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Cranking the semimicroscopic algebraic cluster model

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Initial steps are presented on the cranking of the Semimicroscopic Algebraic Cluster Model (SACM). This permits to treat Hamiltonians which do not present a dynamical symmetry.

Keywords: Cluster models.

Se presentan pasos iniciales para el *cranking* del Modelo Semimicroscópico de Cúmulos Nucleares (SACM). El procedimiento permite tratar Hamiltonianos cuales no presentan una simetría dinámica.

Descriptores: Modelos de cúmulos.

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1. Introduction

The *Semimicroscopic Algebraic Cluster Model* (SACM) was introduced in Refs. 1 and 2. One important advantage of this model is the treatment of the Pauli exclusion principle. The model space is microscopic, observing the Pauli exclusion principle, while the Hamiltonian consists of phenomenological interaction terms. Most applications of the SACM are restricted to *dynamical symmetries*, where the Hamiltonian is a function of the Casimir operators within a given group chain (see, for example Ref. 3). However, more interesting cases appear outside dynamical symmetries, allowing the investigation of phase transitions, *e.g.*, transitions between dynamical symmetries.

In Ref. 4 a geometric mapping of the SACM was given, allowing to consider more general Hamiltonians. This was exploited in Ref. 5 to consider the transition from the $SU(3)$ to the $SO(4)$ dynamical symmetry. Phase transitions were discussed and the persistence of effective symmetries [6]. For example, deformed nuclei exhibit a rotational structure, related to a $SU(3)$ symmetry, though this symmetry is badly broken due to the spin-orbit and pairing interactions. The topic on effective symmetries deals with the understanding on how such symmetries apparently survive. A resumé on the different types of symmetries is given in Ref. 7.

There is an alternative way to treat Hamiltonians within and outside dynamical symmetries. It is the cranking method [8,9] which imposes rotations on the system. It allows to generate rotational bands and the calculation of moment of inertias. Changes in the rotational behavior are related to phase transitions. Thus, the cranking method not only allows to study different phases but also provides additional information on the spectrum and moments of inertia.

Some work was published on the application of the cranking method related to clusterization (see, for example Ref. 10). The one, which is closely related to what we will present is published in Refs. 11 and 12. There, the cranking formalism is applied to the Interacting Boson Approximation (IBA) [13]. An *intrinsic* coherent state was introduced, whose parameters do not obey the properties of a tensor. The method allowed to calculate rotational spectra of Hamiltonians with no dynamical symmetry and the determination of the moment of inertia of the nucleus in consideration. The approximation of small rotational frequencies was applied, which excludes, however, the possibility to treat large angular momenta.

In this contribution we extend the ideas of [11,12] to the SACM. There, the basic excitations are not the quadrupole ones but dipole bosons $\pi^{\dagger}_{m}$ ($m = 0, \pm 1$). The procedure is illustrated for the case of ^{16}O+α, *i.e.* two spherical clusters. The Hamiltonian will be chosen within a dynamical $SU(3)$ symmetry. This is by no means a severe restriction, because first of all the cranking formalism works everywhere, and secondly, for this simple case the cranking of the SACM is very easy to illustrate. For a complete presentation, we refer to a forthcoming publication [14].

The paper is structured as follows: In section 2 the SACM is briefly resumed and the particular Hamiltonian, used for the application to ^{16}O+α, is discussed. In section 3 the intrinsic coherent state is introduced and some particular expectation values are presented. In section 4 the application to ^{16}O+α is given and the results are commented. Finally, in section 5 conclusions are drawn and an outline for future steps is given.

