Matemáticas: Enseñanza Universitaria
ISSN: 0120-6788
reviserm@univalle.edu.co
Escuela Regional de Matemáticas
Colombia

Amster, Pablo; Cárdenas Alzate, Pedro Pablo
An iterative method for a second order problem with nonlinear two-point boundary conditions
Matemáticas: Enseñanza Universitaria, vol. XIX, núm. 2, diciembre, 2011, pp. 3-14
Escuela Regional de Matemáticas
Cali, Colombia

Vol. XIX, Nº 2, Diciembre (2011)
Matemáticas: 3–14

**Matemáticas:**
**Enseñanza Universitaria**


# An iterative method for a second order problem with nonlinear two-point boundary conditions

Pablo Amster
Universidad de Buenos Aires and CONICET

Pedro Pablo Cárdenas Alzate
Universidad Tecnológica de Pereira

Recibido Dec. 21, 2010 Aceptado Apr. 11, 2011

**Abstract**

A semi-linear second order ODE under a nonlinear two-point boundary condition is considered. Under appropriate conditions on the nonlinear term of the equation, we define a two-dimensional shooting argument which allows to obtain solutions for some specific situations by the use of Poincaré-Miranda's theorem. Finally, we apply this result combined with the method of upper and lower solutions and develop an iterative sequence that converges to a solution of the problem.

**Keywords:** Nonlinear two-point boundary conditions; upper and lower solutions; iterative methods.

**MSC(2000):** 34B15

## 1 Introduction

We study the semi-linear second order ODE

$$u''(t) + g(t, u(t), u'(t)) = 0, \qquad 0 < t < T \tag{1}$$

under a nonlinear two-point boundary condition.

Problem (1) under various boundary conditions has been studied by many authors. In the pioneering work of Picard [18], the existence of a solution for the Dirichlet problem was proved by the well-known method of successive approximations, assuming that $g$ is Lipschitz and $T$ is small. These conditions have been improved by Hamel [9], for the special case of a forced pendulum equation (see also [13], [14]). For general $g = g(\cdot, u)$, the variational approach has been employed already in 1915 by Lichtenstein [12]. However, when $g$ depends on $u'$ the problem has non-variational structure, and different techniques are required. As a historical antecedent of the topological methods, we may mention the shooting method introduced in 1905 by Severini [20]; later on, more abstract topological tools have been applied, such as the Leray-Schauder degree theory. For an overview of the use of topological methods to this kind of problems, we refer the reader to [15].

The above-mentioned two-point boundary conditions, as well as some other standard ones, such as the Neumann or the Sturm-Liouville conditions, are *linear*;

it is worthy to mention, however, that the general nonlinear case

$$\phi(u(0), u(T), u'(0), u'(T)) = 0, \tag{2}$$

where $\phi : \mathbb{R}^4 \to \mathbb{R}$ is continuous is very important in applications and, in recent years, a considerable number of works have been developed in this direction.

We shall study the existence of solutions of (1) under a particular case of condition (2): namely, nonlinear boundary conditions of the type

$$u'(0) = f_0(u(0)), \quad u'(T) = f_T(u(T)) \tag{3}$$

where $f_0, f_T : \mathbb{R} \to \mathbb{R}$ are given continuous functions. The special case $f_i(x) = a_i x + b_i$ for $i = 0, T$ corresponds to the Sturm-Liouville conditions, and Neumann conditions when $a_0 = a_T = 0$. Our interest in (3) relies on some models in nonlinear beam theory, usually leading to fourth order problems [7], but that admit second order analogues (see e.g. [19]). The results in the present paper complement and extend those in [1].

The paper is organized as follows. In the second section, we impose a growth condition on $g$, which allows to prove the unique solvability of the associated Dirichlet problem. Furthermore, we prove that the trace mapping $Tr : \mathcal{S} \to \mathbb{R}^2$ given by $Tr(u) = (u(0), u(T))$, where

$$\mathcal{S} := \{u \in H^2(0, T) : u''(t) + g(t, u(t), u'(t)) = 0\} \tag{4}$$

is a homeomorphism for the $H^2$-norm.

