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Role of adjoint equations in estimating monthly mean air surface temperature anomalies

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RESUMEN

Usando la solución de un problema adjunto especialmente formulado se deriva una fórmula integral para estudiar la respuesta de un modelo lineal. La fórmula relaciona directamente cada característica escogida de sensibilidad del modelo con las variaciones de los datos iniciales, y del forzamiento. Por analogía con la bien conocida función de Green, la solución de la ecuación adjunta hace las veces de la función de peso (función de influencia). Un conjunto de tales fórmulas relativamente simples, provee un método efectivo para estimar la sensibilidad del modelo a tipos diferentes de datos de entrada sin tener que resolver cada vez el complicado problema básico.

El modelo simplificado de tres dimensiones de interacción atmósfera-océano para calor global, se toma como ejemplo. El modelo se ha linearizado usando medias mensuales climáticas de viento en la atmósfera y corrientes estacionales climáticas en el océano. Las estructuras espacio-temporales de las funciones de influencia calculadas para las anomalías medias de la temperatura de superficie para diciembre, para la parte europea de la URSS y el territorio de los EEUU son mostradas. Las regiones de máximo local de la función de influencia muestran las zonas de los océanos que son energéticamente activas. Dentro de los intervalos de tiempo en los que los máximos locales existen, sólo las anomalías en el flujo de calor localizados en dichas zonas son las responsables de la magnitud final de la anomalía de la temperatura media considerada.

ABSTRACT

Using the solution of a specially formulated adjoint problem an integral formula is derived for the study of linear model response. The formula relates directly every chosen characteristic of the model sensitivity to variations of initial data and forcing. By analogy with the well-known Green function, the adjoint equation solution performs here the role of a weight function (or influence function). A set of such relatively simple formulas gives an effective method for estimating the model sensitivity to different types of input data without solving every time the complicated basic problem.

The simplified three-dimensional global heat interaction model of atmosphere and ocean is considered as an example. The model has been linearized by using the climatic monthly mean wind in the atmosphere and the climatic seasonal currents in the World Ocean. The time-space structures of the influence functions calculated for the December mean surface temperature anomalies of the European part of USSR and the USA territory, are demonstrated. The regions of local maxima of the influence function show the energetically active zones in the World Ocean. Within the time intervals while these local maxima exists, only the heat flux anomalies located in such zones can be responsible for the final magnitude of the mean temperature anomaly considered.

1. Introduction

Process that forms monthly mean air temperature anomalies in limited areas is undoubtedly of meteorological interest (see, for example, Adem, 1975; Miyakoda and Sirutis, 1985; Lorenz, 1984; Shukla, 1981, 1985; Donn *et al.*, 1986). Besides, this problem can also be considered as a part of more common problem of the sensitivity study of the atmosphere model with respect to variations of its initial data, external sources and internal parameters (Marchuk, 1975, 1979).

With that end in view, a set of linear and nonlinear functionals of perturbations of the basic solution can be taken as the indicators (measure) of the model sensitivity. Different mean values of meteorological fields, the kinetic and the total energies, the enstrophy and other physical characteristics of perturbations are the examples of such indicators. The choice of appropriate functionals is determined by the aims and tasks of each concrete investigation. Needless to say, that the results of the sensitivity analysis of any nonlinear model depend essentially not only on the sensitivity indicators chosen, but also on the stability properties of the basic state and the time-space structure of the forcing and the initial data perturbations considered.

The classical approach to the problem of model sensitivity includes the stability study of particular solutions of the model with respect to variations of their initial data, external sources and internal parameters. In particular, the linear stability problem to initial perturbations means an analysis of the spectral properties of the operator linearized about the basic state. In common case it is necessary to examine a structure of the stable and unstable invariant manifolds of particular solutions, as well as to try to find the Liapunov functions or apply other methods.

In case we want to examine the linear response of the model, an alternative approach based on applying the adjoint method and algorithms of the theory of small perturbations can be used. The adjoint method is universal and constructive enough for estimating the response of mathematical model with respect to small perturbations of the initial data and the external forcing. For the wide class of geophysical problems this approach was suggested and developed by Marchuk (1975, 1982) and Marchuk *et al.* (1985). A set of linear functionals representing the different time-space means of perturbations of the basic state fields are considered here as the main characteristics of the model sensitivity.

The adjoint method enables us to obtain the integral equations relating the sensitivity characteristics to small variations of input data. Besides, the solution of adjoint problem specially formulated can be interpreted as an influence function because it characterizes the influence of such variations on the magnitude of the sensitivity indicators mentioned above. Therefore, the time-space structure of a certain solution of the adjoint equation (influence function), and especially an information about the position and evolution of its local maxima and minima, facilitates better understanding the process that forms the linear response of the system. Besides, this information gives a possibility to estimate the relative contribution of different types of perturbations to the final value of the sensitivity indicator.

Thus the space-time structure of solutions of the adjoint problems specially formulated is of considerable importance in the study of linear response of a model. These solutions play the role which is similar to that of normal modes in the linear instability study (Marchuk and Skiba, 1990).

In this work the adjoint method is used for studying the formation process of the mean air surface temperature anomalies in December above the European part of USSR and the territory of USA in the framework of a simplified atmosphere-ocean-soil heat interaction model.

2. Atmosphere-ocean-soil thermal interaction model

The model domain $D = D_1 + D_2 + D_3$ consists of the spherical layer D_1 of the atmosphere (troposphere), the domain D_2 of upper layer of the World Ocean and the domain D_3 of upper layer of the soil (Fig. 1). Here $D_1 = \{(\lambda, \vartheta, z) : (\lambda, \vartheta) \epsilon S, 0 < z < h_1\}$; $D_2 = \{(\lambda, \vartheta, z) : (\lambda, \vartheta) \epsilon S_2, -h_2 < z < 0\}$; $D_3 = \{(\lambda, \vartheta, z) : (\lambda, \vartheta) \epsilon S_3, -h_3 < z < 0\}$; $S = S_1 + S_2 + S_3$ is the Earth surface, S_1 is the part of S covered by snow and ice, S_2 is the ocean surface, S_3 is the continent surface free of snow and ice, λ is the longitude, ϑ is the colatitude, z is the altitude, $z = h_1$ is the upper boundary of D_1, $z = 0$ is the interface between the atmosphere and the ocean (or the soil), $z = -h_2$ and $z = -h_3$ are the lower boundary of the domains D_2 and D_3 respectively.

