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Uniform blow up of a nonlinear parabolic PDE

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Abstract
In this paper we study the blow up behavior of certain nonlinear PDE with weight on the diffusion term.

Keywords: Nonlinear Parabolic Equation

MSC(2000): 35K55

1 Introduction

In this paper we study the behavior of the solutions $u(x,t)$ to the initial value problem

$$\begin{array}{ll} u_t = \overline{u} u_{xx} + u^2 \quad \text{in} \quad (0,1)\times(0,T) \\ u_x(0,t) = u_x(1,t) = 0 \\ u|_{t=0} = u_0 > 0 \end{array} \tag{1}$$

where

$$\overline{u} = \int_0^1 u(x,t)\, dx$$

is the average of u at time t. It is not difficult to show that this equation has a unique solution for finite time $T < \infty$, and $\limsup_{t\to T} u(x,t) = \infty$. Also, to avoid technicalities, we assume that u_0 satisfies all the necessary compatibility conditions so that u is smooth on $[0,1]\times[0,T)$.

What intuition dictates is that if the weight of the diffusion term is large enough, then solutions to (1) will uniformize in the following sense

$$r(t) := \frac{u_{max}(t)}{u_{min}(t)} \to 1 \quad \text{as} \quad t \to T$$

where $T < \infty$ is the blow up time.

As a first instance we show the following result

Theorem 1.1. *Let $u_0 \in C^\infty([0,1])$ such that*

(i) $(u_0)_x(0) = (u_0)_x(1) = 0$

(ii) $\int_0^1 u\cos(j\pi x)\, dx \geq 0$

Then, there is $\epsilon > 0$ *such that if*

$$\frac{(u_0)_{max}}{(u_0)_{min}} < 1 + \epsilon$$

then

$$\frac{u_{max}(t)}{u_{min}(t)} \to 1 \quad as \quad t \to T$$

Different Theorems (though similar to the previous one) are considered in. We specially note that one can show a uniformization result for an equation with memory (subsection 4.1).

Beyond what is actually proven in this note, we want to propose a new method to prove this type of result for Initial Value Problems of the form,

$$\begin{array}{l} u_t = f(u,t)\,\Delta u + u^2 \quad \text{in} \quad \Omega \times [0,T) \\ \frac{\partial u}{\partial \eta} u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) > 0 \end{array}$$

Of course this is by no means a complete work, but an initial effort. Our heuristic arguments (and what induces us to believe that this method could work in full generality, please see Section 5) rely a lot on some sort of "probabilistic" arguments.

The profile behavior of certain types of nonlinear heat equations have been observed and studied by several authors (see [3] or [4] and the references therein). One of our motivation comes from the study of the Ricci flow with boundary conditions in dimension 2. In this particular case, the evolution equation for the curvature is given by

$$R_t = \exp\left(\int_0^t R(x,\tau)\, d\tau\right) \Delta R + R^2$$

And it is shown in [2] that the curvature uniformizes in the sense described above (our equation with memory is similar to the Ricci flow equation for the scalar curvature).

Let us give an idea of what our methods are. To prove Theorem 1.1 we propose the following argument. First we approximate problem (1) by a discretization in time. Then we show the following estimates

Lemma 1.2. *Let* v *be a solution to the problem (*$k > 0$ *is a constant),*

$$\begin{array}{l} v_t = k u_{xx} \qquad in \quad (0,1) \times (0,\infty) \\ v_x(0,t) = v_x(1,t) = 0 \\ v|_{t=0} = v_0 \end{array}$$

And v_0 *satisfies hypothesis (i) and (ii) of Theorem 1.1, then*

$$r(t) \le 1 + (r_0 - 1)\exp\left(-k\pi^2 t\right)$$

Lemma 1.3. *Let* v *be a solution of*

$$\begin{cases} v_t = v^2 \\ v|_{t=0} > 0 \end{cases}$$

Then if $V_0 = v_{max}(0)$*, the following estimate holds,*

$$r(t) \leq 1 + (r(0) - 1)\exp(2V_0 t)$$

for $0 \leq t \leq \frac{1}{2V_0}$

A combination of these lemmas will produce the following

Proposition 1.4. *There is* $\epsilon > 0$ *such that if* $r(0) < 1 + \epsilon$ *then it remains so for* $0 \leq t < T$

Finally we show the following estimate. There is a $\delta > 0$ such that

$$r(t) \leq 1 + (r(0) - 1)\exp\left(-\delta \int_0^t u_{max}(\tau)\, d\tau\right)$$

From the fact that $u_{max}(t)$ is non integrable, will follow that

$$r(t) \to 1 \quad \text{as} \quad t \to T$$

This paper is organized as follows. In section 2 we describe the discretization procedure which in Numerical Analysis is known as "splitting up" method. We show that if T is the blow up time, for any $0 < \tau < T$ the approximations given by the method converge to a solution of (1) in $[0, \tau]$. In Section 3 we prove some estimates and then Theorem 1.1; in Section 4 we present a couple of different results obtained by following some of the ideas in the proof of Theorem 1.1; in Section 5 we present some of the heuristics that make us believe our methods can be used in more generality.