2. The SACM, a particular Hamiltonian and the geometrical mapping

The SACM is called *semimicroscopic* because its model space is microscopic, while the Hamiltonian is of phenomenological nature. The microscopic space is constructed as follows: The fact is exploited, that the cluster basis is equivalent to the shell model basis [15]. Each individual light cluster is represented by an $SU(3)$ irreducible representation (irrep) of the shell model [16]. In the original version only a definite isospin value is considered. The relative motion is described by the irreps of the harmonic oscillator, depending on the number of relative dipole bosons ($\boldsymbol{\pi}^{\dagger}_m$). The Wildermuth condition [15] is imposed, which requires a minimal number of relative oscillation quanta, otherwise the Pauli exclusion principle is violated. But this condition is not sufficient. The result of the coupling of the $SU(3)$ irreps and the relative motion to irreps of the total $SU(3)$, has to coincide with $SU(3)$ irreps of the shell model. Thus, the list obtained from the coupling is compared to the list of the shell model and only those $SU(3)$ irreps of the cluster basis are retained which coincide with one of the shell model. In this way, the Pauli exclusion principle is observed by taking into account only those irreps which have an overlap with the shell model. Auxiliary scalar bosons ($\boldsymbol{\sigma}^{\dagger}$) are added to the prescription, such that the total number of bosons, dipole plus scalar bosons, is fixed. This is a way to introduce a cutoff into a cluster model and at the same time having Hamiltonians which do not change the total number of bosons $N = n_\pi + n_\sigma$.

In group theoretical notation we have

$$\begin{array}{c} SU_{C_1}(3) \otimes SU_{C_2} \otimes U_R(4) \supset SU_C(3) \otimes SU_R(3) \\ (\lambda_1,\mu_1) \quad (\lambda_2,\mu_2) \quad N \quad (\lambda_C,\mu_C) \quad (n_\pi,0) \\ SU(3) \supset SO(3) \supset SO(2) \\ (\lambda,\mu) \qquad L \qquad M \end{array} \tag{1}$$

Here, the indices C_k refer to the individual clusters, C to the coupling of the two cluster irreps, n_π is the number of relative oscillation quanta, N is the total number of bosons and $\lambda_\alpha, \mu_\alpha$, with $\alpha = C_k, C$ or no index refer to the corresponding $SU(3)$ irreps. A particular feature of the model is that the distance of the two clusters at $0\hbar\omega$ is proportional to $\sqrt{n_0}$ [4], where n_0 is the minimal number of dipole oscillation quanta.

When the Hamiltonian is in a dynamical symmetry, it is a function of the Casimir operators, appearing in the corresponding group chain. For this case, the solutions of the spectrum are analytical and quite complicated systems can be treated by this model. However, as stated in the introduction, most interesting cases are represented by Hamiltonians which are not in a dynamical symmetry. When such Hamiltonians are treated, phase changes can be investigated, as was done in Ref. 5 using an exact diagonalization and the methods of coherent states, related to a geometrical mapping [4].

The cranking method is an alternative way, permitting the determination of at least the Yrast band and the moment of inertia of the nucleus as a function of the angular momentum. Because we only want to illustrate this method, we rather do not start from the most general Hamiltonian. In effect, we will use a Hamiltonian within a dynamical symmetry, which is not a serious restriction but serves to illustrate the general method. Results can be easily compared to the exact one. The Hamiltonian we use is given by

$$\boldsymbol{H} = [\hbar\omega \boldsymbol{n}_\pi - d_1 \boldsymbol{C}_2(\lambda,\mu)] + \gamma \boldsymbol{L}^2. \tag{2}$$

It describes the $SU(3)$ dynamical symmetry. The second order Casimir operator $\boldsymbol{C}_2(\lambda,\mu)$ of $SU(3)$ acquires the form $\boldsymbol{n}_\pi(\boldsymbol{n}_\pi+3)$ for the case of two spherical clusters. For the operators, the phase convention $\boldsymbol{O}^{lm} = (-1)^{l-m}\boldsymbol{O}^{l}_{-m}$ is used.