Let $T(\lambda, \vartheta, z, t)$ be a deviation of the air−, water−, or soil− particle temperature from the basic state value $\tilde{T}(\lambda, \vartheta, z, t)$. Within the time interval $(o, \bar{t})$ we consider the simplified heat transport and diffusion equation

$$\alpha \frac{\partial T}{\partial t} + \text{div}\,(\vec{u}T) - \frac{\partial}{\partial z}(\nu \frac{\partial T}{\partial z}) - \mu \Delta_2 T = 0, \tag{1}$$

in the atmosphere domain D_1 and the ocean domain D_2. Here $\vec{u}(x, t) = \alpha(z)\vec{U}(x, t)$, $x \equiv (\lambda, \vartheta, z)$, $\vec{U}$ is the wind (or current) velocity vector in D_1 (in D_2), $\mu(x, t)$ and $\nu(x, t)$ are the heat turbulent diffusion coefficients in horizontal and vertical directions respectively, div ( ) is the divergence operator and Δ_2 is the spherical part of the Laplace operator, $\alpha(z) = C_p\, \rho(z)$ where $\rho(z)$ is the standard density and C_p is the specific heat. We suppose that $\alpha(z)$, $\vec{u}(x, t)$, $\mu(x, t)$ and $\nu(x, t)$ are the known functions in the time-space domain $D \times (0, \bar{t})$.

The coefficients $\vec{u}(x, t)$ and $\mu(x, t)$ are identically zero for all $x \epsilon D_3$ and therefore in the soil domain the equation (1) is the well-known one-dimensional (in z) heat equation.

As the initial and boundary conditions for Eq. (1) we take:

$$T(x, t) = T^o(x) \quad \text{at} \quad t = 0 \tag{2}$$

$$\nu \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = h_1, \quad z = -h_2 \quad \text{and} \quad z = -h_3 \tag{3}$$

$$\nu \frac{\partial T}{\partial z} = -F(\lambda, \vartheta, t) \quad \text{at} \quad z = 0 \quad \text{on} \quad S_1, \tag{4}$$

$$\mu \frac{\partial T}{\partial n} = 0 \quad \text{on lateral surface } \Omega \text{ of } D_2 \tag{5}$$

where $\vec{n}$ is the unit vector of the outer normal to Ω and $T^o(x)$ and $F(\lambda, \vartheta, t)$ are small deviations of the initial temperature and model heat forcing from the basic state quantities $\tilde{T}^o(x)$ and $\tilde{F}(\lambda, \vartheta, t)$.

On the surface S_2 (or on S_3) at $z = 0$ we prescribe the ocean-atmosphere (or soil-atmosphere) conjunction conditions which are typical for the diffraction problems when two media with dissimilar physical properties interact with each other at the interface (Ladyzhenskaya, 1973):

$$[T] \equiv T(\lambda, \vartheta, -0, t) - T(\lambda, \vartheta, +0, t) = 0, \tag{6}$$

and

$$\left[\nu \frac{\partial T}{\partial z}\right] \equiv \nu \frac{\partial T}{\partial z}(\lambda, \vartheta, -0, t) - \nu \frac{\partial T}{\partial z}(\lambda, \vartheta, +0, t) = F \tag{7}$$

where, by definition, the symbols $[T]$ and $[\nu \frac{\partial T}{\partial z}]$ mean the difference of the limit values (on the interface $z = 0$) of the functions $T(z)$ and $\nu \frac{\partial T}{\partial z}(z)$ obtained from the ocean (or soil) domain ($z = -0$) and from the atmosphere domain ($z = +0$). Thus, according to (6), the function $T(\lambda, \vartheta, z, t)$ is continuous in z everywhere in the domain D. As to the equation (7), it is the heat balance condition at the interface $z = 0$ between two media.

We also assume that the continuity equation

$$\text{div } \vec{u} = 0 \tag{8}$$

holds for the velocity vector $\vec{u}$ in D_1 and D_2. Besides, the normal component of the vector $\vec{u}$ to the lateral surface Ω of the oceanic domain D_2 is supposed to be zero:

$$\vec{u} \cdot \vec{n} = 0. \tag{9}$$

The unique solvability of the generalized solution of the problem (1)−(9) within any finite time interval $(0, \bar{t})$ was proved by Skiba (1978).

3. The adjoint method

The direct method of analysing the sensitivity of our model consists in solving the problem (1)−(7) repetedly for different variations of the initial data $T^0(x)$ and the forcing $F(\lambda, \vartheta, t)$. For complicated problem this procedure takes a lot of time and efforts. More efficient method can be used if we confine the sensitivity study by analysing the behaviour of several linear functionals of the perturbated solutions only (Marchuk, 1975, 1982).

Suppose we want to examine how the functionals

$$\Im_{p^*}(T) = \int_0^{\bar{t}} \int_D p^*(x, t) T(x, t) dx\, dt \tag{10}$$

and

$$\Im_{F^*}(T) = \int_0^{\bar{t}} \int_S F^*(\lambda, \theta, t) T(\lambda, \theta, 0, t) dS\, dt \tag{11}$$

will change if the input data variations $T^0(x)$ and $F(\lambda, \vartheta, t)$ change. Here $P^*(x, t)$ and $F^*(\lambda, \theta, t)$ are the known functions characterizing the sensitivity indicators analyzed. In this connection, in addition to the main problem (1)−(9), let us formulate an adjoint problem based on the concept of the adjoint operator in Hilbert space (Dunford and Schwartz, 1963). To do it we introduce the Hilbert space $L^2(D)$ of real functions $f(x)$ and $g(x)$ defined in the domain D with the inner product

$$< f, g >= \int_D f(x) g(x) dx \tag{12}$$

and the norm

$$\| f \| = < f, f >^{1/2} \tag{13}$$

To formulate the adjoint problem, note that under conditions (3)−(9) the advection operator $A_1 T = \text{div}\,(\vec{u}T)$ is skewsymmetric, and the diffusion operator $A_2 = \frac{\partial}{\partial z}(\nu \frac{\partial T}{\partial z}) + \mu\Delta_2 T$ is symmetric in Hilbert space $L^2(D)$ (Skiba, 1978), i.e.

$$< A_1T,\ g > = < T,\ -A_1 g > \quad \text{and} \quad < A_2 T,\ g > = < T,\ A_2 g >$$

for all functions T and g from the domain of the operator $A_1 + A_2$.