2 Approximation procedure

The scheme we consider is the following. For a given $\tau > 0$ fixed we take a partition,

$$0 = t_0 < t_1 < t_{1+\frac{1}{2}} < t_{2+\frac{1}{2}} < \ldots < t_{k-1} < t_{k-1+\frac{1}{2}} < t_k = \tau < T$$

$$\text{where} \quad t_j = j\theta \quad \text{and} \quad t_{j+\frac{1}{2}} = t_j + \frac{\theta}{2}$$

Define w as follows. On the interval $[t_j, t_{j+\frac{1}{2}})$ define it as the solution of the IVP

$$\begin{cases} w_t = \overline{w}(t_j)\, w_{xx} & \text{in} \quad (0,1) \times (0, \infty) \\ w_x(0,t) = w_x(1,t) = 0 \\ w|_{t=t_j} = w(x, t_j) \end{cases} \tag{2}$$

And then define w in the interval $[t_{j+\frac{1}{2}}, t_{j+1})$ as the solution of

$$\begin{cases} w_t = w^2 \\ w|_{t=t_{j+\frac{1}{2}}} = w\left(x, t_{j+\frac{1}{2}}\right) \end{cases} \tag{3}$$

Before we show that the approximation procedure converges to the unique solution of Problem (1), we need the following Lemma,

Lemma 2.1. *Assume $0 < u_0 < C$, and that u satisfies*

$$\begin{cases} u_t = k u_{xx} \qquad in \quad (0,1) \times (0,\infty) \\ u_x(0,t) = u_x(1,t) = 0 \\ u_{t=0} = u_0 \end{cases}$$

Then, the derivatives of u satisfy,

$$\left|\left(\frac{\partial}{\partial x}\right)^n u(x,t)\right| \leq \max_{x\in[0,1]} \left|\left(\frac{\partial}{\partial x}\right)^n u_0(x)\right|.$$

Proof. We compute evolution equations for the spatial derivatives,

$$\frac{\partial}{\partial t}\left|\left(\frac{\partial}{\partial x}\right)^n u\right|^2 \leq \Delta \left|\left(\frac{\partial}{\partial x}\right)^n u\right|^2$$

and the boundary conditions for n odd and even respectively,

$$\begin{cases} \left|\left(\frac{\partial}{\partial x}\right)^{2k+1} u(0,t)\right|^2 = \left|\left(\frac{\partial}{\partial x}\right)^{2k+1} u(1,t)\right|^2 = 0 \\ \frac{\partial}{\partial x}\left|\left(\frac{\partial}{\partial x}\right)^{2k} u(0,t)\right|^2 = \frac{\partial}{\partial x}\left|\left(\frac{\partial}{\partial x}\right)^{2k} u(1,t)\right|^2 = 0 \end{cases}$$

The estimate follows from the Maximum Principle. □

Remark 2.2. *The previous Lemma shows that as long as w, the solution produced by the approximation procedure, is bounded we can bound any number of derivatives in terms of the derivative of the initial condition and the bound on w.*

Proposition 2.3. *The proposed scheme converges uniformly along a sequence $\theta \to 0$ to the unique solution of (1)*

Proof. We fix ρ to be the interval of existence of the unique solution of

$$\begin{cases} z_t = z^2 \\ z(x,0) = u_0(x) > 0. \end{cases}$$

Then, the maximum principle shows that on $\left[0, \frac{\rho}{2}\right)$, w, the approximation produced by the approximation procedure, is uniformly bounded from above by

$$\max_{(x,t)\in[0,1]\times\left[0,\frac{\rho}{2}\right]} z(x,t)$$

Define,

$$v^{\theta}(x,t) = w(x,t_j) + \frac{2(t-t_j)}{\theta}(w(x,t_{j+1}) - w(x,t_j)) \tag{4}$$

We are going to show that for a sequence $\theta \to 0$ v_θ converges uniformly, and also satisfies an equation of the form

$$v_t^{\theta} = \overline{v}^{\theta}(t)\, v_{xx} + \left(v^{\theta}\right)^2 + \text{Hölder function of} \quad \theta \tag{5}$$

This wherever v_t exists, which is everywhere with the exception of finitely many points. This will show that for a sequence $\theta \to 0$ the sequence of v^{θ}'s thus defined converges uniformly on $\left[0, \frac{\rho}{2}\right)$ to a weak solution of (1). By uniqueness and regularity, the result follows on $\left[0, \frac{\rho}{2}\right]$. It is then clear how to show that indeed this is true for any compact subinterval of $[0,T)$.