3. The geometrical mapping and the intrinsic coherent state

The *geometrical mapping* is achieved, using as a coherent state

$$|a\rangle = \mathcal{N}_{Nn_0} \left(\boldsymbol{a}\cdot\boldsymbol{\pi}^{\dagger}\right)^{n_0} \left[\sigma^{\dagger} + \left(\boldsymbol{a}\cdot\boldsymbol{\pi}^{\dagger}\right)\right]^{N} |0\rangle, \tag{3}$$

with n_0 as the minimal number of oscillation quanta and $(N+n_0)$ the total number of oscillation quanta. This is the same form used as in Ref. 4, with the difference that now the parameters a_m of the coherent state are not necessarily the components of a tensor. This leads to the definition of the *intrinsic coherent state*, as introduced in Refs. 11 y 12. The coherent state in Ref. 3 reflects the fact that there is a minimal number of bosons n_0.

We use the definition

$$\left(\boldsymbol{a}\cdot\boldsymbol{\pi}^{\dagger}\right) = \sum_m a_m \boldsymbol{\pi}^{\dagger}_m. \tag{4}$$

The a_m are in general complex and arbitrary. The complex conjugate is denoted by a^*_m. We also use

$$\tilde{a}_m = (-1)^{1-m} a_{-m}. \tag{5}$$

This is important, when we apply $\boldsymbol{\pi}_m = (-1)^{1-m}\boldsymbol{\pi}^{-m}$ to the coherent state on the right. Applying a boson creation operator $\boldsymbol{\pi}^{\dagger}_m$ to the left results in a^*_m. We use also

$$(\boldsymbol{a}^*\cdot\boldsymbol{a}) = \sum_m a^*_m a_m. \tag{6}$$

As particular examples we list the expression for the normalization and a particular expectation value. The formal expression is the same as in Ref. 4, keeping in mind the distinct feature of the parameters a_m. The normalization coefficient is given by

$$\mathcal{N}^{-2}_{Nn_0} = \frac{N!^2}{(N+n_0)!}\frac{d^{n_0}}{d\gamma_1^{n_0}}\frac{d^{n_0}}{d\gamma_2^{n_0}} \times \left[1+\gamma_1\gamma_2(a^*\cdot a)\right]^{N+n_0} |_{\gamma_1=\gamma_2=1}, \tag{7}$$

taken at $\gamma_1 = \gamma_2 = 1$. A particular expectation value is

$$\langle \left[\boldsymbol{\pi}^\dagger \otimes \boldsymbol{\pi}\right]^{[S]}_m \rangle = (N+n_0)\,[\boldsymbol{a}^* \times \tilde{\boldsymbol{a}}]^S_m \mathcal{N}^2_{Nn_0} \frac{N!^2}{(N+n_0)!} \times \frac{d^{n_0}}{d\gamma_1^{n_0}} \frac{d^{n_0}}{d\gamma_2^{n_0}} \gamma_1\gamma_2 \times \left[1+\gamma_1\gamma_2(a^*\cdot a)\right]^{N+n_0-1} |_{\gamma_1=\gamma_2=1}\,, \quad (8)$$

which includes as a particular case the number operator ($S = 0$). We also used the notation

$$[\boldsymbol{a}^* \times \tilde{\boldsymbol{a}}]^{[S]}_m = \sum_{m_1 m_2} (1m_1, 1m_2|Sm) a^*_{m_1} \tilde{a}_{m_2}, \quad (9)$$

Where the coupling sign "$\times$" instead of "$\otimes$" was used, in order to indicate that no tensors are coupled. It serves just as a notation.

In Ref. 9 an expansion of the complicated factor is applied in terms of powers in a_m, which describe the difference to the relative distance of the clusters. The limit $N \to \infty$ is applied and terms proportional to n_0, the minimal number of dipole oscillation quanta, are neglected. This time we keep also terms in lowest order in n_0.

Finally, some comments are made on the possible symmetries of the cranked cluster model. In fact, for the general case of two different, deformed nuclei, there is no discrete symmetry present. For the static case, however, the time reversal symmetry can be applied, leading to (proved in a similar way as done in Ref. 12)

$$a^*_m = (-1)^{1-m} a_{-m}. \quad (10)$$

Due to the phase choice, this leads to a purely imaginary a_0. Nevertheless, in a parametrization we can assume a_0 to be real, because in the calculation of the expectation value of the Hamiltonian the factor $(i)(-i) = 1$, appears always. In the cranked model, the time reversal symmetry does not exist and in general there are 3 complex variables a_m ($m = 0, \pm 1$), *i.e.*, 6 parameter variables.