In the time-space domain $D \times (0,\ \bar{t})$ we put the adjoint problem:

$$-\alpha\frac{\partial T^*}{\partial t} - \text{div}(\vec{u}T^*) - \frac{\partial}{\partial z}(\nu\frac{\partial T^*}{\partial z}) - \mu\Delta_2 T^* = p^* \tag{14}$$

$$T^*(x,\ \bar{t}) = 0 \quad \text{at} \quad t = \bar{t} \tag{15}$$

$$\nu\frac{\partial T^*}{\partial z} = 0 \quad \text{at} \quad z = h_1, \quad z = -h_2 \quad \text{and} \quad z = -h_3 \tag{16}$$

$$\nu\frac{\partial T^*}{\partial z} = -F^*(\lambda,\ \vartheta,\ t) \quad \text{at} \quad z = 0 \quad \text{on} \quad S_1, \tag{17}$$

$$\mu\frac{\partial T^*}{\partial n} = 0 \quad \text{on lateral surface} \quad \Omega \quad of \quad D_2 \tag{18}$$

where $p^*(x,\ t)$ and $F^*(\lambda,\ \theta,\ t)$ are the functions defined the sensitivity functionals (10) and (11).

The conjunctions conditions on S_2 and S_3 at the interface $z = 0$ are similar to (6) and (7):

$$[T^*] = 0 \tag{19}$$

and

$$\left[\nu\frac{\partial T^*}{\partial z}\right] = F^* \tag{20}$$

Note that Eqs. (8) and (9) are also valid for the coefficients of the adjoint problem.

Let $p^* = 0$. Then it can be shown (Marchuk and Skiba, 1978) that the balance equations

$$\frac{\partial}{\partial t}\int_D \alpha(x)T(x,\ t)dx = \int_S F(\lambda,\ \theta,\ t)dS \tag{21}$$

and

$$-\frac{\partial}{\partial t}\int_D \alpha(x)T^*(x,\ t)dx = \int_S F^*(\lambda,\ \theta,\ t)dS \tag{22}$$

are fulfilled at any time. Besides, the estimates of the solutions T and T^* of the main and adjoint problems have the following form:

$$\frac{\partial}{\partial t}\int_D \alpha(x)T^2(x,\ t)dx \leq \int_S T(\lambda,\ \theta,\ 0,\ t)F(\lambda,\ \theta,\ t)dS \tag{23}$$

$$-\frac{\partial}{\partial t}\int_D \alpha(x)\{T^*(x,\ t)\}^2 dx \leq \int_S T^*(\lambda,\ \theta,\ 0,\ t)F^*(\lambda,\ \theta,\ t)dS \tag{24}$$

The integrals being in the left-hand size of (23) and (24), are equivalent to the norm (13) squared. Suppose that $F = F^* = 0$. Then (23) shows that the solution $T(x,\ t)$ is Liapunov stable. The same is also true for the adjoint solution $T^*(x,\ t)$ provided the problem (14)–(20) is solved from $t = \bar{t}$ to $t = 0$. Thus the adjoint problem is well-posed if it is solved in the opposite direction as compared with the main problem (1)–(9). That is why the initial condition (15) is put at $t = \bar{t}$.

Taking the inner product (12) of Eq. (1) with the function T^* and of Eq. (14) with the function T and subtracting the results obtained, we arrive at

$$\Im(T) = \Im_{F^*}(T) + \Im_{p^*}(T) = \int_0^{\bar{t}}\int_S T^*(\lambda,\ \theta,\ 0,\ t)F(\lambda,\ \theta,\ t)dSdt$$
$$+ \int_D \alpha(x)T^*(x,\ 0)T^0(x)dx \tag{25}$$

Setting $p^* = 0$ we obtain $\Im(T) = \Im_{F^*}(T)$. And if $F^* = 0$ then $\Im(T) = \Im_{p^*}(T)$. Hence, the functionals $\Im_{p^*}(T)$ and $\Im_{F^*}(T)$ can be calculated not only by means of the main problem solution $T(x,\ t)$ using (10) and (11), but also through the adjoint equation solution $T^*(x,\ t)$ using formula (25).

Let us divide the interval $(0,\ \bar{t})$ by N subintervals

$$I_n = (\bar{t} - n\tau,\ \bar{t} - (n-1)\tau) \tag{26}$$

of the sufficiently small length τ where $n = 1, \ldots, N$ and $\bar{t} = N\tau$. Then the formula (25) can be approximated by

$$\Im_\tau(T) \cong \tau\sum_{n=1}^{N} R_n + \int_D \alpha(x)T^*(x,\ 0)T^0(x)dx \tag{27}$$

where

$$R_n = \int_S T_n^*(\lambda,\ \theta)F_n(\lambda,\ \theta)dS \tag{28}$$

and $T_n^*(\lambda,\ \theta)$ and $F_n(\lambda,\ \theta)$ are the mean values of the functions $T^*(\lambda,\ \theta,\ 0,\ t)$ and $F(\lambda,\ \theta,\ t)$ within the interval I_n.

Thus in order to apply the adjoint method for estimating the mean temperature anomalies we have to solve the adjoint problem (14)–(20) and then use (25) or (27), (28). The structure of the formulae (25), and (28), shows that the adjoint equation solution $T^*(x,\ t)$ is the influence function of the heat forcing anomalies $F(\lambda,\ \theta,\ t)$ and the initial temperature anomalies $T^o(x)$

with respect to the magnitude of the functional (25) or (27). By Eq. (28), if the positions and the signs of the local maxima of $T_n^*(\lambda, \theta)$ and $F_n(\lambda, \theta)$ coincide then the integral R_n gives a considerable contribution into the value of $\Im_T(T)$. Conversely, if the spatial structures of $T_n^*(\lambda, \theta)$ and $F_n(\lambda, \theta)$ are orthogonal, i.e. $R_n = 0$, then the anomaly $F(\lambda, \theta, t)$ gives no contribution into the value of $\Im_T$ (T) within the time interval I_n. As a result, the time-space structure of the adjoint problem solutions enables us to analyze the process of forming the linear response of the model. Actually, in each interval I_n, the local maxima of the influence function $T_n^*(\lambda, \theta)$ expose the zones on the Earth surface which give the largest contribution into the value of the sensitivity indicator $\Im_T(T)$. These zones are called the energetically active zones of the oceans and the continents. Note that the time-space structure of $T_n^*(\lambda, \theta)$ depends to a considerable degree on the characteristic $F^*(\lambda, \mu, t)$, i.e. on choice of the sensitivity indicator, and also on the wind and the current fields as well as the vertical turbulent coefficient in the upper layer of the oceans (Marchuk and Lykossov, 1989).