To show the uniform convergence statement, the estimates in Lemma 2.1 and a diagonalization argument show that we can find a dense set $\{t_k\} \subseteq \left[0, \frac{\rho}{2}\right)$, such that $v^{\theta}(x,t_k)$ converges uniformly to a function $v(x,t_k)$ for t_k fixed along a sequence $\{\theta_m\}_m$. We can pick the sequence $\{\theta_m\}_m$ in such a way that $\theta_l = q\theta_m$ for a positive integer q whenever $m > l$. By its definition (given in equation 4), we can bound $v_t^{\theta_l}$ on the interval $[n\theta_l, (n+1)\theta_l]$. This bound does not depend on l, it only depends on the modulus of z. Call this bound M. Using this fact, one can show that the convergence is uniform. Indeed, Let $\epsilon > 0$, choose $N > 0$ such that if $m > l > N$, then $\left|v^{\theta_l}(x,t_k) - v^{\theta_m}(x,t_k)\right| < \frac{\epsilon}{2}$ for all x and k. Given any $t \in \left[0, \frac{\rho}{2}\right)$ we find n such that $t \in [n\theta_m, (n+1)\theta_m)$. Let t_k be such that $|t - t_k| < \frac{\epsilon}{2M}$. Then using the triangle inequality we can bound

$$\left|v^{\theta_l}(x,t) - v^{\theta_m}(x,t)\right| < \epsilon$$

Let us proceed to show that our sequence satisfies equation (5) (for convenience, we will drop the superscript θ). Differentiating (4) with respect to t and applying the mean value theorem we obtain

$$\begin{aligned} v' &= \tfrac{1}{\theta}(w(x,t_{j+1}) - w(x,t_j)) \\ &= \tfrac{2}{\theta}\left(w(x,t_{j+1}) - w\left(x,t_{j+\frac{1}{2}}\right)\right) + \tfrac{2}{\theta}\left(w\left(x,t_{j+\frac{1}{2}}\right) - w(x,t_j)\right) \\ &= w'(x,\tau_1) + w'(x,\tau_2) \end{aligned}$$

with $\tau_2 \in \left(t_j, t_{j+\frac{1}{2}}\right)$ and $\tau_1 \in \left(t_{j+\frac{1}{2}}, t_{j+1}\right)$. Therefore

$$v' = \overline{w}(t_j)\, w_{xx}(x,\tau_1) + w^2(x,\tau_2)$$

On one hand we have

$$\begin{aligned} \left|\overline{w}(t_j)\, w_{xx}(x,\tau_1) - \overline{v}(t)\, v_{xx}(x,t)\right| \leq\ & |\overline{w}(t_j)|\, |w_{xx}(x,\tau_1) - v_{xx}(x,t)| \\ & + |\overline{w}(t_j) - \overline{v}(t)|\, |v_{xx}(x,t)| \end{aligned}$$

We start now bound the term $|\overline{w}(t_j) - \overline{v}(t)|$. First notice that

$$\begin{aligned}\overline{w}(t_j) - \overline{v}(t) &= \tfrac{2(t-t_j)}{\theta}\left(\overline{w}(t_{j+1}) - \overline{w}(t_j)\right)\\ &= \tfrac{2(t-t_j)}{\theta}\left(\overline{w}(t_{j+1}) - \overline{w}\left(t_{j+\frac{1}{2}}\right)\right).\end{aligned}$$

From this follows that (by (3), notice also that we are using the fact that the solution to (2) has constant average in space),

$$|\overline{w}(t_j) - \overline{v}(t)| \leq C\theta$$

On the other hand,

$$\begin{aligned}v_{xx}(x,t) - w_{xx}(x,\tau_1) &= w_{xx}(x,t_j) - w_{xx}(x,\tau_1)\\ &\quad + \tfrac{2(t-t_j)}{\theta}\left(w_{xx}(x,t_{j+1}) - w_{xx}\left(x,t_{j+\frac{1}{2}}\right)\right)\\ &\quad + \tfrac{2(t-t_j)}{\theta}\left(w_{xx}\left(x,t_{j+\frac{1}{2}}\right) - w_{xx}(x,t_j)\right)\end{aligned}$$

By the Schauder estimates, we have the following bounds

$$|w_{xx}(x,t_j) - w_{xx}(x,\tau_1)| < C\theta^{\alpha}$$
$$\left|w_{xx}\left(x,t_{j+\frac{1}{2}}\right) - w_{xx}(x,t_j)\right| < C\theta^{\alpha}$$

where C depends on $\overline{w}(t_j)$ and a bound on $|w_{xxx}(x,t_j)|$ independent of the patition, which indeed we can produce by the observation following Lemma 2.1.

