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Advances on the algebraic approach to non-perturbative description of QCD

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Starting from QCD and applying several approximations, like restricting to $SU(2)$ for color, a system of quarks in more than one orbital are considered. Restricting to quarks in the lowest s-orbit results in an analytic solution. We give a recursive method how to include all orbitals and only restrict to s- and p-orbitals. The structure of the Hamiltonian is discussed. Under certain approximations for the potential interaction, an analytic result is obtained. We discuss possible methods to treat the problem and explain the state of the investigation.

Keywords: QCD at low energy; analytic solutions; algebraic model.

Partiendo de la QCD y aplicando varias aproximaciones, como restringir a $SU(2)$ en el sector de color, investigamos un sistema de muchos quarks en más que un orbital. Restregiendo solo a quarks en el orbital s más bajo, se obtiene una solución analítica. Damos un método recursivo para incluir todos los demás orbitales y posteriormente nos restringimos a los orbitales s y p. Se discute la estructura del Hamiltoniano. Bajo ciertas aproximaciones para la interacción potencial, se obtiene otra vez una solución analítica. Mostramos posibles procedimientos y explicamos el estado de la investigación.

Descriptores: QCD a bajas energías; soluciones analíticas; modelo algebraico.

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1. Introduction

At low energy, the QCD coupling constant becomes large and the theory is highly non-perturbative. Significant advances have been made using numerical simulations within the lattice gauge theory. It requires, however significant numerical effort and obtaining the complete spectrum is not feasible. Several other approaches have recently been proposed that map QCD onto a low energy effective theory, e.g. using canonical coordinates of the Coulomb gauge [1]. Several fundamental aspects of low-energy QCD can be explained within the Coulomb gauge framework, including a mechanism for confinement. The resulting Hamiltonian is nevertheless complicated and approximate methods have to be employed to resolve it.

In parallel, algebraic models have been developed where one attempts to isolate a few relevant degrees of freedom and to model their interactions. In Refs. 2 to 4 the basic assumption was to use only quarks and antiquarks confined to a hadronic bag in the s-orbital and interacting with a single level interpreted as describing a gluon pair with total spin and color zero. With only a few parameters the model is quite successful in describing the hadron spectrum without running into the problem of generating too many low-energy states. The latter is the consequence of mixing between quark-antiquark pairs and gluon pairs which reproduces the expected content of hadrons, including the vacuum, in terms of quark and gluon constituents [5]. The model is even capable of reproducing highly nontrivial aspects of the spectrum, like the position of the Roper resonance [4].

A connection between a first principle approach to the phenomenological model discussed above, has been attempted in Ref. 6. Starting from the $SU(2)$ QCD Hamiltonian in Coulomb gauge a set of low-energy effective degrees of freedom has been proposed based on the following assumptions: The Faddeev-Popov term is approximated by a simple potential interaction, eliminating in this way the dynamic contribution of gluons and the quarks. The quarks were confined to a finite volume and restricted to the lowest orbital state, i.e. the s-orbital. Under these approximations the effective Hamiltonian was obtained, with a single parameter describing the interaction. The effective Hamiltonian can be diagonalized analytically. For zero effective quark mass all colorless states were found to be degenerate, while the colored states were raised in energy in a way proportional to the coupling constant. This is a remnant effect of confinement and its origin is the Faddeev-Popov term. Emergence of breaking of the chiral symmetry was observed.

In this contribution we relax some of the approximations made so far in order to build a model that is more realistic. We do this by including quarks in the p-orbital.

The paper is organized as follows. In Sec. 2 the general structure of the Hamiltonian is presented. In Sec. 3 we
show how all higher orbitals can be included in a recursive way. Finally, only the first s- and p-orbital are included and in Sec. 4 future strategies needed to solve the extended model are discussed. Again an analytic solution is obtained, when the potential interaction is constant. Finally, in Sec. 5 conclusions are drawn.

2. The Hamiltonian

In this section we discuss the general structure of the kinetic and potential energy terms, for any number of orbitals. We will finally restrict to interactions considering only the lowest s- and p-orbital.