Since $\Im(T)$ can be defined by two equivalent ways (see (10), (11) and (25)), it is useful to distinguish two opposite situations. If a lot of functionals are taken as the indicators in the sensitivity study, but the number I of the pairs $\{F_i(\lambda, \theta, t), T_i^o(x)\}$ analyzed is rather small $(i = 1, \ldots, I)$ then it is more efficient way to solve I times the main problem (1)–(9) and after that apply the formulas like (10) and (11). In another case when I is large, but the number of the sensitivity indicators is small then it is better to use the adjoint method and the formula (25) (Marchuk, 1975).

4. Calculation of the influence functions

We now consider the results of two numerical experiments which have been carried out on purpose to find the time-space structure of the influence function $T^*(x, t)$ in case when the climatic December means of the air surface temperature anomalies for two limited areas are taken as the sensitivity functionals (11). The limited area is the European USSR (hereafter the region B_1) in the first experiment, and the USA territory (the region B_2) in the second one. In other words, the functional (11) is equal to

$$\Im_{F^*}(T) = \frac{1}{\Delta t \text{ mes } B_m} \int_{\bar{t}-\Delta t}^{\bar{t}} \int_{B_m} T(\lambda, \theta, 0, t) dS\, dt, \tag{29}$$

in the $m - th$ experiment where mes B_m is the area of $B_m (m = 1, 2)$, and Δt is the one-month period, besides, the interval $(\bar{t} - \Delta t, \bar{t})$ coincides with December in our calculations. The functional (29) corresponds to

$$F^*(\lambda, \theta, t) = \begin{cases} \frac{1}{\Delta t \text{ mes } B_m}, & \text{if } (\lambda, \theta, t) \epsilon B_m \times (\bar{t} - \Delta t, \bar{t}) \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Here the simbol x means the product of two sets B_m and $(\bar{t} - \Delta t, \bar{t})$.

In accordance with the adjoint method, the functional (29) can be also estimated through the Eq. (27) if the adjoint problem (14)–(20) is solved with $p^* = 0$ and F^* defined by (30).

Analysis of the adjoint solutions indicates that T^* varies sufficiently slowly and smoothly in time, so that the approximate relation (27) can be used. The "information source" (30) is non-zero within the interval $(\bar{t} - \Delta t,\ \bar{t})$ and equal to the zero in the interval $(0,\ \bar{t} - \Delta t)$. Therefore, while the adjoint equation is solved with F^* in operation, the quantity $T^*(x,\ t)$ increases continuously on the interval $(\bar{t} - \Delta t,\ \bar{t})$ from the zero at $t = \bar{t}$ to a certain value at $t = \bar{t} - \Delta t$, and then begins to decrease monotonically because of large-scale diffusion. Hence if the time interval $(0,\ \bar{t})$ is large enough then the quantity $T^*(x,\ 0)$ will be rather small and the last integral in (25) and (27) becomes negligible. In this case Eq. (27) is reduced to

$$\Im_\tau(T) = \tau \sum_{n=1}^{N} \int_S T_n^*(\lambda,\ \theta) F_n(\lambda,\ \theta) dS \tag{31}$$

It follows from (31) that within each interval I_n, the heat flux anomaly $F_n(\lambda,\ \theta)$ has the weight $T_n^*(\lambda,\ \theta)$. Thus, for each $n (n = 1,\ 2, \ldots, N)$ the integration over the whole sphere S in the formula (31) is reduced to that over the subset of S, which is an intersection of two areas containing the non-zero (practically, only significant) values of $F_n(\lambda,\ \theta)$ and $T_n^*(\lambda,\ \theta)$. Therefore, the large values of $F_n(\lambda,\ \theta)$ are of importance only in such subsets of the surface of the sphere S where they are accompanied by sufficiently large values of $T_n^*(\lambda,\ \theta)$. That is why positions of the local maxima of $T_n^*(\lambda,\ \theta)$ are of great interest for us.

The scheme time step equal to 6 hr, was chosen on the basis of many tests with the numerical scheme used. Seven levels in z were considered: the levels 1–3 were in the atmosphere, the fourth level corresponded to the interface $z = 0$, and the levels 5–7 were in the ocean or soil. Three months interval, October-December, was taken as $(0,\ \bar{t})$ and divided into 9 ten-day subintervals $I_n (n = 1, \ldots, 9$; see (26)). Thus $\tau = 10$ days and $N = 9$ in the formula (27). Besides, the intervals I_n were numerated such a way that I_1 corresponded to the last 10-day period of December whereas I_9 coincided with the 1st 10 days of October. The adjoint problems were integrated in backward direction of time with the monthly mean climatic velocities in the atmosphere and the seasonal mean climatic velocities in the World Ocean. Values of the vertical turbulent coefficient in the ocean upper layer have been specified by the method that takes into account a variability of the sea surface temperatures (Marchuk and Lykossov, 1989).

The correctness of the computation was monitored by checking the satisfaction of the balance relation (22) for T^*. According to (30), the integral

$$\int_D \alpha(x) T^*(x,\ t) dx$$

must be equal to the unit within the interval $(0,\ \bar{t} - \Delta t)$ where $F^* = 0$.