The estimate

$$\left|w_{xx}(x,t_{j+1}) - w_{xx}\left(x,t_{j+\frac{1}{2}}\right)\right| < C\theta$$

is obtained from (3) without difficulty. □

Remark 2.4. *We can apply the same scheme to a problem of the form*

$$\begin{cases} u_t = f(u)\,\Delta u + u^2 & \text{in} \quad (0,1)\times(0,T) \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

under the assumption that f is smooth.

3 The Estimates

In order to show our first estimate, the following fact will be useful

Lemma 3.1. *Let u be a solution to*

$$\begin{cases} u_t = u_{xx} & \text{in} \quad (0,1)\times(0,\infty) \\ u_x(0,t) = u_x(1,t) = 0 \end{cases}$$

in $[0,1]\times(t_0,t_1)$. Let x_t be an interior relative minimum point of $u(\cdot,t)$. There is $\epsilon > 0$ and $\delta(\epsilon) > 0$ such that $u(\cdot,\tau)$ is increasing on $(x_t-\epsilon, x_t+\epsilon)$ for $\tau \in [t, t+\delta)$.

Proof. Since u_{xx} satisfies a heat equation, its zeroes are isolated. Hence, using the fact that x_t is a relative minimum of $u(\cdot, t)$, we can find $\epsilon > 0$ and $\delta > 0$ such that if $x \in (x_t - \epsilon, x_t + \epsilon)$ then $u_{xx}(x,t) \geq 0$ and $u_{xx}(x+\epsilon, \tau), u_{xx}(x-\epsilon, \tau) \geq 0$ for $0 \leq \tau \leq \delta$. By the Maximum Principle we must have $u_{xx}(x,t) \geq 0$ for $(x,t) \in (x_t - \epsilon, x_t + \epsilon) \times [t, t+\delta)$.

□

Lemma 3.2. *Let v be a solution to the problem*

$$\begin{cases} v_t = k u_{xx} & in \quad (0,1) \times (0,\infty) \\ v_x(0,t) = v_x(1,t) = 0 \\ v|_{t=0} = v_0 > 0 \end{cases} \tag{6}$$

Assume v_0 satisfies hypothesis (i) and (ii) of Theorem 1.1, then

$$r(t) \leq 1 + (r_0 - 1)\exp\left(-k\pi^2 t\right) \tag{7}$$

Proof. We will show that the time interval where (7) holds is open and closed. Since it is valid at $t = 0$ we get the result. Closedness is easily verified. Let us show that this interval is also open.

Let t_0 be such that

$$r(t_0) = 1 + (r(0) - 1)\exp\left(-k\pi^2 t_0\right)$$

We will show that

$$r(t) \leq 1 + (r(t_0) - 1)\exp\left(-k\pi^2 (t - t_0)\right)$$

for a short time $t \in [t_0, t_0 + \delta)$ as this implies the result.

Let $U(t) = \max_{x\in[0,1]} u(x,t)$. Write

$$u(x,t) = n_0 + \sum_{j\geq 1} n_j \exp\left(-k\pi^2 j^2 (t - t_0)\right)\cos(k\pi j x)$$

By hypothesis (ii) we have

$$U(t) = n_0 + \sum_{j\geq 1} n_j \exp\left(-k\pi^2 j^2 (t - t_0)\right)$$

and this yields,

$$U(t) - u(x,t) = \sum_{j\geq 1} (n_j - \alpha_j n_j)\exp\left[-k\pi^2 j^2 (t - t_0)\right]$$

since $|\alpha_j| \leq 1$ the previous inequality together with hypothesis (ii) shows that

$$(U(t) - u(x,t))\exp\left[-k\pi^2 (t - t_0)\right]$$

is decreasing.