For each orbital, the quarks can occupy a positive energy level, which defines a pseudo-spin projection of 1/2, and a negative energy level, with pseudo-spin projection -1/2. This is illustrated in Fig. 1. For simplicity the s- and p-orbitals are plotted at the same energy, even though the p-orbital is expected to have higher energy. We use the Dirac picture, i.e. real quarks correspond to occupied states in the upper level while anti-quarks correspond to holes in the lower energy level. When more orbitals are considered, one has to add two levels for each orbital additional, one at positive and the other at negative energy.

The Hamiltonian consist of two parts. The first is the kinetic energy

\[ K = \int dx \psi^\dagger(x) \left[-i\nabla \cdot \alpha \right] \psi(x) \]

\[ = \int dx \left[ \psi_1^\dagger(x, \lambda, c, f) \left[-i\nabla \cdot \sigma \right] \chi \psi_2(x, \lambda, c, f) \right] + [\psi_2^\dagger(x, \lambda, c, f) \left[-i\nabla \cdot \sigma \right] \chi \psi_1(x, \lambda, c, f)], \]

with \( \psi_1 \) (\( \psi_2 \)) being the upper (lower) component of the Dirac four-spinor and \( \sigma \) denotes the Pauli matrices. The other part is the potential interaction given by \( \psi = [\psi_1, \psi_2] \)

\[ V = \int dx d\mathbf{y} \psi^\dagger(x) T^a \psi(x) V(|x - \mathbf{y}|) \psi^\dagger(\mathbf{y}) T^a \psi(\mathbf{y}), \]

where the contribution from gluons has been approximated by a static potential. The upper (index 1) and lower component (index 2) of the wave function \( \psi \) are given in the decoupled representation of the orbital angular momentum and spin, i.e.,

\[ \psi_1(x, \sigma, c, f) = \sum_{Nl} b_N^{\dagger l} Y_{lm}(x) \chi_{lm} \]

\[ \psi_2(x, \sigma, c, f) = \sum_{Nl} b_N^{\dagger l} Y_{lm}(x) \chi_{lm}, \]

where contributions from different orbitals are given by the spherical harmonics; \( N \) refers to the principal quantum number, which for the harmonic oscillator gives the number of oscillation quanta described by the wave functions, \( R_{Nl} \), that represents the radial motion of quarks in a hadronic bag; \( \chi_{lm} \) are the spin wave functions and finally, \( b \) and \( b^\dagger \) are annihilation and creation operators, respectively. In the case of annihilation operators the +1/2 (-1/2) indicates the upper (lower) level, \( t \) the orbital momentum, \( m \) its projection, \( \sigma \) the projection of the spin, \( c \) the color component and finally \( f \) the flavor component. These creation and annihilation operators can be transformed to the basis in which spin an orbital angular momentum are coupled to the total spin \( j \),

\[ b_{\alpha(\frac{1}{2})j \lambda, cf} = \sum_{m} (\frac{1}{2} \sigma | j \lambda \rangle b_{\alpha lm, cf}. \]

We define the operation of raising indices by,

\[ b_{\alpha(\frac{1}{2})j \lambda, cf} = (-1)^{\frac{1}{2} - \alpha} (-1)^{\frac{1}{2} - \lambda} (-1)^{\frac{1}{2} - c} \]

\[ (-1)^{\frac{1}{2} - f} b_{\alpha(-\frac{1}{2})j - \lambda, c - f} \]

With the help of this expansion, the kinetic part can be written as

\[ K = \sum_{NN'} K(NN'j) \left\{ b_{\frac{1}{2}(Nl=j+\frac{1}{2})j \lambda, cf}^\dagger b_{\frac{1}{2}(N'l'=j'-\frac{1}{2})j' \lambda, cf}^\dagger + b_{\frac{1}{2}(Nl=j+\frac{1}{2})j \lambda, cf} b_{\frac{1}{2}(N'l'=j'-\frac{1}{2})j' \lambda, cf} \right\} \]

\[ + \sum_{NN'} K^*(NN'j) \left\{ b_{\frac{1}{2}(N'l'=j-\frac{1}{2})j' \lambda, cf}^\dagger b_{\frac{1}{2}(Nl=j+\frac{1}{2})j \lambda, cf}^\dagger + b_{\frac{1}{2}(N'l'=j-\frac{1}{2})j' \lambda, cf} b_{\frac{1}{2}(Nl=j+\frac{1}{2})j \lambda, cf} \right\}, \]

with

\[ K(NN'j) = -i \int d^2r \left[ R_{Nj+\frac{1}{2}}(r) \left( \frac{d}{dr} - \frac{j - \frac{1}{2}}{r} \right) R_{N'j-\frac{1}{2}}(r) \right]. \]