4. Application to ^{20}Ne$\to^{16}$O+α

In this case, both clusters are spherical, *i.e.*, the quantum numbers of the clusters in Eq. (1) are

$$(\lambda_1, \mu_1) = (\lambda_2, \mu_2) = (\lambda_C, \mu_C) = (0, 0). \quad (11)$$

This simplifies the calculation because only the relative motion enters here as an independent degree of freedom. The minimal number of $\boldsymbol{\pi}$ bosons is $n_0 = 8$. The Hamiltonian of Eq. (2) is too simple in order to describe the spectrum of ^{20}Ne, though, the ground state band and the position of the first excited 0^+ band can be easily fitted. Here, we are not interested to obtain a perfect fit but only illustrate the cranking method. Thus, we do not present detailed fits.

The calculation of the expectation value of the Hamiltonian is simplified, when the following very particular parametrization of the parameters a_m is used (it is not the most general one but still permits to obtain transparent formulas for the expectation values)

$$a_0 = \delta\,\cos(\theta)$$
$$a_1 = \frac{\delta}{\sqrt{2}}\sin(\theta) \;=\; a_{-1}.$$
$$(\lambda_1, \mu_1) = (\lambda_2, \mu_2) = (\lambda_C, \mu_C) = (0, 0). \quad (12)$$

It is not the most general one, which include a_m to be complex and also $a_1 \neq a_{-1}$. But this parametrization will make our point clearer, though, a more general parametrization would give more details.

To the Hamiltonian (2) the term $-\Omega \boldsymbol{L}_x$ is added, in order to arrive at the cranking Hamiltonian with the rotational frequency Ω, *i.e.*,

$$\boldsymbol{H}' = \boldsymbol{H} - \Omega \boldsymbol{L}_x \quad , \quad (13)$$

describing a rotation around the x-axis, which is perpendicular to the z-axis, defined as the one which connects both clusters.

With this, the geometrical mapped Hamiltonian is given by

$$\begin{aligned} E = {} & n_0\left[\hbar\omega - d_1 n_0 - 4d_1 - \Omega\sin(2\theta)\right] \\ & + n_0 N\left[\hbar\omega - 2d_1 n_0 - 4d_1 - \Omega\sin(2\theta)\right]\delta^2 \\ & \frac{1}{2} n_0^2 N^2\left[-\hbar\omega + 2d_1 n_0 + 4d_1 + \Omega\sin(2\theta)\right]\delta^4 \\ & + \gamma n_0^2 \sin^2(2\theta)\left(1 + 2N\delta^2 - N^2 n_0 \delta^4\right) \\ & + 2\gamma n_0\left(1 + N\delta^2 - \frac{1}{2}N^2 n_0 \delta^4\right), \end{aligned} \quad (14)$$

where δ^2 is assumed to be small (remember that δ refers to the *change with respect to the zero order distance* [4]). This assumption is not severe, because also the general result can be used, with arbitrary δ. Note also, that we do not assume small Ω, thus permitting a future discussion of high spin states.

Differentiate V with respect to θ and setting the result to zero, leads to the equation

$$\sin(2\theta) \approx \frac{\Omega}{2\gamma n_0} \times \left(1 - N\delta^2 + \left[2N^2 + \frac{1}{2}n_0 N^2\right]\delta^4\right). \quad (15)$$

Note, that for $\Omega = 0$ the $\theta = 0$, as it should be. In this case $a_{+1} = 0 = a_{-1}$ too.

