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A categorical characterization of non-distributive lattices

Gonzalo Medina Arellano
Universidad Nacional de Colombia

Yuri Alexander Poveda Quiñones
Universidad Tecnológica de Pereira

Leonardo Prieto Sanabria
Universidad Tecnológica de Pereira

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Abstract
In this paper we give a categorical characterization of non-distributive lattices and non-distributive modular lattices, in terms of finite limits and finite colimits.

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MSC(2000): MSC(2000): 03G10.

1 Introduction

The following characterization of the non-distributive and modular lattices is well known (see Theorems 3.5 (Dedekind) and 3.6 (Birkhoff)): a lattice is non-distributive if and only if it contains a sublattice isomorphic to one of the lattices M_3 or N_5. A lattice is modular if and only if it does not contain a sublattice isomorphic to N_5.

It is our goal to give an alternative and, in some sense, more geometric characterization for the non-distributive and modular lattices using a categorical approach.

This paper is organized in the following way: in section 2 we give some definitions and examples; in section 3 we state and prove our main results: a categorical characterization for non-distributive lattices (theorem 1) and for non-distributive modular lattices (theorem 2).

2 Basic notions

We begin by introducing the basic definitions and the notation that will be used throughout this paper. We will denote the pullback and pushout of two morphisms f and g by $pullback(f, g)$ and $pushout(f, g)$, respectively.

Definition 1. *We say that the pullback $fx = gy$ (the pushout $xh = yl$, respectively) is* trivial *if $x \approx Id$ or $y \approx Id$.*

We will denote by $\mathcal{C}_{\mathbb{P}}$ the category associated to the poset $(\mathbb{P}, \leq)$ (see [1]) and we will refer to a poset $(\mathbb{P}, \leq)$ simply by its underlying set $\mathbb{P}$.

Definition 2. *A non-distributive lattice* $\mathbb{P}$ *will be called* exact *if it contains a sublattice* $\widehat{\mathbb{P}}$ *isomorphic to* M_3 *or* N_5*, such that the following condition are satisfied:*

1. *Every* $p \in \mathbb{P}$ *lies between two elements of* $\widehat{\mathbb{P}}$ *and*

2. *Every pair of elements of* $\mathbb{P}$ *is comparable if it lies between two consecutive elements of* $\widehat{\mathbb{P}}$ *or if the closed interval defined by them contains some element of* $\widehat{\mathbb{P}}$*.*

Now we give the definitions of non-distributive category, and of modular category and give some examples to illustrate that our definitions lead to the expected results in some familiar cases.

Definition 3. *A category* $\mathcal{C}_{\mathbb{P}}$ *will be called* non-distributive *if there exist three morphisms* f_1, f_2 *y* f_3 *with the same codomain, such that at least two of the pullbacks* $f_i x_{ij} = f_j x_{ji}$ *are non trivial, and furthermore every non trivial pullback having one of the following forms pullback*$(f_i x_{il}, f_j x_{jk})$ *or pullback*(f_i, f_j) *is a pushout. Notice that in this situation,* $x_{ii} = id$ *for every* i*.*

Definition 4. *A category* $\mathcal{C}_{\mathbb{P}}$ *will be called* distributive *if it does not satisfy the conditions of definition 3, and* $\mathcal{C}_{\mathbb{P}}$ *will be called* modular *if for every morphisms* f_1, f_2, f_3 *with the same codomain, satisfying the conditions of definition 3, if two of the pullbacks pullback*(f_i, f_j) *are non trivial then the third pullback is non trivial* $(i \neq j)$*.*

Remark 1. *It is a straightforward verification that if pullback*(f_i, f_j) *is non trivial then the corresponding pushout*(x_{ij}, x_{ji}) *is also non trivial. Additionally, it follows easily that if pullback*(f_i, f_j) *is non trivial then* f_i, f_j *are not isomorphic. Consequently, if two of the pullbacks of the above definition are non trivial, then none of the morphisms* f_1, f_2, f_3 *is the identity morphism.*

In the following examples we illustrate that our definitions are minimal, in the sense that if one of the conditions of definition 3 is not satisfied, then it is possible to give an example of distributive lattice whose whose associated category satisfies the remaining conditions of the definition.