Similarly for the potential part we obtain

\[ V = \sum_{N, i, j, c, \alpha, f, f', L, M, c} \frac{(-1)^{L-M}}{2^{L+1}} (-1)^{L-c} \left[ b_{\alpha(N, l_{1}^{j}), L_{1}, j_{1}, c_{1}}^{\dagger} b_{\alpha', (N', l_{2}^{j'}, L_{2}, j_{2}, c_{2})} \langle j_{1}, j_{2}, \lambda_{2} | LM \rangle \left( \frac{1}{2} c_{1}, \frac{1}{2} c_{2} \right) \right] \]

\times \left[ b_{\alpha', (N, l_{1}^{j}), L_{1}, j_{1}, c_{1}}^{\dagger} b_{\alpha''', (N', l_{2}^{j'}, L', j_{2}, c_{2})}^{\dagger} \langle j_{3}, j_{4}, \lambda_{4} | L - M \rangle \left( \frac{1}{2} c_{3}, \frac{1}{2} c_{4} \right) \right] V_{N, l_{1}, j_{1}, L} \right] (8) \]

The indices \( \alpha \) are used to label pseudo-spin components, and \( l_{1}, f, c_{1}, j_{1}, \lambda_{1} \) label angular momentum, flavor, color, total spin and its projection, respectively. The \( V(N, l_{1}, j_{1}, L) \)'s give the intensity of each component of the interaction, and are given by

\[ V(N, l_{1}, j_{1}, L) = -\frac{1}{2(L+1)} (-1)^{2l_{1} + \frac{1}{2} + j_{1} + \frac{1}{2}} \frac{1}{4} \int \frac{d(\hat{x} \cdot \hat{y})}{V(\langle |x|, |y|, \hat{x} \cdot \hat{y} \rangle)} \]

\[ \times \int_{-1}^{1} |d(\hat{x} \cdot \hat{y})| P_{\lambda}(\langle |x|, |y|, \hat{x} \cdot \hat{y} \rangle) \]

\[ \times \langle l_{1} 0, l_{2} 0 |L 0 \rangle \langle l_{3} 0, l_{4} 0 |L 0 \rangle \left\{ \begin{array}{c} j_{1} \\l_{1} \\j_{2} \\l_{2} \\ j_{3} \\l_{3} \\j_{4} \\l_{4} \end{array} \right\} , \]

(9)

3. A recursive procedure to include higher orbitals

In the following two subsections we will treat independently the kinetic and the potential part of the Hamiltonian. Each one is simple in an appropriate basis, but unfortunately, there is no simple basis that diagonalizes both of them simultaneously.

3.1. The kinetic part

Coupling the orbital angular momentum with the spin allows us to use Eq. (4) as new fermion creation operators (the same for the annihilation operators). If \( l = 0 \) we will use occasionally for \( b^{\dagger} \) the notation \( s^{l} \) for \( l \) equal to 1 (p-orbital) \( p^{l} \), for \( l \) equal to 2 (d-orbital) \( d^{l} \), etc.

