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L^2 harmonic forms and the structure of complete manifolds

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Abstract

In this paper we discuss the relation between the existence of L^2 harmonic one-forms on a complete noncompact manifold and its structure. In particular, we prove an improved Bochner inequality for L^2 harmonic one-forms and demonstrate that in the equality case the set of L^2 harmonic one-forms spans a totally geodesic submanifold.

Keywords: Bochner Technique, Hodge Laplacian, Harmonic Forms

MSC(2000): Primary: 58J05

1 Introduction

There is a strong relationship between the Ricci curvature of a noncompact complete manifold M^n and the spectrum of the Dirichlet Laplacian on it. S. Y. Cheng has proved that if the Ricci curvature of the manifold is bounded below, $Ric_M \geq -(n-1)$, then the lower bound of the essential spectrum of the Dirichlet Laplacian $\lambda_1(M)$ satisfies $\lambda_1(M) \leq \frac{(n-1)^2}{4}$ [1]. A restatement of his Theorem says that there are no noncompact manifolds that satisfy the properties $Ric_M \geq -\frac{4\lambda_1(M)}{n-1} + \epsilon$ for some $\epsilon > 0$ and $\lambda_1(M) > 0$.

P. Li, J. Wang and X. Wang have investigated the case of manifolds for which $\lambda_1(M)$ attains an optimal positive value. We include a brief survey of such results:

Theorem 1.1. [6] *Let M^n $(n \geq 3)$ be a complete conformally compact manifold with $Ric_M \geq -(n-1)$. If $\lambda_1(M) \geq n-2$, then either*

(i) $H^1(L^2(M)) = 0$, or

(ii) M splits into a warped product $M = \mathbb{R} \times N$, where N is a compact manifold.

Theorem 1.2. [4] *Let M^n $(n \geq 3)$ be a complete Riemannian manifold with $Ric_M \geq -\frac{n\lambda_1(M)}{n-1} + \epsilon$ for some $\epsilon > 0$. If $\lambda_1(M) \geq 0$, then the first cohomology group of M satisfies $H^1(L^2(M)) = 0$.*

In a restatement of Theorem 1.2 one could replace the assumptions on the manifold as $Ric_M \geq -(n-1)$ and $\lambda_1(M) = \frac{(n-1)^2}{n} + \epsilon$ for some $\epsilon > 0$.

Theorem 1.3. [5] *Let M^n $(n \geq 3)$ be a complete Riemannian manifold with $Ric_M \geq -(n-1)$. If $\lambda_1(M) \geq n-2$, then either* *(i) M has only one end with infinite volume, or*

(ii) M splits into a warped product $M = \mathbb{R} \times N$, where N is a compact manifold. In this case $\lambda_1(M) = n - 2$.

Their results are based on finding optimal Bochner-type of inequalities for harmonic functions and one-forms on the manifold. In this paper we continue to investigate the implications of such optimal Bochner inequalities and their relationship to the structure of the manifold.

The author would like to thank Professor Peter Li, for bringing this topic to my attention and for the many useful discussions.

2 Main Results

We will be considering the Hodge Laplacian on smooth differential k-forms on a complete noncompact manifold M^n with $n \geq 3$. It is given by

$$\Delta^k = \delta d + d\delta$$

where d is covariant differentiation and δ is the adjoint operator in the L^2 pairing.

For ω, η C^∞ k-forms with compact support:

$$\int (\Delta^k \omega, \eta) = \int (d\omega, d\eta) + \int (\delta\omega, \delta\eta)$$

So Δ can be extended to a positive definite, self-adjoint operator on the space $L^2(\Lambda^k)$, of k-forms with bounded L^2 norm.

Note that in this context the Laplacian Δ on functions has the opposite sign of the standard Dirichlet Laplacian. We keep this sign in order to maintain a consistency between the Laplacians on functions and forms even if this is not the one usually used by geometers. We denote by $\lambda_1(M)$ the lower bound of the essential spectrum of the Dirichlet Laplacian. Recall that the essential spectrum consists of the points in the spectrum that are either accumulation points or have infinite multiplicity.

We begin by stating the standard Bochner Weitzenböck formula for a harmonic form ω. For a coordinate orthonormal frame field $\{V_i\}_i$ with dual coframe $\{dx^j\}_j$

$$-\Delta|\omega|^2 = 2\sum_i |D_{V_i}\omega|^2 - 2(\omega\ ,\ \mathcal{W}\omega) \tag{1}$$

where $\mathcal{W} = -\sum_{i,j} dx^i \wedge i(V_j) R_{V_i V_j}$ is the Weitzenböck tensor [7].

We have used the notation $D^2_{XY} = D_X D_Y - D_{D_X Y}$, whereas $R_{XY} = D_X D_Y - D_Y D_X - D_{[X,Y]}$ is the curvature tensor. The Ricci curvature of the manifold is given by $Ric(X,Y) = \sum_i (R_{V_i X} Y, V_i)$.