Then, we define a two-dimensional shooting argument, which proves to be successful with the aid of the Poincaré-Miranda theorem (see e.g. [11]) in some particular situations, which include the Sturm-Liouville boundary conditions. This generalizes some of the results in [2], and constitutes the main tool for our iterative method for problem (1)-(3), developed in the third section.

Our method, based on the existence of an ordered couple $(\alpha, \beta)$ of a lower and an upper solution, has been successfully applied to different boundary value problems when $g$ does not depend on $u'$. For general $g$, existence results can still be obtained if one assumes a Nagumo-Bernstein type condition (see [3], [16])). However, these results are usually proved by fixed point or degree arguments and, in consequence, they are non-constructive.

We shall assume instead a Lipschitz condition on $u'$, which is more restrictive, but allows the construction of a non-increasing (resp. non-decreasing) sequence of upper (lower) solutions that converges to a solution of the problem. Our method is slightly different from the monotone techniques known in the literature for linear boundary conditions, see e.g. [4], [17] among others (for upper and lower solutions in the reversed order, see [10]).

## 2 A continuum of solutions of (1)

For simplicity, let us assume that $g$ is continuous, and write it as

$$g(t, u(t), u'(t)) = r(t)u'(t) + h(t, u(t), u'(t)),$$

with $r \in W^{1,\infty}(0,T)$. We shall assume that $h$ satisfies a global Lipschitz condition on $u'$, namely

$$\left|\frac{h(t,u,A) - h(t,u,B)}{A - B}\right| \leq k < \frac{\pi}{T} \qquad \text{for} \quad A \neq B. \tag{5}$$

Furthermore, in this section we shall assume the following one-side growth condition on $u$:

$$\frac{h(t,u,A) - h(t,v,A)}{u - v} \leq c \tag{6}$$

for $u \neq v$, where the constant $c \in \mathbb{R}$ satisfies

$$c + \frac{k\pi}{T} < \left(\frac{\pi}{T}\right)^2 + \frac{1}{2}\inf_{0\leq t\leq T} r'(t). \tag{7}$$

Under these assumptions, the set $\mathcal{S}$ of solutions of (1) defined by (4) is homeomorphic to $\mathbb{R}^2$. More precisely,

**Theorem 1.** *Assume that (5) and (6) hold and let $x, y \in \mathbb{R}$. Then there exists a unique solution $u_{x,y}$ of (1) satisfying the non-homogeneous Dirichlet condition*

$$u_{x,y}(0) = x, \qquad u_{x,y}(T) = y$$

*Furthermore, the mapping $Tr : \left(\mathcal{S}, \|\cdot\|_{H^2}\right) \to \mathbb{R}^2$ given by $Tr(u) = (u(0), u(T))$ is a homeomorphism.*

*Proof.* For fixed $v \in H^1(0,T)$, let $u := \mathcal{T}v$ be defined as the unique solution of the linear problem

$$u'' = -[rv' + h(\cdot, v, v')]$$

$$u(0) = x, \quad u(T) = y.$$

It is immediate that $\mathcal{T} : H^1(0,T) \to H^1(0,T)$ is compact. Moreover, if $S_\sigma : H^2(0,T) \to L^2(0,T)$ is the semilinear operator defined by $S_\sigma u := u'' + \sigma[ru' + h(\cdot, u, u')]$, with $\sigma \in [0,1]$, then using (6) and (7) it is seen that the following a *priori* bound holds for any $u, v \in H^2(0,T)$ with $u - v \in H^1_0(0,T)$:

$$\|u' - v'\|_{L^2} \leq \mu\|S_\sigma u - S_\sigma v\|_{L^2} \tag{8}$$

for some constant $\mu$ independent of $\sigma$.

Hence, if $u = \sigma\mathcal{T}u$ for some $\sigma \in [0,1]$, then setting $l_{x,y}(t) = \frac{y-x}{T}t + x$ we obtain:

$$\|u' - \sigma l'_{x,y}\|_{L^2} \leq \mu\|S_\sigma(\sigma l_{x,y})\|_{L^2} \leq C$$

for some constant $C$ depending only on $x$ and $y$. Existence of solutions follows from the Leray-Schauder Theorem. Uniqueness is an immediate consequence of (8) for $\sigma = 1$.