For $n = 3,\ 6,\ 9$ the contours of the function $T_n^*(\lambda,\ \theta)$ are presented in Figures 1 – 3 for the first experiment and in Figures 4 – 6 for the second one. The figures chosen reflect the most characteristic features in the evolution of $T^*(\lambda,\ \theta,\ 0,\ t)$. Within the intervals $I_1 - I_3$ the non-zero values of the influence function $T_n^*(\lambda,\ \theta)$ $(n = 1 - 3)$ are only in the small neighbourhood of the European USSR (Fig. 1) and the USA-territory (Fig. 4). Starting from the time moment $\bar{t} - \Delta t (n \geq 4)$ the forcing F^* is equal to zero, and the distribution of $T^*(x,\ t)$ depends on such physical processes as the advection and the trubulent diffusion. It results in generating the local maxima of $T_n^*(\lambda,\ \theta)$. For $n = 6$, for example, there are three local maxima of this function in the 1st experiment (Fig. 2) and two maxima in the 2nd one (Fig. 5). Positions of some local maxima within the intervals $I_4 - I_9$ demonstrate the role of the well-known energetically active zones (EAZO) in the World Ocean (Marchuk, 1989). In particular, during the intervals $I_7 - I_9$, maximal values of $T^*(\lambda,\ \theta)$ are mostly located in the Norwegian EAZO, Newfaundland's EAZO and Gulfstream's EAZO in the 1st experiment (Fig. 3) and the Northern Pacific EAZO in the 2nd experiment (Fig. 6).

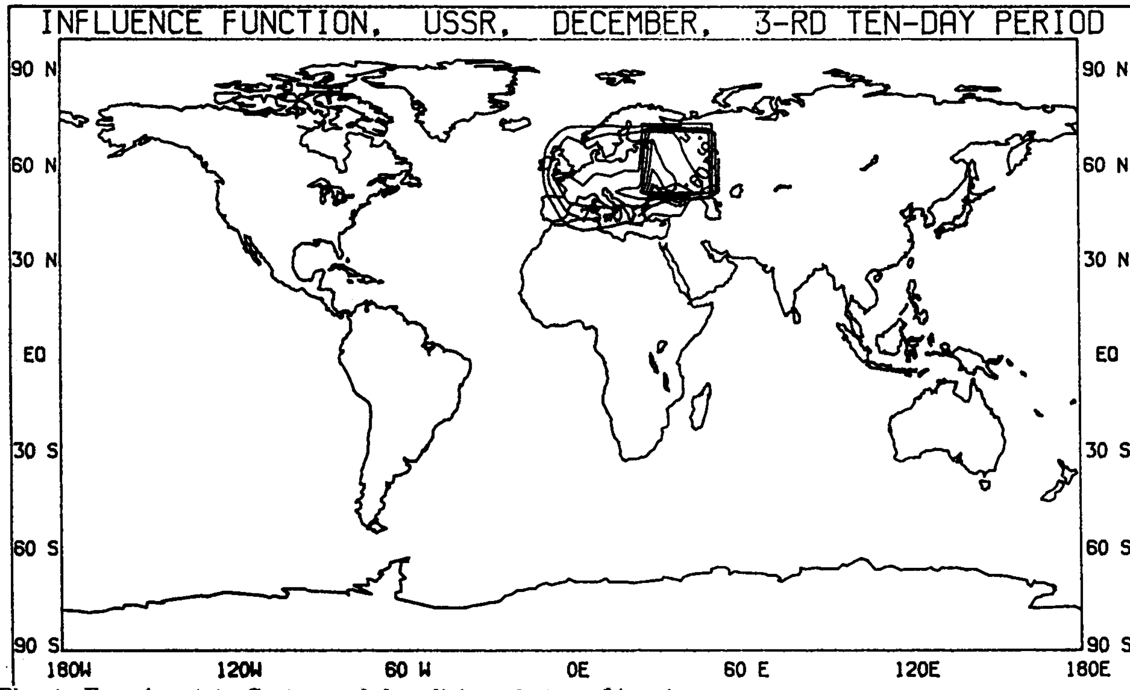

Fig. 1. Experiment 1. Contours of the adjoint solution $T_3^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_1.

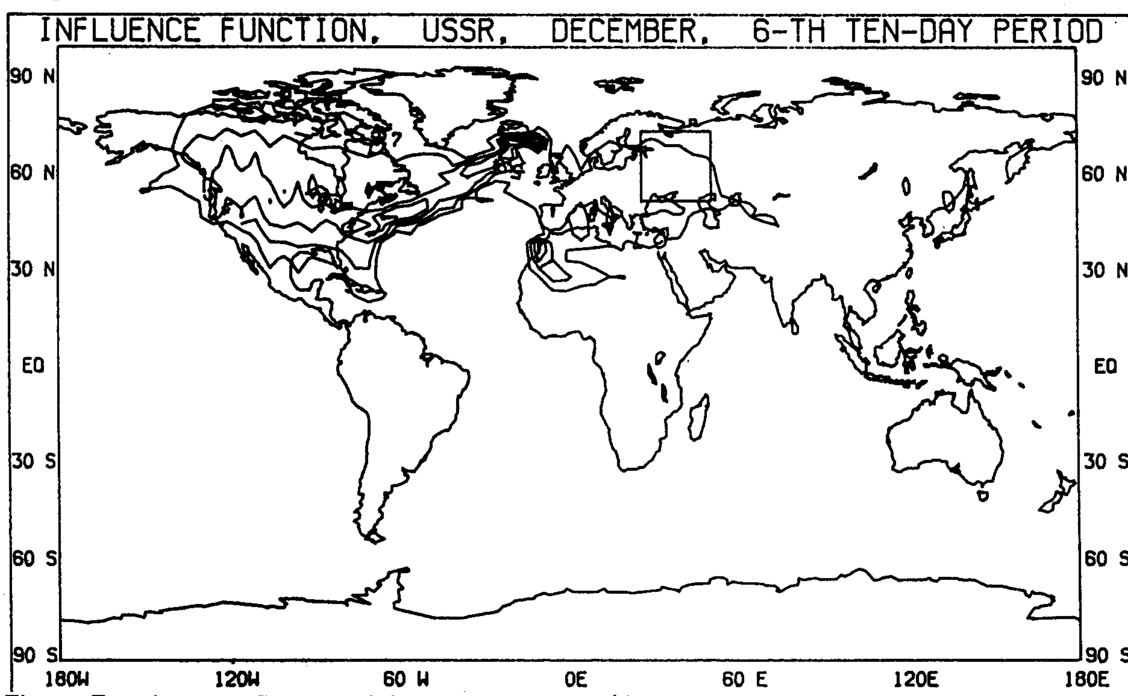

Fig. 2. Experiment 1. Contours of the adjoint solution $T_6^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_1.