Therefore we have the estimate

$$U(t) - u(x,t) \leq (U(t_0) - u(x,t_0)) \exp\left[-k\pi^2 (t - t_0)\right]$$

From this we get

$$\frac{U(t)}{u(x,t)} \leq 1 + \left(\frac{U(t_0)}{u(x,t)} - \frac{u(x,t_0)}{u(x,t)}\right) \exp\left[-k\pi^2 (t - t_0)\right]$$

which produces the inequality

$$\frac{U(t)}{u(x,t)} \leq 1 + \left(r(t_0) - \frac{u(x,t_0)}{u(x,t)}\right) \exp\left[-k\pi^2 (t - t_0)\right]$$

Now we have to be a little bit careful (but not much). If x is such that $u_t(x,t) > 0$, then we obtain the inequality

$$\frac{U(t)}{u(x,t)} \leq 1 + (r(t_0) - 1) \exp\left[-k\pi^2 (t - t_0)\right]$$

Let x_0 be such that $u(x_0, t_0)$ is the minimum of u at time t_0. Let V be a small neighborhood of x_0. Define

$$M_\delta = \{x : u_t(x,t) > 0 \quad \text{for} \quad t \in [t_0, t_0 + \delta)\}$$

We can choose V such that $x_t \in V$, $u(x_t, t)$ is the minimum of u at time t and $x_t \in M_\delta$ for $\delta > 0$ small enough (this follows from Lemma 3.1). Then we have

$$\sup_{M_\delta} \frac{U(t)}{u(x,t)} \leq 1 + (r(t_0) - 1) \exp\left[-k\pi^2 (t - t_0)\right]$$

But $r(t) = \sup_{M_\delta} \frac{U(t)}{u(x,t)}$, and this shows the lemma.

□

Lemma 3.3. *Let v be a solution of*

$$\begin{cases} v_t = v^2 \\ v|_{t=0} > 0 \end{cases}$$

Then if $V_0 = v_{max}(0)$, the following estimate holds,

$$r(t) \leq 1 + (r(0) - 1) \exp(2V_0 t) \quad \text{for} \quad 0 \leq t \leq \frac{1}{2V_0}$$

Proof. We can solve explicitly for $v_{max}(t)$,

$$v_{max}(t) = \frac{V_0}{1 - V_0 t} \quad \text{where} \quad V_0 = \max v(\cdot, 0)$$

and also,

$$v_{min}(t) = \frac{v_0}{1 - v_0 t} \quad \text{where} \quad v_0 = \min v(\cdot, 0)$$

Then all we must show is that

$$r(t) = \frac{V_0}{v_0}\left(\frac{1 - v_0 t}{1 - V_0 t}\right) \leq 1 + (r(0) - 1)\exp(2V_0 t)$$

Let $t = \frac{\delta}{2V_0}$ with $0 \leq \delta < 1$. Then

$$r(t) = r_0\left(\frac{2 - \frac{\delta}{r_0}}{2 - \delta}\right)$$

So we must verify that

$$r_0\left(\frac{2 - \frac{\delta}{r_0}}{2 - \delta}\right) \leq 1 + (r_0 - 1)\exp(\delta)$$

which is equivalent to the following inequality

$$0 \leq r_0\left[(2 - \delta)\exp(\delta) - 2\right](r_0 - 1)$$

Everything reduces then to show that $(2 - \delta)\exp(\delta) - 2 \geq 0 \quad \text{if} \quad \delta \leq 1$. To do so, consider the function $f(x) = (2 - x)\exp(x) - 2$. A simple calculation shows that

$$f'(x) = \exp(x)(1 - x) \geq 0 \quad \text{if} \quad x \leq 1$$

Since $f(0) = 0$, the proof of the lemma is complete. □

Notation. We will adopt the following conventions

$$r_k := r(t_k), \quad W_k = \max W(\cdot, t_k)$$

Remark 3.4. *If w_0 satisfies conditions (i) and (ii) in Theorem 1.1, then $w(x,t)$ the solution to*

$$\begin{cases} w_t = w^2 \\ w(\cdot, 0) = w_0 \end{cases} \tag{8}$$

continues to satisfy these conditions for all $t > 0$ (as long as the solution exists).