The expectation value of the $\boldsymbol{L}_x$ operator is given by

$$\langle L_x \rangle = n_0 \sin(2\theta)\left(1 + N\delta^2 - \frac{1}{2}n_0 N^2 \delta^4\right). \quad (16)$$

The value L is the angular momentum and the last equation relates θ to L, when we set $\langle L_x\rangle \approx L$ (more correctly it is $\sqrt{L(L+1)}$). When $\theta = 0$, then L=0. The maximum value is obtained, when we set $\sin(2\theta) = 1$, *i.e.* $\theta = \pi/4$. In this case and setting $\delta = 0$, the expectation value of $\langle L_x\rangle$ is n_0. Setting δ^2 to zero, implies to exclude intershell excitations, *i.e.*, we explore the $0\hbar\omega$ space only. This is a very nice result, because for the ground state at $0\hbar\omega$ the orbital irrep is given by $(n_0, 0)$, *i.e.*, the maximal spin possible is n_0! The contributions from the part, depending on powers of δ^2, refers to intershell excitations, which increase the maximal spin possible.

Substituting in $\langle \boldsymbol{L}_x\rangle$ the equation for $sin(2\theta)$, we arrive at

$$\langle L_x\rangle \approx \frac{\Omega}{2\gamma}\left(1 + N^4 n_0\delta^4 + N^2\delta^4\right) \qquad (17)$$

We have three equations: $\partial V/\partial\delta = 0$, with V taken from Ref. 14, $\partial V/\partial\theta = 0$ and Eq. (16). Note, that from the first two equations we get δ and θ as functions in Ω. Therefore, setting in the expression for δ and θ, the third equation gives us Ω as a function in L. The L can then be set to the desired value, *e.g.*, 0, 2, 4, etc.

The potential is of the form

$$E = F_1(\theta) + F_2(\theta)\delta^2 + F_3(\theta)\delta^4. \qquad (18)$$

Finally, we derive the moment of inertia to lowest order in δ^2. It is just given by the trivial result

$$\frac{\partial\langle \boldsymbol{L}_x\rangle}{\partial\Omega} \;=\; I \approx \frac{1}{2\gamma}\left(1 + N^4 n_0\delta^4 + N^2\delta^4\right). \qquad (19)$$

To lowest order, it is just $1/2\gamma$, which we could have guessed, setting in the Hamiltonian $\gamma\boldsymbol{L}^2 = (1/2I)\boldsymbol{L}^2$.

Thus, the cranking formalism leads to consistent results, using as a trial state the coherent state proposed. The correction, of course, takes into account contributions from higher shell excitations.

The results change, when non-spherical clusters are taken into account. How to proceed is indicated in Ref. 4. There the contributions of operators, acting on the clusters, are listed.

5. Conclusions and outlook

In this contribution we illustrated the procedure on how to crank a general Hamiltonian of the SACM, which is not necessarily within a dynamical symmetry. We used, however, for illustrative reasons a particular Hamiltonian in a dynamical symmetry $SU(3)$ and a particular parametrization of the coherent state parameters. With this, we reproduce obvious results, demonstrating the correctness of our procedure.

The next steps are to consider more general Hamiltonians in the simplified parametrization of a_m, in terms of δ and θ. This suffices already for the investigation of possible phase transitions, which should change the rotational behavior of the system.

The simplified parametrization refers to axial symmetric systems, as can be seen by the following argument: Because the geometrical mapping of $\boldsymbol{L}_z = \left(\boldsymbol{\pi}^\dagger_{+1}\boldsymbol{\pi}^{+1} - \boldsymbol{\pi}^\dagger_{-1}\boldsymbol{\pi}^{-1}\right)$ leads to an expression proportional to $\left(|a_{+1}|^2 - |a_{-1}|^2\right)$, it disappears in our simplified parametrization ($a_{+1} = a_{-1}$). Axial symmetric systems comprise two spherical clusters or one or two deformed clusters, with axial symmetry, aligned along the z-axis. Thus, the removal of the condition $a_{+1} = a_{-1}$ leads to the description of cluster systems with arbitrary relative orientations.

The cranking formalism is general enough to allow the description of quite complicated Hamiltonians. The complete formalism is in preparation [14].

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