Example 1. *(A) If* $\mathbb{P}$ *is a chain then* $\mathcal{C}_{\mathbb{P}}$ *is a distributive category since every pullback is trivial. (B) If* $\mathbb{P}$ *is the lattice* 2^2 *(associated to the power set of a two element set) then* $\mathcal{C}_{\mathbb{P}}$ *is a distributive category since there exists only one non trivial pullback.*

Example 2. *(A) The category of Figure 1 is distributive since pullback*(f_1, f_2) *is not a pushout; however, pullback*(f_i, f_j) $(i, j \in \{1, 2, 3\}, i \neq j)$ *are three non trivial pullbacks. (B) The category of Figure 2 is distributive eventhough two of the three non trivial pullback*(f_i, f_j) $(i \neq j)$ *are pushouts, since we have that pullback*$(f_1 x_{12},\ f_3)$ *is not a pushout. (C) The category* $\mathcal{C}_{N_5}$ *(See Figure 3) is* *non-distributive and non-modular since pullback(f_1, f_2) and pullback(f_1, f_3) are two non trivial pullbacks which are pushouts. (D) The category associated to M_3 is non-distributive and modular. In particular, $\mathcal{C}_{M_3}$ has exactly three non trivial pullbacks which are pushouts.*

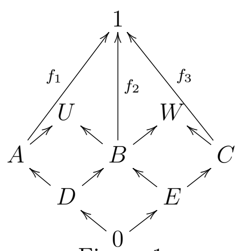


Figure 1

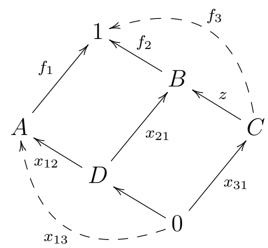


Figure 2

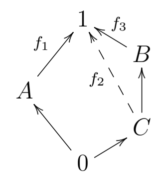


Figure 3

3 Non-distributive lattices and non-distributive categories

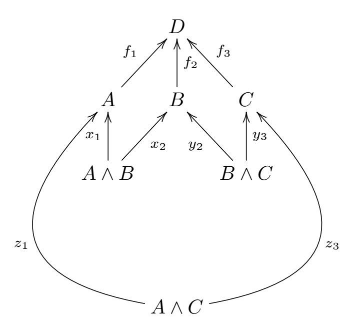
First, let us recall that a category $\mathcal{C}$ is *cartesian* if it has finite limits and finite colimits. In particular, it has final object 1 and initial object 0 (see [1]).

Now we come to the first one of our main results: the categorical characterization of non-distributive lattices.

Theorem 1. *A cartesian category $\mathcal{C}_{\mathbb{P}}$ is non-distributive if and only if $\mathbb{P}$ is a non-distributive lattice.*

Demostración. (⇒) The finite coproducts and finite products in $\mathcal{C}_{\mathbb{P}}$ correspond to the finite supremum and infimum in $\mathbb{P}$, thus $\mathbb{P}$ is a lattice. We will consider now two cases:

Case 1: Let f_1, f_2, f_3 be three morphisms satisfying the conditions of definition 3 and assume that the following three non trivial pullbacks are pushouts:



(1)

A is not comparable to B; otherwise *pullback*(f_1, f_2) would be trivial. Analogously, B is not comparable to C, and A is not comparable to C. Let us prove now that $A \wedge B \wedge C$ must be equal to at least two elements of the set

$$\{A \wedge B, A \wedge C, B \wedge C\}.$$

Indeed, if $A \wedge B \wedge C \neq A \wedge B$ and $A \wedge B \wedge C \neq B \wedge C$, from the above diagram we have that $A \wedge B \wedge C < A \wedge B$ and $A \wedge B \wedge C < B \wedge C$.

Without loss of generality, if $A \wedge B \wedge C \neq A \wedge B$ and $A \wedge B \wedge C \neq B \wedge C$ then *pullback*$(f_1 x_1,\ f_3 y_3)$ would be non trivial; taking into account that $\mathcal{C}_{\mathbb{P}}$ is non-distributive, this pullback would be a pushout and this would imply that B and D are equal which contradicts that f_2 is not an identity. Therefore $A \wedge B \wedge C$ must be equal to at least two elements of the set $\{A \wedge B, A \wedge C, B \wedge C\}$.