Thus, $Tr$ is bijective, and its continuity is clear. On the other hand, if $(x, y) \to (x_0, y_0)$, then applying (8) to $u = u_{x,y} - l_{x,y}$ and $v = u_{x_0,y_0} - l_{x_0,y_0}$ it is easy to see that $u_{x,y} \to u_{x_0,y_0}$ for the $H^1$-norm. As $u_{x,y}$ and $u_{x_0,y_0}$ satisfy (1), we conclude from (5) that also $u''_{x,y} \to u''_{x_0,y_0}$ for the $L^2$-norm and so completes the proof. $\square$

It is worth noticing that the previous result allows to define a two-dimensional shooting argument as follows: let $\Theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\Theta(x, y) = (u'_{x,y}(0) - f_0(x), u'_{x,y}(T) - f_T(y)).$$

From the previous theorem, we deduce that $\Theta$ is continuous, and it is clear that, if $\Theta(x, y) = (0, 0)$, then $u_{x,y}$ is a solution of the problem.

**Example 1.** *Assume that (5) and (6) hold, and that*

$$h(t, u, 0)sgn(u) < 0 \quad \text{for } |u| \geq M, \tag{9}$$

$$f_0(M^+) \geq 0 \geq f_0(M^-), \qquad f_T(M^+) \leq 0 \leq f_T(M^-) \tag{10}$$

*for some constants $M^- \leq -M < M \leq M^+$. Then (1)-(3) admits at least one solution.*

*In particular, the result holds for the Sturm-Liouville conditions*

$$u'(0) = a_0 u(0) + b_0, \quad u'(T) = a_T u(T) + b_T, \qquad a_0 > 0 > a_T. \tag{11}$$

*Furthermore, in this case the solution is unique, provided that $c < 0$ in (6).*

*Indeed, it follows from (9) that $u_{x,y}$ cannot attain in $(0, T)$ neither a maximum value larger than $M$ , nor a minimum value smaller than $-M$. Moreover, for $M^- \leq y \leq M^+$ we obtain:*

$$u_{M^+,y}(0) = M^+ \geq y = u_{M^+,y}(T), \qquad u_{M^-,y}(0) = M^- \leq y = u_{M^-,y}(T).$$

*Thus, $u'_{M^+,y}(0) \leq 0 \leq u'_{M^-,y}(0)$, and hence $\Theta_1(M^+, y) \leq 0 \leq \Theta_1(M^-, y)$. In the same way, it follows that $\Theta_2(x, M^+) \geq 0 \geq \Theta_2(x, M^-)$ for $M^- \leq x \leq M^+$. By the Poincaré-Miranda's generalized intermediate value theorem, we conclude that $\Theta$ has at least one zero $(x, y) \in [M^-, M^+] \times [M^-, M^+]$.*

*On the other hand, if $u$ and $v$ are solutions of (1)-(11), then*

$$(u - v)'' + (r + \psi)(u - v)' + h(\cdot, u, v') - h(\cdot, v, v') = 0$$

*where*

$$\psi = \frac{h(\cdot, u, u') - h(\cdot, u, v')}{u' - v'} \in L^\infty(0, T).$$

*Next, take* $p(t) = e^{\int_0^t (r(s)+\psi(s))\,ds}$, *multiply the previous equality by* $(u-v)p$ *and integrate. We obtain:*

$$0 = p(u'-v')(u-v)\Big|_0^T - \int_0^T p(u'-v')^2 + \int_0^T p[h(\cdot,u,v') - h(\cdot,v,v')](u-v)$$

$$\leq p(T)a_T(u-v)^2(T) - a_0(u-v)^2(0) - \int_0^T p(u'-v')^2 + c\int_0^T p(u-v)^2.$$

*Hence, for* $c < 0$ *it is seen that* $u = v$.

## 3 Iterative sequences of upper and lower solutions

In this section we shall construct solutions of (1) under the two-point boundary condition (3) by an iterative method, based upon the existence of upper and lower solutions.