Lemma 3.5. *There is $\epsilon > 0$ such that if $r(0) < 1 + \epsilon$ then $r(t) < 1 + \epsilon$*

Proof. Let $\{t_0, \dots, t_k\}$ be the partition of the approximation. Assume that up to time t_j it holds that

$$\frac{w_{max}(t)}{w_{min}(t)} \leq 1 + \epsilon$$

Since in the interval $[t_j, t_{j+\frac{1}{2}})$ w satisfies

$$\begin{cases} w_t = \overline{w}(t_j)\, w_{xx} \\ w_x(0) = w_x(1) = 0 \end{cases}$$

then, Lemma 3.2 yields for $t \in \left[t, t_{j+\frac{1}{2}}\right]$

$$\begin{aligned} \tfrac{w_{max}(t)}{w_{min}(t)} &\leq 1 + (r_j - 1)\exp\left(-\tfrac{\pi^2}{1+\epsilon} W_j (t - t_j)\right) \\ &\leq 1 + \epsilon \end{aligned}$$

On the other hand, on the interval $[t_{j+\frac{1}{2}}, t_{j+1})$, w satisfies (8). Since by the Maximum Principle $W_{j+\frac{1}{2}} \leq W_j$, we obtain for $t \in \left[t_{j+\frac{1}{2}}, t_{j+1}\right]$

$$\begin{aligned} \tfrac{w_{max}(t)}{w_{min}(t)} &\leq 1 + \left(r_{j+\frac{1}{2}} - 1\right)\exp\left[2W_j\left(t - t_{j+\frac{1}{2}}\right)\right] \\ &\leq 1 + \left((r_j - 1)\exp\left[-\tfrac{\pi^2}{1+\epsilon} W_j\left(t_{j+\frac{1}{2}} - t_j\right)\right]\right)\exp\left[2W_j\left(t - t_{j+\frac{1}{2}}\right)\right] \\ &\leq 1 + (r_j - 1)\exp\left[\left(2 - \tfrac{\pi^2}{1+\epsilon}\right) W_j\left(t - t_{j+\frac{1}{2}}\right)\right] \\ &\leq 1 + \epsilon \end{aligned}$$

Hence, for any $t \in [t_j, t_{j+1}]$ we have the estimate

$$r(t) \leq 1 + \epsilon, \tag{9}$$

as long as $\frac{\pi^2}{1+\epsilon} > 2$. This finishes the proof of the lemma.

□

As a corollary from the previous proof, we get the following estimate

Corollary 3.6. *There is* $\delta > 0$

$$r(t) \leq 1 + (r(0) - 1)\exp\left(-\delta \int_0^t u_{max}(\tau)\, d\tau\right)$$

Proof. Fix $t \in [0, T)$, and pick a partition $\{t_0, \dots, t_{m+1}\}$ of $[0, t]$. Then, iterating inequality (9) yields

$$r(t) \leq 1 + (r_0 - 1)\exp\left(-\delta \sum_{j=0}^{m} W_j \frac{\Delta t}{2}\right)$$

where Δt is the size of the partition, i.e., $\Delta t = t_{j+1} - t_j$.

Since as $\Delta t \to 0$

$$\sum_{j=0}^{m} W_j \left(t_{j+1} - t_j\right) \to \frac{1}{2} \int_0^t u_{max}\left(\tau\right) \, d\tau$$

the estimate follows.

□

We have the following simple fact

Proposition 3.7. *We have*

$$\int_0^T u_{max} \, d\tau = \infty$$

Proof. Let $f\left(t\right)$ be a function equal to u_{max} at $t = 0$, and solving

$$\frac{df}{dt} = 2u_{max} f$$

Then we have

$$\begin{cases} \frac{\partial}{\partial t}\left(u - f\right) \leq u\Delta\left(u - f\right) + 2u_{max}\left(u - f\right) \\ \frac{\partial}{\partial \eta}\left(u - f\right) = 0 \end{cases}$$

Since $u_{max} \to \infty$ as $t \to T$, we have $f \to \infty$. But

$$\log \frac{f\left(t\right)}{f\left(0\right)} = 2 \int_0^t u_{max}\left(\tau\right) \, d\tau$$

and the statement of the proposition follows.

□

Finally, Corollary 3.6 and Proposition 3.7 imply Theorem 1.1.

4 Miscellaneous Results

In this section we illustrate another type of result that can be obtained by the methods introduced before.

Proposition 4.1. *Let*

$$u_0\left(x\right) = M + \sum_{k=1}^{\infty} \lambda_k \cos\left(k\pi x\right)$$

Assume that we can find $\beta > 0$ and $\alpha > 0$ such that

(i) $|\lambda_k| \leq \beta \alpha^{k-1}$

(ii) $\frac{\beta}{\alpha}+\frac{\beta\alpha}{1-\alpha^2}<M\left(\pi^2-1\right)$
Then for a solution of (1) we have

$$\lim_{t\to T}\frac{u\left(x,t\right)}{\overline{u}\left(t\right)}=1.$$

We call conditions (i) and (ii) pinching conditions.