If $A \wedge B \wedge C = A \wedge B = B \wedge C = A \wedge C$ then the lattice in (1) becomes the one shown in Figure 4 and is therefore isomorphic to M_3. If now $A \wedge B \wedge C = A \wedge C = A \wedge B \neq B \wedge C$, then $A \wedge B \wedge C < B \wedge C$, and consequently, the lattice in (1) reduces to the one shown in Figure 5 and is now therefore isomorphic to N_5. The cases $A \wedge B \wedge C \neq A \wedge B$, and $A \wedge B \wedge C \neq A \wedge C$, are completely analogous to the previous situation.

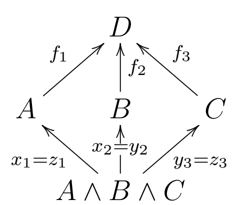


Figure 4

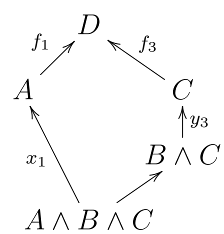


Figure 5

Case 2: We have the non trivial lattice of Figure 6 and the trivial *pullback*(f_1, f_3).

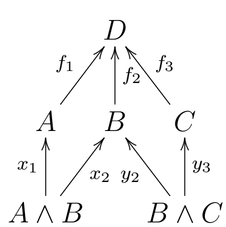


Figure 6

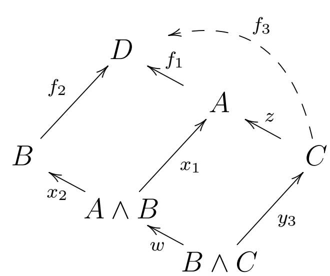


Figure 7

This implies $A < C$ or $C < A$. We can assume, without loss of generality, $C < A$ and now the diagram in Figure 6 has the shape shown in Figure 7 (Notice that $A \wedge B \wedge C = B \wedge C$ since $A \wedge C = C$; $f_1 z = f_3$.)

Since $A \wedge B \wedge C = B \wedge C$, we have that $pullback(f_3, f_2x_2)$ is $f_2x_2w = f_1zy_3$. If this pullback is non trivial then, by our hypothesis, the above diagram corresponds also to $pushout(w, y_3)$, and consequently $(A \wedge B) \vee C = D$, but since $(A \wedge B) \vee C \leq A$, we have $A = D$ and this contradicts the fact that f_1 is not an isomorphism. Therefore the pullback is trivial; now, since $pullback(f_2, f_3)$ is non trivial, we have $B \wedge C < C$ which in turn yields $A \wedge B = B \wedge C$ and the sublattice $\{B \wedge C, B, C, A, D\}$

is isomorphic to N_5.

($\Leftarrow$) This follows at once from the definitions. □

Now we are ready for our second main result: a categorical characterization of modular lattices.

Theorem 2. *The non-distributive category $\mathcal{C}_{\mathbb{P}}$ is modular if and only if $\mathbb{P}$ is a non-distributive and modular lattice (i.e., it does not contain a sublattice isomorphic to N_5).*

Demostración. It is a straightforward consequence of Theorem 1 and definition 4). □

Corollary 2. *The category $\mathcal{C}_{\mathbb{P}}$ is modular if and only if $\mathbb{P}$ is a modular lattice.*

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Authors' address

Gonzalo Medina Arellano — Departamento de Matemáticas y Estadística, Universidad Nacional de Colombia - Sede Manizales, Manizales-Colombia

e-mail: `gmedinaar@unal.edu.co`

Yuri Alexander Poveda Quiñones — Departamento de Matemáticas, Universidad Tecnológica de Pereira, Pereira-Colombia

e-mail: `yapoveda@utp.edu.co`

Leonardo Prieto Sanabria — Departamento de Matemáticas, Universidad Tecnológica de Pereira, Pereira-Colombia

e-mail: `lprieto@utp.edu.co`