Let us recall that $(\alpha, \beta)$ is an ordered couple of a lower and an upper solution for (1) if $\alpha \leq \beta$ and

$$\alpha'' + g(\cdot,\alpha,\alpha') \geq 0 \geq \beta'' + g(\cdot,\beta,\beta').$$

Existence results under various boundary conditions in presence of an ordered couple of a lower and an upper solution are known (see e. g. [6]). In our particular case, we shall assume the boundary constraints:

$$\alpha'(0) - f_0(\alpha(0)) \geq 0 \geq \beta'(0) - f_0(\beta(0)),$$

$$\alpha'(T) - f_T(\alpha(T)) \leq 0 \leq \beta'(T) - f_T(\beta(T))$$

and a Nagumo type condition adapted from [5]:

$$|g(t,u,v)| \leq \psi(|v|), \qquad \text{for } \alpha(t) \leq u \leq \beta(t), m \leq |v| \leq M \tag{12}$$

where $\psi : [0,+\infty) \to (0,+\infty)$ is continuous and satisfies:

$$\int_m^M \frac{1}{\psi(t)}\,dt > T,$$

and

$$m = \min\left\{\frac{|\alpha(0)-\beta(T)|}{T}, \frac{|\alpha(T)-\beta(0)|}{T}, \max_{\alpha(0)\leq s\leq\beta(0)} |f_0(s)|, \max_{\alpha(T)\leq s\leq\beta(T)} |f_T(s)|\right\}$$

$$M > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, m\}.$$

Then, the following existence result can be obtained as in [1]:

**Theorem 2.** *Assume there exists an ordered couple* $(\alpha, \beta)$ *of a lower and an upper solution as before, and that (12) holds. Then the boundary value problem (1)-(3) admits at least one solution* $u$, *with* $\alpha \leq u \leq \beta$.

*Sketch of the proof.* The proof follows the outline of the standard results on the subject. Let $P(t,u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$ and $Q(v) = sgn(v)\min\{|v|, M\}$, and apply Schauder's Theorem in order to obtain a solution of the problem

$$u''(t) - \lambda u(t) = -g(t, P(t,u(t)), Q(u'(t))) - \lambda P(t,u(t)),$$

$$u'(0) = f_0(P(0,u(0)), \qquad u'(T) = f_T(P(T,u(T))$$

for some fixed $\lambda > 0$. It is easy to see that $\alpha \leq u \leq \beta$, and hence $P(t,u(t)) = u(t)$ for every $t$. Furthermore, if we suppose that for example $u'(t_1) = M$, then there exists $t_0$ such that $u'(t_0) = m$ and $m < u'(t) < M$ for $t$ between $t_0$ and $t_1$. Hence

$$T < \int_m^M \frac{1}{\psi(s)} ds = \int_{t_0}^{t_1} \frac{u''(t)}{\psi(u'(t))} dt \leq |t_1 - t_0|,$$

a contradiction. The same conclusion holds if we suppose $u'(t_1) = -M$; thus, $|u'(t)| < M$ and the proof is complete.

□

**Example 2.** *The previous result applies when (9) and (10) hold: indeed, in this case it is clear $(M^-, M^+)$ is an ordered couple of a lower and an upper solution. Thus, conditions (5) and (6) in example 1 can be dropped.*

*Also, we may consider the forced pendulum equation with friction*

$$u'' + ru' + \sin u = \theta,$$

*and assume that the forcing term $\theta$ is a measurable function satisfying:*

$$-1 \leq \theta(t) \leq 1 \qquad \forall\, t \in [0,T].$$

*Then $\alpha \equiv \frac{\pi}{2}$ and $\beta \equiv \frac{3}{2}\pi$ are respectively a lower and an upper solution. Hence, (1)-(3) has a solution for any continuous $f_0$ and $f_T$ such that*

$$f_0\left(\frac{\pi}{2}\right) \leq 0 \leq f_0\left(\frac{3\pi}{2}\right)$$

*and*

$$f_T\left(\frac{\pi}{2}\right) \geq 0 \geq f_T\left(\frac{3\pi}{2}\right).$$

Our last result is concerned with the construction of solutions by iteration, provided that $h$ and $f$ satisfy some stronger assumptions.