Proof. The idea of the proof is as follows: Consider a small interval of time $[0,\Delta t]$ Then, on the interval $\left[0,\frac{\Delta t}{2}\right]$ we find an approximate equation satisfied by the Fourier coefficients of the solution of the equation

$$\begin{cases} v_t=v^2 \\ v\left(x,0\right)=u_0 \end{cases}$$

The explicit solution of this equation (up to first order in t) is given by,

$$v\left(x,t\right)=\frac{u_0\left(x\right)}{1-u_0\left(x\right)t}\sim u_0\left(x\right)+u_0\left(x\right)^2t$$

From this, and the first pinching hypothesis, we get the following estimate for the Fourier coefficients of u,

$$\left|\lambda_p\left(\frac{\Delta t}{2}\right)\right|\leq\beta\alpha^{p-1}+\beta\alpha^{p-1}\left[\frac{\beta\alpha}{1-\alpha^2}+\frac{p\beta}{\alpha}+M\right]\frac{\Delta t}{2}$$

For a solution of

$$\begin{cases} u_t=\overline{u}\Delta u \\ u_x\left(0,t\right)=u_x\left(0,t\right)=0 \\ u\left(x,\frac{\Delta t}{2}\right)=v\left(x,\frac{\Delta t}{2}\right) \end{cases}$$

the Fourier coefficients are given by the formula

$$\lambda_p\left(\Delta t\right)=\lambda_p\left(\frac{\Delta t}{2}\right)\exp\left(-\overline{u}p^2\pi^2\frac{\Delta t}{2}\right)\leq\lambda_p\left(\frac{\Delta t}{2}\right)\exp\left(-Mp^2\pi^2\frac{\Delta t}{2}\right)$$

Now, combining these results, we obtain the formula for the Fourier coefficients of a solution to (1) using the approximation scheme for t small

$$\lambda_p\left(\Delta t\right)\leq\beta\alpha^{p-1}\exp\left(\left[\frac{\beta\alpha}{1-\alpha^2}+\frac{p\beta}{\alpha}+M-Mp^2\pi^2\right]\frac{\Delta t}{2}\right)$$

From this, using a limiting process as before, we conclude that the pinching conditions are satisfied for all time. Since $\overline{u}\to\infty$ the proposition follows. □

Remark 4.2. *Notice that from the previous proof we can conclude that* $\lim_{t\to T}\left\|u-\overline{u}\right\|_{L^\infty}=0$.

An example of a u_0 satisfying the pinching conditions of Proposition 4.1 is

$$u_0\left(x\right)=2+\sum_{n=1}^{\infty}\epsilon_k\left(\frac{1}{2}\right)^k\cos\left(n\pi x\right),\epsilon_k=\pm1$$

4.1 An Equation with memory

To exemplify our methods we work with a type equation that has been seldom treated. Consider the following problem

$$\begin{cases} u_t = \exp\left(2\int_0^t \overline{u}(\tau)\, d\tau\right)\Delta u + u^2 \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = u_0 > 0 \end{cases}$$

One can show the following fact about solutions to this equation

Proposition 4.3. *Under the pinching conditions of Proposition 4.1 plus* $u_0 <$ 1*, there is an* $\epsilon > 0$ *such that if*

$$\frac{u_{max}(0)}{u_{min}(0)} < 1 + \epsilon$$

then $\frac{u_{max}(t)}{u_{min}(t)} \to 1$ *when* $t \to T$.

Proof. Assuming that a splitting up method works for this equation (and indeed, it can be proved it does), then one can show that (and here is where we use the hypothesis $u_0 < 1$),

$$\exp\left(2\int_0^t \overline{u}(\tau)\, d\tau\right) \geq \overline{u}(t).$$

The proposition follows, for conveniently chosen pinching conditions as proposition 4.1.

□

4.2 More General Estimates

So far our estimates on the diffusion rates for the nonlinear problem, need some technical hypothesis. The following estimate is always true.

Proposition 4.4. *A solution of (1) satisfies*

$$\begin{aligned} r(t) \leq\ & 1 + (r(0) - 1)\exp\left(\int_0^t -\pi^2 \tfrac{U(\tau)}{r(\tau)} + 2U(\tau)\, d\tau\right) \\ & + \pi^2 \int_0^t \tfrac{U(\tau)}{r(\tau)}(\tau)(r(\tau) - 1)\exp\left(\int_\tau^t -\pi^2 \tfrac{U(\sigma)}{r(\sigma)} + 2U(\sigma)\, d\sigma\right) d\tau \end{aligned}$$

where $U = \max_{x\in[0,1]} u(x,t)$.