The tensor $\mathcal{W}$ is zero on functions and the Ricci curvature on one-forms, as $(\mathcal{W}\omega, \eta) = \mathrm{Ric}(\omega^*, \eta^*)$, where ω^*, η^* are the pointwise dual vectors corresponding to the one-forms ω, η.

If the curvature operator $\rho(X,Y) = (R_{XY}Y, X)_{x_o} \geq \lambda\,|X^* \wedge Y^*|^2$ then, $(\mathcal{W}\eta, \eta)_{x_o} \geq k(N-k)\lambda|\eta|^2$ for any k-form η . For a proof of this relationship and for a further exploration of the Weitzenböck tensor we refer the interested reader to [3].

In previous literature, inequality (1) was combined with the standard Kato's inequality

$$\sum_i |D_{V_i}|\omega||^2 \leq \sum_i |D_{V_i}\omega|^2 \tag{2}$$

to prove vanishing theorems for harmonic forms on compact and noncompact manifolds, depending on $\lambda_1(M)$ (see for example [2], [7]). The computation of stronger Bochner inequalities has allowed for the proof of the Theorems we mentioned in the introduction.

We first prove an improved Bochner inequality for L^2 harmonic one-forms.

Lemma 2.1. *If ω is an L^2 harmonic one-form on a complete manifold, then the Bochner-type inequality*

$$-\Delta|\omega| \geq \frac{1}{|\omega|}\, Ric(\omega, \omega) + \frac{1}{n-1}\frac{|\nabla|\omega||^2}{|\omega|} \tag{3}$$

is true pointwise.

Proof. We now assume that the coordinate frame $\{V_i\}_i$ is normal at a fixed point p. In these normal coordinates $D_{V_i}\omega = (D_{V_i}\omega_j)dx^j$ at p. Then,

$$-\Delta|\omega|^2 = \sum_i D_{V_iV_i}|\omega|^2 = -2|\omega|\;\Delta|\omega| + 2\sum_i \left|D_{V_i}|\omega|\right|^2$$

at p. Combining this equality with the original Bochner-Weitzenböck formula (1) we obtain

$$-|\omega|\,\Delta|\omega| = \sum_i [\;|D_{V_i}\omega|^2 - \left|D_{V_i}|\omega|\right|^2\;] - (\omega\;,\,\mathcal{W}\omega). \tag{4}$$

If we use Kato's inequality, $\sum_i |D_{V_i}\omega|^2 \geq \sum_i \left|D_{V_i}|\omega|\right|^2$, then we get the usual Bochner formula, which allows us to conclude that the norm $|\omega|$ of a harmonic k-form is a subharmonic function whenever the Weitzenböck tensor is nonnegative. For L^2 harmonic one-forms however, this inequality can be improved. We use the known fact due to S. T. Yau, which states that any L^2 harmonic form ω on a complete manifold is both closed and co-closed, in other words $d\omega = \delta\omega = 0$.

The one-form ω is given locally by $\omega = \omega_i\, dx^i$, with the summation index suppressed. Since $\delta\omega\big|_p = 0$, it follows that $\sum_{i=1}^n D_{V_i}\omega_i\big|_p = 0$. Furthermore,

$|\sum_{i=2}^{n} D_{V_i}\omega_i|^2 \leq (n-1)\sum_{i=2}^{n}|D_{V_i}\omega_i|^2$. Combining these two results we obtain $|D_{V_1}\omega_1|^2 \leq (n-1)\sum_{i=2}^{n}|D_{V_i}\omega_i|^2$ and

$$\sum_{i=1}^{n}|D_{V_i}\omega_i|^2 = |D_{V_1}\omega_1|^2 + \sum_{i=2}^{n}|D_{V_i}\omega_i|^2 \geq (1+\frac{1}{n-1})|D_{V_1}\omega_1|^2.$$

It follows that

$$\begin{aligned}\sum_i |D_{V_i}\omega|^2 &= \sum_{i,j}(D_{V_i}\omega_j, D_{V_i}\omega_j) = \sum_i |D_{V_i}\omega_i|^2 + \sum_{i\neq j}(D_{V_i}\omega_j, D_{V_i}\omega_j)\\ &\geq (1+\frac{1}{n-1})|D_{V_1}\omega_1|^2 + 2\sum_{i>j}|D_{V_i}\omega_j|^2\\ &\geq \frac{n}{n-1}\Big[\,|D_{V_1}\omega_1|^2 + \sum_{i>1}|D_{V_i}\omega_1|^2\,\Big] = \frac{n}{n-1}\sum_i |D_{V_i}\omega_1|^2 \qquad (5)\end{aligned}$$