Using as radial functions the three dimensional harmonic oscillator wave functions, the kinetic energy can be written in terms of the sum over the spin \( j \) and the oscillator quantum numbers \( N, N' \), each term in the sum being a sum of the following two operators

\[ K^{j}_{(N, l, N+1)} = -2\sqrt{2j+1} \]

\[ \times \left( \left[ b^{\dagger}_{N+1, \frac{1}{2}} \otimes b_{N, \frac{1}{2}} \right]^{000} + \left[ b^{\dagger}_{N, \frac{1}{2}} \otimes b_{N+1, \frac{1}{2}} \right]^{000} \right) \]

\[ K^{j}_{(N, l, N+1)} = -2\sqrt{2j+1} \]

\[ \times \left( -\left[ b^{\dagger}_{N+1, \frac{1}{2}} \otimes b_{N, \frac{1}{2}} \right]^{000} + \left[ b^{\dagger}_{N, \frac{1}{2}} \otimes b_{N+1, \frac{1}{2}} \right]^{000} \right) \]

(10)

where the spin-color-flavor coupling is indicated by "\( \otimes \)". The first index in the creation and annihilation operators refers to the pseudo-spin, (1/2) for the upper level and (-1/2) for the lower level. When \( l = 1 \), quarks in the s-level are combined with quarks in the p-level, coupled to total spin \( j=1 \). When \( l = 2 \), quarks in the p-level, each coupled to total spin \( j=3 \), are combined with quarks in the d-level, also coupled to total spin \( 3/2 \). The spin values \( j \) are given by \( l \pm 1/2 \). The radial number, \( N \), starts from \( j + 1/2 \) and acquires odd values, while \( N' \) has even values only and satisfies \( N' = N - 1, N + 1 \). This selection rule for radial quantum numbers is true only for the harmonic oscillator. The \( K \)-operators satisfy the commutation relations

\[ \left[ K^{j}_{(N, l, N+1)} + 1, K^{j}_{(N', l, N+1)} - 1 \right] = 2K^{2j}_{(N, l, N+1)} \]

\[ \left[ K^{j}_{(N, l, N+1)} + 1, K^{j}_{(N', l, N+1)} + 1 \right] = \pm K^{2j}_{(N, l, N+1)} \]

(11)

i.e., for each combination of \( N, N' \) and \( l \) we get a \( SU^{j}_{NN'} \) pseudo-spin group. Further, operators with different spins commute. The operator \( K^{j}_{(l, l+1)} \) is given by

\[ K^{j}_{(l, l+1)} = \frac{1}{2} \left( \left( N^{N+1, j} + N^{N+1, j} \right) \right) \]

(12)

It gives half of the difference between the number of quarks in the upper and lower level. The quarks are coupled to total spin \( j \) and are either in the \( (N') \) or \( (N + 1) \) level. Note that the pseudo-spin group defined by the \( K \)-operators is different from the pseudo-spin group in the original definition, which has been given with reference to lowering and raising operators within the same orbital. Since the \( K \)-operators raise and lower quarks from one orbital to a different one.

Figure 1. Schematic representation of the s and p orbitals at positive and negative energy. The upper levels are indicated by the pseudo-spin projection \( \frac{1}{2} \) while the lower levels are indicated by the pseudo-spin projection \(-\frac{1}{2}\).

Figure 2. Illustration of the structure of the kinetic energy. The operators, which appear in the kinetic energy, can be divided into columns of operators, related to a given spin \( j \), which commute with those of different columns. Only the positive energy states are plotted.

The just discussed structure is illustrated in Fig. 2. There is one column at the far left, which connects s-orbital states with spin 1/2 to p-orbital states with the same spin. However, the combination may involve different quantum numbers. For example, the s-orbital with \( N = 0 \) is combined with the p-orbital with \( N = 1 \) to form a pseudo-spin algebra. Analogous coupling scheme occurs for the s-orbital with \( N = 2 \) which is combined with two p-orbitals with \( N = 1 \) or \( N = 3 \), etc. A similar structure arises for the other states with spin 3/2, 5/2, etc. It is worth mentioning that all operators for a given spin commute with operators with a different spin, which implies a division into separate columns as given in Fig. 2. The commutation relations given by Eq. (11) imply a division of the kinetic part of the Hamiltonian into pseudo-spin operators acting independently on different parts of the pseudo-spin space, i.e., one part is acting in the pseudo-spin space given by the \((s, p)\) \( j = \frac{1}{2} \) orbital combination, a second one acts only on the \((p, d)\) \( j = \frac{3}{2} \) orbital combination, etc.