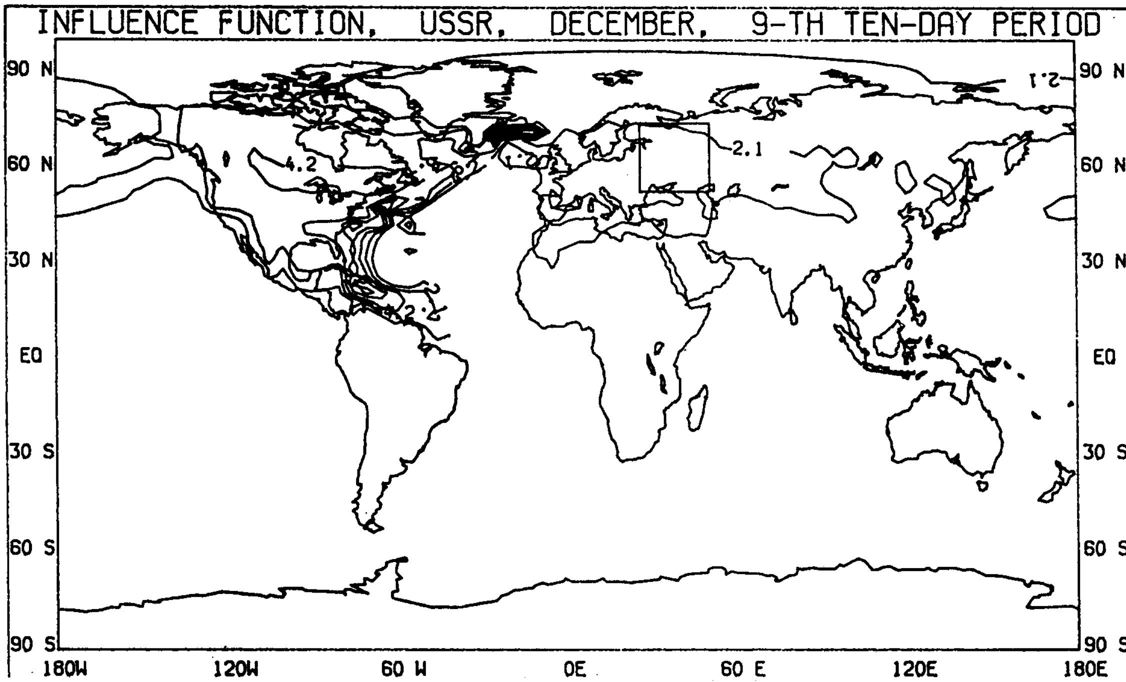

Fig. 3. Experiment 1. Contours of the adjoint solution $T_9^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_1.

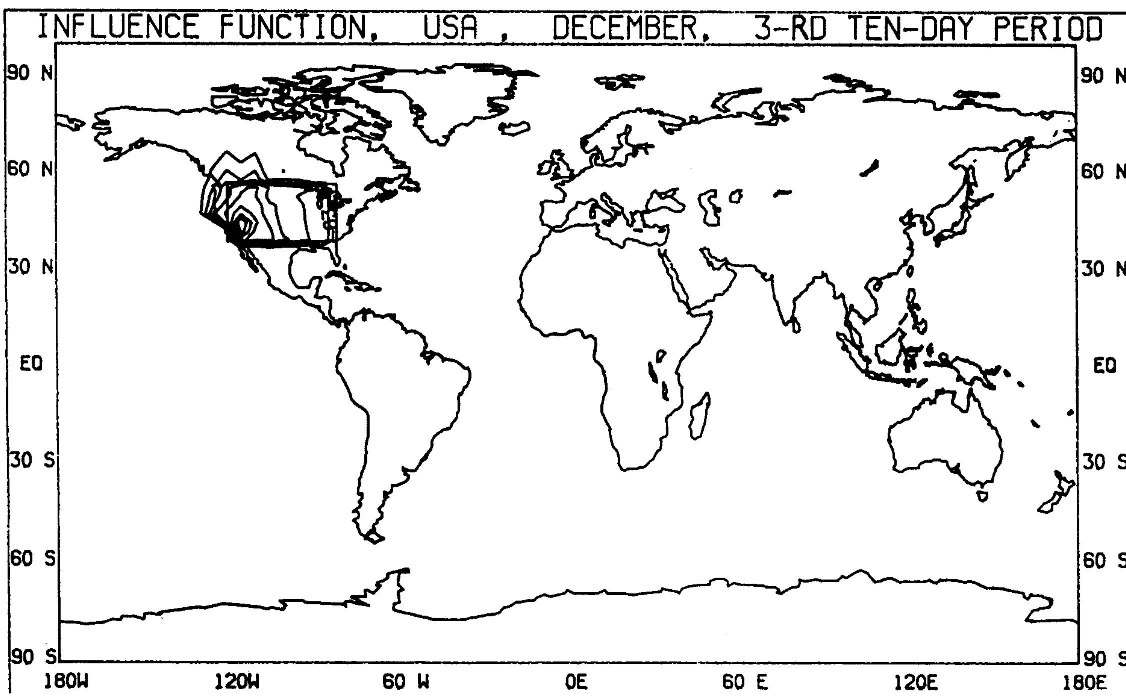

Fig. 4. Experiment 2. Contours of the adjoint solution $T_3^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_2.