since $\frac{n}{n-1} \leq 1$, for $n \geq 2$.
Without loss of generality we may choose the local frame such that $\omega(p) = |\omega|dx^1(p)$, i.e. $\omega_j(p) = 0$ for $j \neq 1$. Then, for each i

$$\begin{aligned}D_{V_i}|\omega| &= \sum_{j,k} D_{V_i}(\omega_j dx^j, \omega_k dx^k)^{1/2} = \sum_{j,k}((D_{V_i}\omega_j)dx^j\,, \frac{\omega_k dx^k}{|\omega|}) =\\ &= \sum_j (D_{V_i}\omega_j)\,\delta_{j1} = D_{V_i}\omega_1 \qquad (6)\end{aligned}$$

at p.
Inequalities (5) and (6) yield the improved Kato's inequality

$$\sum_i |D_{V_i}\omega|^2 \geq \frac{n}{n-1}\sum_i \big|D_{V_i}|\omega|\big|^2. \qquad (7)$$

The computations we made were for the normal coordinates at p, but (7) holds at every point of the manifold for a general coordinate frame $\{V_i\}$, since it is independent of the coordinate system used.
The Lemma follows by combining inequalities (4) and (7), and noting that the Weitzenböck tensor is the Ricci curvature on one-forms.

□

The proof of Theorem 1.2 strongly relies on the above Lemma (see [4]). If $Ric_M \geq -(n-1)$ and ω is an L^2 harmonic one-form on M, then by inequality (3)

$$-\Delta|\omega| \geq -(n-1)|\omega| + \frac{1}{n-1}\frac{|\nabla|\omega||^2}{|\omega|}.$$

By slightly modifying the argument in Theorem 1.2 one can get that for a cut-off function ϕ which is constant on $B_p(R)$ and vanishes outside $B_p(2R)$:

$$(\lambda_1 - C_\alpha) \int_M \phi^2 |\omega|^2 \leq C' \int_M |\nabla \phi|^2 |\omega|^2$$

for $C_\alpha = \frac{(1+\alpha)(n-1)^2}{n-1+\alpha n}$ and for all $\alpha \in (0, \infty)$. Observe that the right-hand-side of the above inequality goes to zero as $R \to \infty$, whereas the left-hand-side converges to the L^2 norm of ω. Furthermore, $C_\alpha < \frac{(n-1)^2}{n}$ and $\lim_{\alpha \to \infty} C_\alpha = \frac{(n-1)^2}{n}$. If $\lambda_1 > \frac{(n-1)^2}{n}$ we conclude that $\int_M |\omega|^2 = 0$, in other words, $\omega = 0$ everywhere on the manifold. Conversely, if we know that there exists a non-zero L^2 harmonic one-form on the manifold, then $\lambda_1 \leq \frac{(n-1)^2}{n}$.

It would be interesting to see what happens in the borderline case when there exists a harmonic L^2 one-form for which we have an equality in the improved Bochner inequality (3). Note that $\lambda_1 = \frac{(n-1)^2}{n}$ does not necessarily imply such an equality in (3).

Lemma 2.2. *Let M^n be a complete manifold with $Ric_M \geq -(n-1)$. Suppose that ω is a harmonic one-form on M with bounded L^2 norm such that*

$$-\Delta|\omega| = \ -(n-1)|\omega| + \frac{1}{n-1}\frac{|\nabla|\omega||^2}{|\omega|} \tag{8}$$

at a point p.
Then the gradient of ω has the following representation at p in normal coordinates:

$$D_{V_i}\omega_k = \begin{pmatrix} -(n-1)\mu & 0 & & 0 \\ 0 & \mu & & 0 \\ & & \ddots & \\ 0 & 0 & & \mu \end{pmatrix} \tag{9}$$

Proof. As in the proof of Lemma 2.1, we use normal coordinates $\{V_i\}$ at a point p with dual coframe $\{dx^j\}$.
The improved Bochner inequality (3) in Lemma 2.1 implies that ω must in general satisfy the inequality

$$-\Delta|\omega| \geq \ -(n-1)|\omega| + \frac{1}{n-1}\frac{|\nabla|\omega||^2}{|\omega|}$$

at p, but which needs to be an equality under our assumptions. From the proof of Lemma 2.1, it follows that the improved Kato's inequality (7) should also be an equality at p for ω. In fact,

$$\sum_{i=2}^{n} \left|D_{V_i}\omega_i\right|^2 = \frac{1}{n-1}\sum_{i=2}^{n}\left|D_{V_i}\omega_i\right|^2,$$

which is true if and only if

$$D_{V_i}\omega_i = D_{V_j}\omega_j, \quad \forall i, j > 1 \tag{10}$$

at p.
Since ω is an L^2 harmonic form it follows that $d\omega = \delta\omega = 0$ everywhere on the manifold [8].
Combining the fact that $\delta\omega = 0$ and equation (10) we obtain

$$D_{V_1}\omega_1 = -(n-1)\, D_{V_i}\omega_i \quad \forall i > 1$$

at p. The Lemma follows by denoting $D_{V_i}\omega_i$ for $i > 1$ by μ. □

The above Lemma is of similar nature to what happens in the equality case of the more familiar Kato's inequality (2). It is a well known fact that if ω is an L^2 harmonic form such that

$$\sum_i |D_{V_i}|\omega||^2 = \sum_i |D_{V_i}\omega|^2 \tag{11}$$

on M, then ω is parallel and $|\omega|$ =constant.