Let us firstly establish the following auxiliary lemmas:

**Lemma 1.** *Assume that (5) holds and let $\lambda$ be a positive constant satisfying $\lambda \geq k\frac{\pi}{T} - \left(\frac{\pi}{T}\right)^2 - \frac{1}{2}\mathrm{inf} r'$. Then for any $z, \theta \in C([0,T])$ the equation*

$$u'' + ru' + h(\cdot, z, u') - \lambda u = \theta$$

*is uniquely solvable under the Sturm-Liouville conditions (11). Furthermore, the mapping $\mathcal{K} : C([0,T])^2 \to C([0,T])$ given by $\mathcal{K}(z,\theta) = u$ is compact.*

*Proof.* Existence and uniqueness follow as a particular case of example 1, with $\overline{g}(\cdot,u,u') = ru' + \overline{h}(\cdot,u,u')$, where

$$\overline{h}(\cdot,u,u') = h(\cdot,z,u') - \lambda u - \theta.$$

Indeed, it is clear that $\overline{h}$ satisfies (5) and (6) with $c = -\lambda$. Moreover,

$$\overline{h}(t,u,0)sgn(u) = (h(t,z(t),0) - \theta(t))sgn(u) - \lambda|u| < 0$$

when $|u| > \|h(\cdot,z,0) - \theta\|_\infty$. Thus, (9) is also satisfied.

Let $(z,\theta)$ tend to $(z_0,\theta_0)$, and set $u = \mathcal{K}(z,\theta)$, $u_0 = \mathcal{K}(z_0,\theta_0)$. Then

$$(u-u_0)'' + (r+\psi)(u-u_0)' - \lambda(u-u_0) = h(\cdot,z,u_0') - h(\cdot,z_0,u_0') + \theta - \theta_0$$

where $\psi = \frac{h(\cdot,z,u') - h(\cdot,z,u_0')}{u'-u_0'}$. Hence, continuity of $\mathcal{K}$ is a consequence of the following estimate, which is valid for any $w$ satisfying (11) with $b_0 = b_T = 0$ and some constant $c$ depending only on $k$:

$$\|w\|_{H^1} \leq c\|w'' + (r+\psi)w' - \lambda w\|_{L^2}.$$

Indeed, this is easily deduced by applying the Cauchy-Schwartz inequality to the integral $\int_0^T pLw.w$, where $Lw = w'' + (r+\psi)w' - \lambda w$ and $p(t) = e^{\int_0^t (r(s)+\psi(s))\,ds}$, and the fact that $0 < m \leq p \leq M$ for some $m$ and $M$ depending only on $k$. Finally, compactness of $\mathcal{K}$ follows from the imbedding $H^1(0,T) \hookrightarrow C([0,T])$.

□

**Remark 1.** *In the previous proof, an analogous estimate can be also obtained for the $H^2$-norm of $w$. This implies the compactness of $\mathcal{K}$, but now regarded as an operator from $C([0,T])^2$ to $C^1([0,T])$. More generally, one might consider also $a_i$ and $b_i$ as variables for $i = 0,T$: in this case, $\mathcal{K}$ could be seen as a compact operator from $\mathbb{R}^4 \times C([0,T])^2$ to $C^1([0,T])$.*

**Lemma 2.** *Let $\phi \in L^\infty(0,T)$ and assume that $w'' + \phi w' - \lambda w \geq 0$ (in the weak sense) for some $\lambda \geq 0$, and*

$$w'(0) - a_0 w(0) \geq 0 \geq w'(T) - a_T w(T)$$

*with $a_0 > 0 > a_T$. Then $w \leq 0$.*

*Proof.* If $w(0), w(T) \leq 0$, the result is the well-known maximum principle for Dirichlet conditions.

If for example $w(0) > 0$, then restricting $w$ up to its first zero if necessary, it suffices to consider only the case $w \geq 0$. Taking $p(t) = e^{\int_0^t \phi(s)\,ds}$, it is observed that $(pw')' \geq \lambda pw \geq 0$. Thus, $pw'$ is nondecreasing on $[0,T]$, and hence

$$0 \geq p(T)a_T w(T) \geq p(T)w'(T) \geq p(0)w'(0) \geq p(0)a_0 w(0) > 0,$$

a contradiction. The proof is similar when $w(T) > 0$. □

In order to define our iterative scheme, we shall assume that $f_0$ and $f_T$ satisfy a one-side Lipschitz condition:

$(F)$ There exists a positive constant $R$ such that

$$f_0(y) - f_0(x) \leq R(y-x)$$

if $\alpha(0) \leq x < y \leq \beta(0)$, and

$$f_T(y) - f_T(x) \geq -R(y-x)$$

if $\alpha(T) \leq x < y \leq \beta(T)$.