With these definitions the general structure of the kinetic energy part of the Hamiltonian is given by

\[
\sum_{j=\frac{1}{2}}^{\infty} \sum_{N=j+\frac{1}{2}}^{N'=N-1, N+1} c_{NN'}^j \left( K_{+}^{N'j} + K_{-}^{N'j} \right),
\]

with even steps \((\Delta N = 2)\) in \( N \). This expression shows a clear division between different spin values. Because of the pseudo-spin algebra of Eq. (11), which allows an analytical solution for a fixed \( N, N' \) and \( l \), we suspect that there is a good chance also to find an analytically solution for the spectrum of the kinetic energy for any number of levels. This has to be proven yet. For example, an analytic result can be easily obtained when we restrict to the s-orbital with \( N = 0 \) and the p-orbital with \( N = 1 \). The kinetic energy is then of the form

\[
c_1 \left( K_+^{\frac{1}{2}} + K_-^{\frac{1}{2}} \right) = 2c_1 K_+^{\frac{1}{2}},
\]

where we skipped the indices referring to the number of oscillation quanta. The sum of the two operators is proportional to the \( x \)-component of the \( SU(2) \) algebra. This can be diagonalized immediately, giving

\[
2c_1 M_j,
\]

where the index \( j \) tracks the spin and \( M_j \) is the pseudo-spin related to the algebra. Unfortunately, in order to diagonalize the whole Hamiltonian, we have also to take into account the potential contribution of the Hamiltonian, which will we discuss in the next section.

3.2. The potential part

In order to obtain a simple representation for the interaction Hamiltonian we assume that the potential is a constant in space. This is justified as long as we are concerned with the long range part of the interactions which drives confinement. This implies that the largest contributions come from small
relative momenta between color-charges and by choosing a
constant interaction we approximate them by a \( \delta \)-function in
momentum space. For a constant potential in Eq. (9) the
integral over \( \theta \) gives a contribution proportional to \( \delta_{L0} \).
Furthermore, the integrals over the \( x \) and \( y \) factorize, resulting in a
particularly simple form,

\[
V(N_i l_j, j_i, L = 0) = -\frac{V_0}{2} \delta_{L0} \\
\times \delta_{L_1 L_2} \delta_{j_3 j_1} \delta_{L_3 L_4} \delta_{N_1 N_2} \delta_{N_3 N_4}.
\tag{16}
\]

For the potential term in the Hamiltonian we eventually find,

\[
V = \frac{V_0}{2} \sum_{j_1, j_2} [S(j_1) \cdot S(j_2)],
\tag{17}
\]

which is nothing but the total color operator! Here \( S \) are the
generators of color charge for quarks in spin-\( j \) orbitals. Ex-
plicitly, these operators are given by,

\[
S(j) = S_{(0, c, 0)}(j),
\]

\[
= 2\sqrt{2} \sum_{N l} [b_{(N l j)}^{+} \otimes b_{(N l j)}]_{0, 0}^{1, 0},
\tag{18}
\]

where the operators are coupled to color one, pseudo-spin,
spin and flavor zero. Each term in Eq. (18) represents a color
generator for a given \( N, l \) and \( j \).

4. Strategies to continue

Because the \( K \) operators commute with the \( S(J) \) operators,
the main problem is how to diagonalize the kinetic energy.
This is true for a constant potential. However, if deviations
from a constant potential are taken into account, more gen-
eral interactions arise. Then, the strategy on how to obtain
the eigenvalues has to be changed.

We have considered three different approaches:

4.1. Complete treatment

Using only the \( s \frac{1}{2} \) and \( p \frac{3}{2} \) orbitals, counting the number of
states which can be connected by the kinetic and potential
part, we reach dimensions of the order of 80000, for the case
of spin-color-flavor zero! This is still a relative small space,
comparing it to shell model calculations. Still, it is not very
attractive. Nevertheless, we are currently designing a FOR-
TRAN routine, using the Lanczos method to diagonalize such
a large matrix. The main motivation is that the matrix is finite
(no cut-off involved) and, thus, the used Hamiltonian can be
exactly diagonalized, no matter how large the interaction pa-
rameters are. The interaction parameters depend on the size
of the finite volume. Increasing them, will modify the cur-
rently used interaction parameters. When applying approxi-
mate methods, this exact diagonalization gives us thus a han-
dle to compare the results with the exact ones.