The Theorem below gives a geometric description of the equality case in our improved Bochner inequality.

Theorem 2.3. *Let M^n be a complete manifold with $Ric_M \geq -(n-1)$. Suppose that ω is a harmonic one-form on M with bounded L^2 norm such that*

$$-\Delta|\omega| = \ -(n-1)|\omega| + \frac{1}{n-1}\frac{|\nabla|\omega||^2}{|\omega|} \tag{12}$$

on M. Then ω spans a totally geodesic submanifold on M whose smoothness depends on the smoothness of ω.

Proof. We again consider normal coordinates at a point p of M and assume that $\omega = |\omega|dx^1 = \omega_1 dx^1$ at p as in the proof of Lemma 2.1.
From the computation in (6) and Lemma 2.2 we obtain

$$D_{V_i}|\omega| = D_{V_i}\omega_1 = -(n-1)\mu\, \delta_{i1}$$

Let ω^* be the well-defined vector field dual to ω on M, such that $\imath(\omega^*)\omega = (\omega^*, \omega) = 1$ where $\imath$ is convolution. This definition implies that $|\omega^*| = \frac{1}{|\omega|}$ whenever $|\omega| \neq 0$.
Choosing p such that $|\omega| \neq 0$ at p, we may choose $V_1(p) = |\omega|\omega^*(p)$.
Wherever $|\omega| \neq 0$ we define the unit vector field $W = |\omega|\omega^*$. In a coordinate neighborhood of p as above $W = \sum_i \frac{1}{|\omega|}\omega_i\, V_i$.

Furthermore, in the coordinate system with $\omega(p) = |\omega|\, dx_1(p)$

$$\begin{aligned} D_W W\Big|_p &= D_{V_1}[\sum_i \frac{1}{|\omega|}\omega_i\, V_i] = \\ &= \sum_i \big[(D_{V_1}(\frac{1}{|\omega|}))\,\omega_i\, V_i + \frac{1}{|\omega|}\,(D_{V_1}\omega_i)\, V_i + \frac{\omega_i}{|\omega|}\, D_{V_1} V_i\big]\Big|_p = \\ &= -\sum_i [(D_{V_1}\omega,\omega)\frac{\omega_i}{|\omega|^3}\, V_i]_p - (n-1)\mu\,\frac{1}{|\omega|(p)}\, V_1 = \\ &= (n-1)\mu\omega_1 \frac{\omega_1}{|\omega|^3}\, V_1\Big|_p - (n-1)\mu\,\frac{1}{|\omega|}\, V_1\Big|_p \\ &= 0 \end{aligned}$$

In other words W is a geodesic vector field in M. By the existence and uniqueness theorem of geodesics, the field can be extended indefinitely if ω is continuous. The Theorem follows. □

References

[1] Cheng, S. Y.: Eigenvalue Comparison Theorems and its Geometric Application. Math. Z., 143:289-297, 1975.

[2] Elworthy, K. D. and Rosenberg, S.: Generalized Bochner Theorems and the Spectrum of Complete Manifolds. Acta Appl. Math., 12:1-33, 1988.

[3] Gallot, S. and Meyer, D.: Opérateur de Courbure et Laplacien des Formes Différentielles dÚne Variété Riemannienne. J. Math. Pures et Appl., 54:259-284, 1975.

[4] Li, P. and Wang, J.: Complete Manifolds with Positive Spectrum. J. Diff. Geom., 58(3):501-534, 2001.

[5] Li, P. and Wang, J.: Complete Manifolds with Positive Spectrum, II. J. Diff. Geom., 62(1):143-162, 2002.

[6] Wang, X.: On the Geometry of Conformally Compact Einstein Manifolds. Ph.D, Thesis, Stanford University, 2001.

[7] Wu, H.: The Bochner Technique. In Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, (Beijing, 1980), Vol. 1,2,3, pages 929-1071. Science Press, Beijing, 1982.

[8] Yau, S. T.: Some Function-Theoretic Properties of Complete Riemannian Manifold and Their Applications to Geometry. Indiana Univ. Math. J., 25(7):659-670, 1976.

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