In virtue of Lemma 1, if (5) holds then for $\lambda = \min\{R, k\frac{\pi}{T} - \left(\frac{\pi}{T}\right)^2 - \frac{1}{2}\mathrm{inf} r'\}$, we may define the compact operator $\mathcal{T} : C([0,T]) \to C([0,T])$ given by $\mathcal{T}v = u$, where $u$ is the unique solution of the problem

$$u'' + ru' + h(\cdot, v, u') - \lambda u = -\lambda v$$

satisfying the following Sturm-Liouville condition:

$$u'(0) - Ru(0) = f_0(v(0)) - Rv(0), \quad u'(T) + Ru(T) = f_T(v(T)) + Rv(T).$$

From Remark 1, we observe, moreover, that the set $\mathcal{T}(\{v : \alpha \leq v \leq \beta\})$ is bounded for the $C^1$-norm. In particular, this implies the existence of a constant $M = M(R)$ such that if $u = \mathcal{T}v$ for some $v$ lying between $\alpha$ and $\beta$, then $\|u'\|_\infty \leq M$. This suggests to consider the following Lipschitz condition on $h$:

$(H)$

$$|h(t,u,A) - h(t,v,A)| \leq R|u-v|$$

for $u, v$ such that $\alpha(t) \leq u < v \leq \beta(t)$ and $|A| \leq M(R)$.

**Remark 2.** *Conditions $(F)$ and $(H)$ are trivially satisfied if $f_0$, $f_T$ and $h$ are $C^1$ functions, and $\frac{\partial h}{\partial u}$ is bounded with respect to $u'$.*

**Theorem 3.** *Assume there exists an ordered couple $(\alpha, \beta)$ of a lower and an upper solution as before. Further, assume that (5), $(H)$ and $(F)$ hold. Set $\lambda$ as before, and define the sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ recursively by*

$$\underline{u}_0 = \alpha, \qquad \overline{u}_0 = \beta$$

*and*

$$\overline{u}_{n+1} = \mathcal{T}\overline{u}_n, \qquad \underline{u}_{n+1} = \mathcal{T}\underline{u}_n$$

*Then $(\underline{u}_n\, \overline{u}_n)$ is an ordered couple of a lower and an upper solution. Furthermore, $\{\overline{u}_n\}$ (resp. $\{\underline{u}_n\}$) is non-increasing (non-decreasing) and converges to a solution of the problem.*

*Proof.* Let us firstly prove that $\alpha \le \overline{u}_1 \le \beta$. From the definition,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \beta, \overline{u}_1') - \lambda\overline{u}_1 = -\lambda\beta \ge -\lambda\beta + \beta'' + r\beta' + h(\cdot, \beta, \beta').$$

Hence, setting

$$\psi = \frac{h(\cdot, \beta, \overline{u}_1') - h(\cdot, \beta, \beta')}{\overline{u}_1' - \beta'} \in L^\infty(0, T)$$

we deduce that

$$(\overline{u}_1 - \beta)'' + (r + \psi)(\overline{u}_1 - \beta)' - \lambda(\overline{u}_1 - \beta) \ge 0.$$

On the other hand,

$$\overline{u}_1'(0) - R\overline{u}_1(0) = f_0(\beta(0)) - R\beta(0)$$

and

$$\overline{u}_1'(T) + R\overline{u}_1(T) = f_T(\beta(T)) + R\beta(T).$$

Thus,

$$(\overline{u}_1 - \beta)'(0) - R(\overline{u}_1 - \beta)(0) = 0 = (\overline{u}_1 - \beta)'(T) - R(\overline{u}_1 - \beta)(T),$$

and from Lemma 2 we obtain that $\overline{u}_1 \le \beta$.