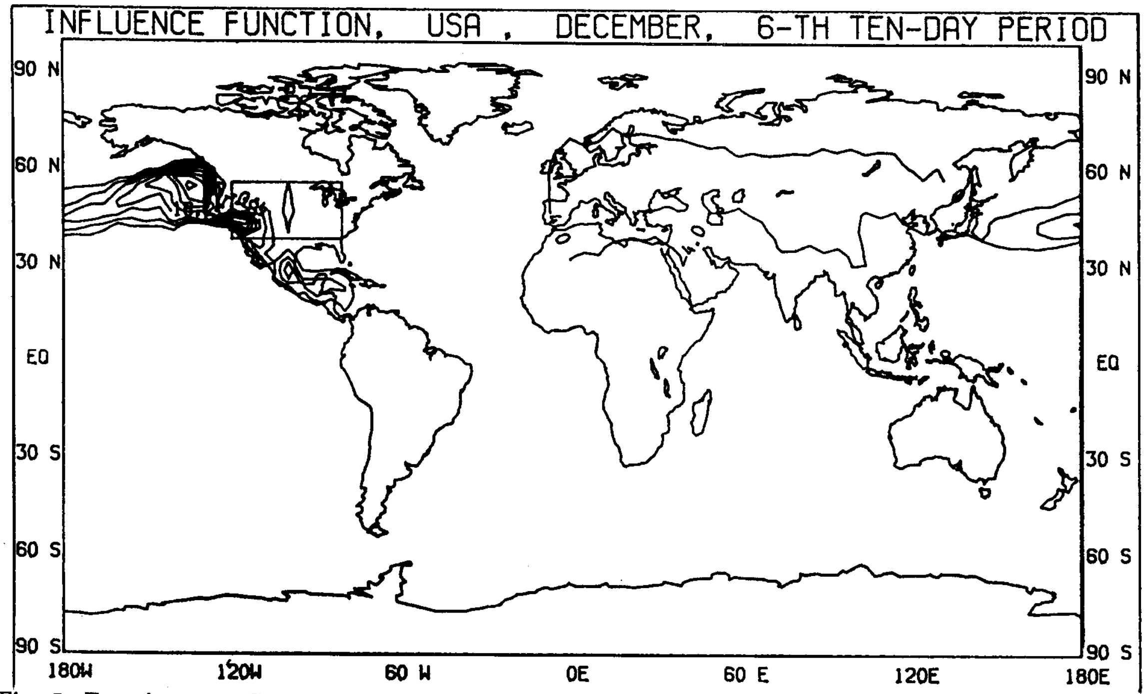

Fig. 5. Experiment 2. Contours of the adjoint solution $T_6^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_2.

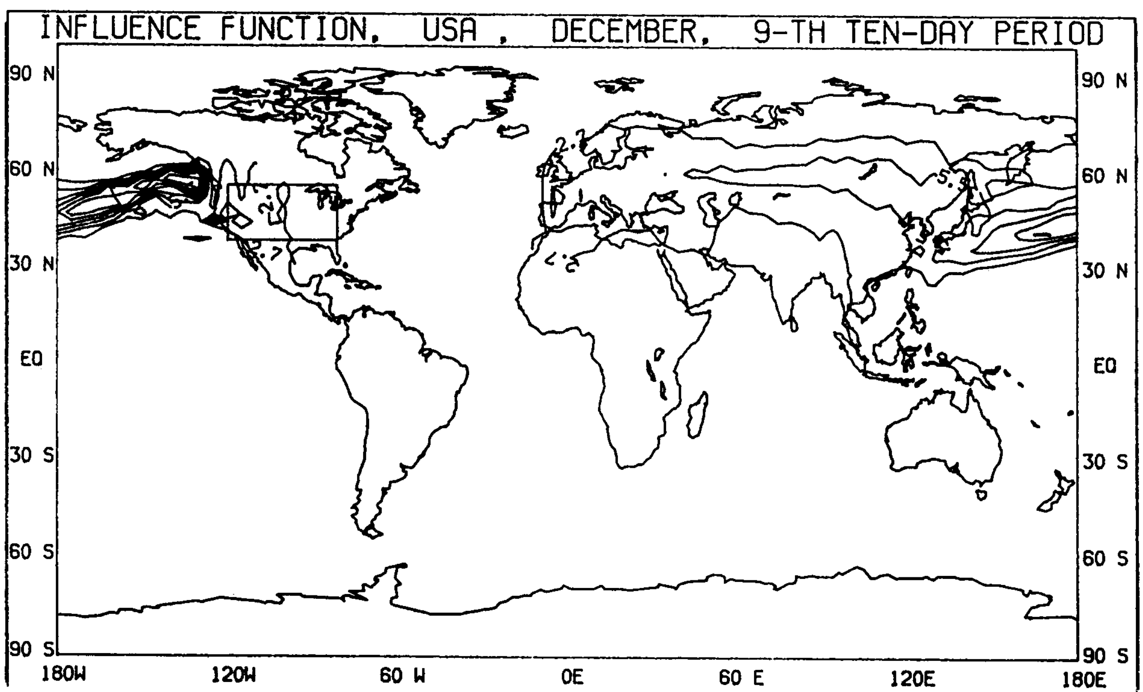

Fig. 6. Experiment 2. Contours of the adjoint solution $T_9^*(\lambda, \theta)$. The rectangular frame shows the forecast region B_2.

According to the formula (31), an information about values of the heat flux anomalies $F(\lambda, \theta, t)$ is of considerable importance in the zones of local maxima of $T^*(\lambda, \theta, t)$. Thus, within the framework of the present model it is possible to study the mechanism of the relation between heat flux anomalies F and the ultimate changes in the monthly mean temperature for various regions of the Earth. Of course, to get more exact information about the time-space structure of the influence functions T^* we should especify the wind and current velocity fields as well as the turbulent diffusion coefficients in the atmosphere and ocean.

APPENDIX

FACTORIZATION METHOD IN THE ATMOSPHERE IN MERIDIONAL DIRECTION

The three-dimensional non-stationary problems (1)–(9) and (14)–(20) have been reduced to a set of one-dimensional simple problems by using the splitting-up method (Peaceman and Rachford, 1955; Yanenko, 1959; Marchuk, 1982). These simple problems are successively solved by the direct (i.e. exact numerical) methods without any iterative procedures, besides, the Crank-Nicholson scheme is applyed for the time approximation on each splitted step. As a result, the finite-difference scheme is of the second order in spatial and time variables. All the numerical algorithms are described in detail in Skiba (1978) and Skiba and Tandon (1990). The total numerical algorithm is stable, independently of the choice of the scheme time step. The direct methods used are the factorization method in the oceanic domain D_2 and in the z-direction in D (Marchuk, 1982); the periodical factorization method in the λ-direction in the atmospheric domain D_1 (Samarskiy, 1971) and a special factorization method in the θ-direction in the domain D_1. We now shortly describe the last one. To a not inconsiderable degree it is based on Swartztrauber (1974) method.