The previous proposition is just a curious estimate. The method of proof is the same as before. However, this estimate is not very useful. Indeed, the righthand side of the inequality is nondecreasing, so it doesn't show whether the ratio improves or does not improve with time, and the reason for this is the π^2 in the second integral. Would it be possible under certain hypothesis to improve the value of that constant?

5 Some Heuristics

The scheme proposed to solve equations with a diffusion term plus a nonlinear term, reduces the study of the uniformization properties of such equation to the study of two separate problems. We can then say that there is a conflict between two operators, one the diffusion term which tries to uniformize, and the nonlinear term which tries to concentrate. What we did in our previous arguments was to quantify this uniformization and concentration rates with some technical assumptions (these are the two main estimates in the proof), and then "compare" these rates so that we are able to decide which operator wins the conflict. To set some terminology for the upcoming digression, we set the following notation

Definition 5.1. *Given* $u(x,t)$*, let*

$$M(t) = \max_{x\in[0,1]} u(x,t) \quad m(t) = \min_{x\in[0,1]} u(x,t).$$

As we have seen before, the delicate part of our proof, and the one that requires those frustrating and technical assumptions, is to estimate the rate of uniformization of a solution of the heat equation. More specifically, let u be a solution of

$$\begin{cases} u_t = f(u,t)\,\Delta u \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = u_0(x) > 0, \end{cases}$$

if the following estimate were true without any assumptions on u_0,

$$M(t) - m(t) \leq (M(0) - m(0)) \exp(-\nu_1 F(t_0)\, t) \tag{10}$$

where ν_1 is the first nonzero eigenvalue of the operator Δ with Neumann boundary condition and $F(t_0) = \max_{x\in[0,1]} f(u(x,t_0), t_0)$, then we could expect to show, under some hypothesis on $f(\cdot,t)$, uniformization results as those shown in this note for Boundary Value Problems of the form

$$\begin{cases} u_t = f(u,t)\,\Delta u + u^2 \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = u_0(x) > 0. \end{cases} \tag{11}$$

However estimate (10) is flatantly false. It is easy to construct counterexamples to show that this is not the case when $f = 1$ (i.e., the case of the heat equation). for instance, let

$$u(x,t) = 3 + \exp\left(-\pi^2 t\right)\cos(\pi x) - \frac{1}{9}\exp\left(-9\pi^2 t\right)\cos(3\pi x).$$

But this is not the point where we want to stop. We could ask if for a given f estimate (10) is generically true, i.e, if it is true for a "big" set of initial conditions. If such is the case, could then be true that under certain hypothesis on f (say for instance $f(u,t) = u$), and for most positive initial conditions, solutions of (11) uniformize? Could then a density argument be used to prove that for any positive initial condition, solutions of (11) uniformize? We want to add a grain of faith in our believes with the following result, which we include without a proof, and the last remark.

Proposition 5.2. *Fix* $0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$ *Let* $A_n = \{(\lambda_j) \in [-1,1]^n : \sum_j \lambda_j \geq 0, \sum_k \nu_j^2 \lambda_j \geq 0\}$. *Then,*

$$\mu \left\{ (\lambda_j)_{j \in \mathbf{N}} \in A_n : \sum \nu_j^2 \lambda_j < \nu_1^2 \sum \lambda_j \right\} \to 0 \quad as \quad n \to \infty.$$

Proposition 5.2 would show that if we randomly pick $\lambda_1, \ldots, \lambda_n$, and then consider the function

$$u(x,t) = \sum_{k=1}^{n} \lambda_k \exp\left(-\nu_k^2 t\right) \cos(\nu_k x),$$

then with high probability, for a short time,

$$M(t) - m(t) \leq (M(0) - m(0)) \exp(-\nu_1 t).$$

Remark 5.3. *Consider the boundary value problem*

$$\begin{cases} u_t = u\Delta u + u^2 & in \quad (0,1) \times (0,T) \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = u_0(x) > 0 \end{cases} \tag{12}$$

We can study problem (12) using the methods in [1] (or in [4]). In particular, if $\Delta u_0 + u_0 > C$ *for a constant* C, *then it holds for the solution of (12),i.e., for all time. Then we can show that the measure of the blow up set has positive measure. Finally one can show that for any* $t_k \to T$ *the blow-up time, there is a subsequence of times such that the profiles of the solution converge in* L^2 *to a solution of the equation* $\Delta u + u = 1$ *with Neumann boundary conditions. Since we are working on the interval* $[0,1]$, *then the only possible profile is the constant function 1.*

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