In the same way,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \beta, \overline{u}_1') - \lambda\overline{u}_1 \le -\lambda\beta + \alpha'' + r\alpha' + h(\cdot, \alpha, \alpha')$$

and hence

$$(\overline{u}_1 - \alpha)'' + (r + \psi)(\overline{u}_1 - \alpha)' - \lambda(\overline{u}_1 - \alpha) \ge 0$$

where

$$\psi = \frac{h(\cdot, \alpha, \overline{u}_1') - h(\cdot, \alpha, \alpha')}{\overline{u}_1' - \alpha'} \in L^\infty(0, T).$$

Also

$$\overline{u}_1'(0) - R\overline{u}_1(0) = f_0(\beta(0)) - R\beta(0) \le f_0(\alpha(0)) - R\alpha(0)$$

and

$$\overline{u}_1'(T) + R\overline{u}_1(T) = f_T(\beta(T)) + R\beta(T) \ge f_T(\alpha(T)) + R\alpha(T),$$

and we conclude that $\overline{u}_1 \ge \alpha$.

On the other hand,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \overline{u}_1, \overline{u}_1') = (\lambda - R)(\overline{u}_1 - \beta) + [h(\cdot, \overline{u}_1, \overline{u}_1') + R\overline{u}_1] - [h(\cdot, \beta, \overline{u}_1') + R\beta] \le 0,$$

and we deduce that $\overline{u}_1$ is an upper solution of the problem. Inductively, it follows that $\overline{u}_n$ is an upper solution for every $n$, with $\alpha \le \overline{u}_{n+1} \le \overline{u}_n$, which by Dini's theorem implies that $\overline{u}_n$ converges uniformly to a function $\overline{u}$. From the definition of $\{\overline{u}_n\}$,

$$\overline{u}''_{n+1} + r\overline{u}'_{n+1} + h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) \to 0$$

uniformly. Moreover, from Lemma 1 and Remark 1 we know that $\{\overline{u}_n\}$ is bounded in $H^2(0,T)$, and it follows easily that

$$\overline{u}'' + r\overline{u}' + h(\cdot, \overline{u}, \overline{u}') = 0.$$

Thus, $\overline{u}$ is a solution of the problem. The proof for $\underline{u}_n$ is analogous. Moreover, if we assume as inductive hypothesis that $\underline{u}_n \leq \overline{u}_n$, then

$$\begin{aligned} &\overline{u}''_{n+1} + r\overline{u}'_{n+1} + h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) - \lambda\overline{u}_{n+1} = -\lambda\overline{u}_n \\ &\leq -\lambda\underline{u}_n = \underline{u}''_{n+1} + r\underline{u}'_{n+1} + h(\cdot, \underline{u}_n, \underline{u}'_{n+1}) - \lambda\underline{u}_{n+1}. \end{aligned}$$

In the same way as before, we may define

$$\psi = \frac{h(\cdot, \underline{u}_n, \overline{u}'_{n+1}) - h(\cdot, \underline{u}_n, \underline{u}'_{n+1})}{\overline{u}'_{n+1} - \underline{u}'_{n+1}} \in L^\infty(0,T),$$

and hence for $w = \overline{u}_{n+1} - \underline{u}_{n+1}$ we deduce:

$$w'' + (r+\psi)w' - \lambda w \leq h(\cdot, \underline{u}_n, \overline{u}'_{n+1}) - h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) \leq -R(\underline{u}_n - \overline{u}_n) \leq -Rw.$$

From Lemma 2, we conclude that $w \geq 0$, i.e. $\underline{u}_{n+1} \leq \overline{u}_{n+1}$. □

**Remark 3.** *It is interesting to observe that, even if (5) is somewhat too restrictive, some condition regarding the growth of $h$ with respect to $u'$ is required. We may recall, for instance, the following example by Habets and Pouso [8] for the mean curvature operator:*

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = u + a,$$

*where the function $a \in L^\infty(0,T)$ is defined by*

$$a(t) = 2[\chi_{[0,\frac{T}{2}]}(t) - \chi_{(\frac{T}{2},T]}(t)] = \begin{cases} 2 & 0 \leq t \leq \frac{T}{2} \\ -2 & \frac{T}{2} < t \leq T \end{cases}$$

*Under conditions (11) with $b_0 = b_T = 0$, it is seen that $\alpha = -3$ and $\beta = 3$ is an ordered couple of a lower and an upper solution, but the equation has no solutions when $T > 2\sqrt{2}$. However, here*

$$h(\cdot, u, u') = (u+a)\left(\sqrt{1+u'^2}\right)^{3/2},$$

*and (9) is satisfied. This explains the need of the Nagumo condition, or at least a similar one, in Theorem 2.*

**Acknowledgement**

This work has been partially supported by projects UBACyT 20020090100067 and PIP 11220090100637, CONICET.