For k-th vertical level ($k = 1, \ldots, K$), the finite difference splitted problem along the θ-direction in D_1 is represented by the following system of equations:

$$a_{ij}T_{i,\,j-1} - b_{ij}T_{ij} + c_{ij}T_{i,\,j+1} = -f_{ij}$$

$$(i = 1, \ldots, I; \qquad j = 1, \ldots, J-1),$$

$$-\bar{b}T_o + \sum_{i=1}^{I} \bar{c}_i T_{i,\,1} = -f_o \qquad (j = 0),$$

$$-\tilde{b}T_J + \sum_{i=1}^{I} \tilde{a}_i T_{i,\,J-1} = -f_J \qquad (j = J). \tag{A.1}$$

The indices i and j indicate the grid points in λ and θ directions respectively. The points $j = 0$ and $j = J$ correspond to the North and South Poles. We now introduce the notation

$$\vec{T}^{\,T} = (T_o, \vec{T}_1^{\,T}, \ldots, \vec{T}_I^{\,T}, T_J), \qquad \vec{f}^{\,T} = (f_o, \vec{f}_1^{\,T}, \ldots, \vec{f}_I^{\,T}, f_J) \tag{A.2}$$

where the superscript T means the transposition of a vector and $\vec{T}_i^{\,T} = (T_{i1}, T_{i2}, \ldots, T_{i,\,J-1})$ for $i = 1, 2, \ldots, I$.

Then we can write the system (A.1) in terms of vectors:

$$D\vec{T} = -\vec{f} \tag{A.3}$$

where the matrix D can be represented in the block form:

$$D = \begin{Vmatrix} -\bar{b} & \vec{X}_1 & \vec{X}_2 & \dots & \vec{X}_I & 0 \\ \vec{Y}_1 & P_1 & 0 & \dots & 0 & \vec{W}_1 \\ \vec{Y}_2 & 0 & P_2 & \dots & 0 & \vec{W}_2 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ \vec{Y}_I & 0 & 0 & \dots & P_I & \vec{W}_I \\ 0 & \vec{U}_1 & \vec{U}_2 & \dots & \vec{U}_I & -\tilde{b} \end{Vmatrix} \tag{A.4}$$

Here

$$\vec{X}_i = (\bar{c}_i,\ 0, \dots, 0), \qquad \vec{U}_i = (0, \dots, 0,\ \tilde{a}_i),$$

$$\vec{Y}_i^T = (a_{i,\,1},\ 0, \dots, 0), \qquad \vec{W}_i^T = (0, \dots, 0,\ c_{i,\,J-1}) \tag{A.5}$$

are vectors of the dimension $J-1$ and each $((J-1)*(J-1))$- matrix P_i is tridiagonal.

The characteristic feature of this system is that all the meridians meet at the pole points $(j = 0,\ J)$. Thus each pole point connects all meridians in the problem (A.1). Therefore, we should partition a matrix (A.4) of the system (A.3) by such a way that all the internal blocks of this matrix were tridiagonal. The decomposition of the matrix enables us to construct the numerical algorithm so that to separate all meridians from two pole points and apply the well-known factorization method for each inner diagonal block of the matrix (Skiba, 1978). Then the system (A.3) can be written as

$$-\bar{b}T_o + \sum_{i=1}^{I} \vec{X}_i \vec{T}_i = -f_o,$$

$$T_o \vec{Y}_i + P_i \vec{T}_i + T_J \vec{W}_i = -\vec{f}_i, \ (i = 1, \dots, I)$$

$$\sum_{i=1}^{I} \vec{U}_i \vec{T}_i - \tilde{b}\, \vec{T}_J = -f_J. \tag{A.6}$$

We will try to find a solution of the system (A.6) of the form

$$\vec{T}_i = \vec{G}_i + T_o \vec{V}_i + T_J \vec{Z}_i \tag{A.7}$$

Then the vectors $\vec{G}_i$, $\vec{V}_i$ and $\vec{Z}_i$ have to be solutions of the following systems:

$$P_i \vec{G}_i = -\vec{f}_i, \quad P_i \vec{V}_i = -\vec{Y}_i, \quad P_i \vec{Z}_i = -\vec{W}_i. \qquad (A.8)$$

Since P_i is a tridiagonal matrix, all these problems are solved by the 3-point factorization method.

The problems (A.8) can be solved simultaneously using parallel processors. Note that for each $i(i = 1, \dots, I)$ the only one inverse matrix P_i^{-1} is required to be found in order to solve all three problems (A.8). As soon as vectors $\vec{G}_i$, $\vec{V}_i$ and $\vec{Z}_i$ are known, we can calculate values of T_o and T_J at the pole points:

$$T_o = R_o/R, \qquad T_J = R_J/R \qquad (A.9)$$

where

$$R_o = \left(\sum_{i=1}^{I} \vec{U}_i\vec{Z}_i - \tilde{b}\right)\left(f_o + \sum_{i=1}^{I} \vec{X}_i\vec{G}_i\right) - \left(\sum_{i=1}^{I} \vec{X}_i\vec{Z}_i\right)\left(f_J + \sum_{i=1}^{I} \vec{U}_i\vec{G}_i\right)$$

$$R_J = \left(\sum_{i=1}^{I} \vec{X}_i\vec{V}_i - \tilde{b}\right)\left(f_J + \sum_{i=1}^{I} \vec{U}_i\vec{G}_i\right) - \left(\sum_{i=1}^{I} \vec{U}_i\vec{V}_i\right)\left(f_o + \sum_{i=1}^{I} \vec{X}_i\vec{G}_i\right)$$

$$R = \left(\sum_{i=1}^{I} \vec{X}_i\vec{Z}_i\right)\left(\sum_{i=1}^{I} \vec{U}_i\vec{V}_i\right) - \left(\sum_{i=1}^{I} \vec{X}_i\vec{V}_i - \tilde{b}\right)\left(\sum_{i=1}^{I} \vec{U}_i\vec{Z}_i - \tilde{b}\right).$$

Note that due to relations (A.5) $\vec{U}_i\vec{Z}_i = \tilde{a}_i Z_{i,\,J-i}$, The same is true for another products of vectors in the last three formulae. Finally (A.7) solves the problem.

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