ then $F \in \mathcal{S}^*$.

The class $\mathcal{S}^*$ has another property, related to collective risk theory. If $F \in \mathcal{S}^*$ then $F \in \mathcal{S}$ and also $F_I \in \mathcal{S}$, where F_I is the integrated tail function

$$F_I(x) := \frac{1}{\mu}\int_0^x \bar{F}(y)dy,\ x \geq 0$$

([7], th. 3.3). The basic model of collective risk theory is defined as follows. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common cdf F

and mean μ. This sequence represents the succesive amounts of individual claim payments made by an Insurance Company. And let $N(t),\ t \geq 0$, be a Poisson process with parameter $\rho > 0$, representing the number of claims up to time t (years). Every time a claim arrives a payment is done. Define the annual premium as the quantity Π such that $\Pi > \mu\rho$. This is the total payment made by the insurers in one year. And define the amount of reserves at the beginning of the period $[0, \infty)$ as the variable x. In the presence of a heavy tailed distribution, for instance, if $F \in \mathcal{S}^*$, it is a key question to estimate the probability of ruin defined as follows

$$\psi(x) := 1 - \lim_{t\to\infty} P\left(\sum_{i=1}^{N(s)} X_i \leq x + \Pi s,\ \forall s \in (0, t]\right).$$

Then, theorem 4.6 by Embrechts and Veraverbecke [5], provides the following asymptotic estimative of $\psi(x)$

$$\psi(x) \sim \frac{\rho}{\Pi - \rho\mu} \int_x^\infty \bar{F}(t)dt,\ x \to \infty.$$

Consider again F the cdf of $X = C^D$ with $k > 1$. Applying Karamata's theorem 5 together with Proposition (3.*ii*) leads to

$$\int_x^\infty \bar{F}(t)dt \sim \frac{c}{k-1} x^{1-k}(\ln x)^{-\lambda},\ x \to \infty,$$

and the following estimative for the probability of ruin

$$\psi(x) \sim \frac{c\rho}{(\Pi - \mu\rho)(k-1)} x^{1-k}(\ln x)^{-\lambda},\ x \to \infty.$$

4 Conclusions

The distribution of the random variable $X = C^D$ posseses several properties related to the fact that it is a heavy tailed distribution. The representation of the tail and the hazard rate functions as expectations or quotients of expectations provides a way of evaluating them through extensive simulation of the random variable C.

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