## References

[1] Amster P. and Cárdenas Alzate P. P.: Existence of solutions for some nonlinear beam equations. Portugaliae Matemática 63, fasc. 1 (2006), 113-125.

[2] Amster P. and Cárdenas Alzate P. P.: Sturm-Liouville boundary conditions for a second order ODE. Matemáticas: Enseñanza Universitaria (Colombia) XV N° 1 (2007), 3-12.

[3] Bernstein S.: Sur les équations du calcul des variations, Ann. Sci. Ecole Norm. Sup., 29 (1912), 431-485.

[4] Cherpion M., De Coster C. and Habets P.: A constructive monotone iterative method for second-order BVP in the presence of lower and upper solutions, Appl. Math. Comput. 123 (2001), no. 1, 7591.

[5] Franco D. and O'Regan D.: Existence of solutions to second order problems with nonlinear boundary conditions. Proc. of the Fourth Int. Conf. on Dynamical Systems and Diff. Equations, Discrete and Continuous Dynamical Systems (2003), 273-280.

[6] Gaines R. and Mawhin J.: Coincidence Degree and Nonlinear Differential Equation. Lecture Notes in Math. 568 (1977).

[7] Grossinho M. and Ma T. F.: Symmetric equilibria for a beam with a nonlinear elastic foundation. Portugaliae Mathematica, 51 (1994), 375-393.

[8] Habets P. and Pouso R.: Examples of the nonexistence of a solution in the presence of upper and lower solutions. ANZIAM J. 44 (2003), 591-594.

[9] Hamel G.: Über erzwungene Schwingungen bei endlichen Amplituden. Math. Ann., 86 (1922), 1-13.

[10] Jiang D., Fan M., Wan A.: A monotone method for constructing extremal solutions to second-order periodic boundary value problems. Journal of Computational and Applied Mathematics 136 (2001) 189-197.

[11] Kulpa W.: The Poincaré-Miranda Theorem. The American Mathematical Monthly, Vol. 104, No. 6 (1997), 545-550.

[12] Lichtenstein L.: Über einige Existenzprobleme der Variationsrechnung. Methode der unendlichvielen Variabeln, J. Reine Angew. Math. 145 (1915), 24-85.

[13] Mawhin J.: Periodic oscillations of forced pendulum-like equations. Lecture Notes in Math., Springer, 964 (1982), 458-476.

[14] Mawhin J.: The forced pendulum: A paradigm for nonlinear analysis and dynamical systems. Expo. Math., 6 (1988), 271-287.

[15] Mawhin J.: Boundary value problems for nonlinear ordinary differential equations: from successive approximations to topology. Development of mathematics 1900-1950 (Luxembourg, 1992), Birkhäuser, Basel (1994), 443-477.

[16] Nagumo M.: Uber die differentialgleichung $y'' = f(t, y, y')$. Proc. Phys-Math. Soc. Japan 19 (1937), 861-866.

[17] Omari P., A monotone method for constructing extremal soltions of second order scalar boundary value problems, Appl. Math. Comput. 18 (1986), 257-275.

[18] Picard E.: Sur l'application des méthodes d'approximations succesives à l'étude de certaines équations différentielles ordinaires, J. Math. Pures Appl. 9 (1893), 217-271.

[19] Rebelo C. and Sanchez L.: Existence and multiplicity for an O.D.E. with nonlinear boundary conditions. Differential Equations and Dynamical Systems, Vol. 3, Number 4, October 1995, 383-396.

[20] Severini C.: Sopra gli integrali delle equazione differenziali del secondo ordine con valori prestabiliti in due punti dati, Atti R. Acc. Torino 40 (1904-5), 1035-1040.

*Dirección de los autores*
Pablo Amster — Departamento de Matemáticas, Facultad de Ciencias Exactas y Naturales, Buenos Aires - Argentina
e-mail: `pamster@dm.uba.ar`
Pedro Pablo Cárdenas Alzate — Departamento de Matemáticas, Universidad Tecnológica de Pereira, Risaralda - Colombia
e-mail: `ppablo@utp.edu.co`