4.2. Applying many-body approximation

The methods most relevant for this problem are the BCS and
RPA approximation. For the BCS treatment we have selected
all terms which are quadratic in the creation and annihilation
operators and solved the Hamiltonian within an BCS ansatz.
The remaining terms, up to fourth order in the creation and
annihilation operators, will be treated within the RPA ap-
proximation. Up to now, we have got the BCS solution, which
will be discused in a forthcoming publication.

4.3. Map to another effective Hamiltonian

The idea is to approximate the potential interaction in average
by a color-spin operator, as done in Sec. 3 for the potential
interaction. For example, considering only the \( j = 1/2 \) sector,
containing the

\[
S^\uparrow_{\frac{1}{2}, \lambda c f} p^\uparrow_{(l=1 \frac{1}{2})} \lambda c f
\]

fermion creation (and annihilation) operators, the total color-
spin operator is given by

\[
S^2(s_{\frac{1}{2}} p_{\frac{1}{2}}) = \left( S^2(s_{\frac{1}{2}}) + S^2(p_{\frac{1}{2}}) \right)^{\frac{1}{2}}.
\tag{19}
\]

The effective Hamiltonian \( H_{sp} \) is then given by

\[
H_{s\frac{1}{2} p_{\frac{1}{2}}} = c_1 \left( K\uparrow_{\frac{1}{2}} + K\downarrow_{\frac{1}{2}} \right) + d_1 S^2(s_{\frac{1}{2}} p_{\frac{1}{2}})
\]

\[
= 2c_1 K\uparrow_{\frac{1}{2}} + d_1 S^2(s_{\frac{1}{2}} p_{\frac{1}{2}}).
\tag{20}
\]

This Hamiltonian can be diagonalized analytically, with the
eigenvalue

\[
E = 2c_1 M\uparrow_{\frac{1}{2}} + d_1 S(s_{\frac{1}{2}} p_{\frac{1}{2}}) (S(s_{\frac{1}{2}} p_{\frac{1}{2}}) + 1).
\tag{21}
\]

Thus in this approximation it is possible to include the first p-orbital and still have an analytical solution. In Eq. (21), the \( S(s_{\frac{1}{2}} p_{\frac{1}{2}}) \) is the total color of the sector with spin \( j = 1/2 \).
The kinetic energy is diagonalized changing the basis where
\( K\uparrow \) is diagonal to the one where \( K\downarrow \) is diagonal. The index
\( J \) denotes the pseudo-spin in this basis. This diagonalization
does not affect the color part because they act on a differ-
space. The \( K \) operators are coupled to total color spin zero
and, thus, commute with the color operators.

Due to the last term in Eq. (21), particles with a total color
different from zero are lifted to larger energies, the same was
observed in Ref. [6]. The difference with [6] is that even if
\( S(s_{\frac{1}{2}}) \) is different form zero, the total color spin may be still
zero.

Further trivial analytic solutions are found by adding
terms to (20) of the form

\[
c_j \left( K\uparrow_{\frac{1}{2}} + K\downarrow_{\frac{1}{2}} \right) + d_j S^2 \left( l_1 = j - \frac{1}{2}, \frac{1}{2} \right) \left( l_2 = j + \frac{1}{2}, \frac{1}{2} \right)
\]

\[
= 2c_j K\uparrow_{\frac{1}{2}} + d_j S^2 \left( l_1 = j - \frac{1}{2}, \frac{1}{2} \right) \left( l_2 = j + \frac{1}{2}, \frac{1}{2} \right),
\tag{22}
\]

with $j = 3/2, 5/2$, etc. (For example, for $j = 1/2$ we have $l_1 = 0$, the $s_1$ orbital, and $l_2 = 1$, the $p_1$ orbital, which is the case studied in Eq. (20)). Using the abbreviation

$$S_j = S \left( \left( l_1 = j - \frac{1}{2} \right)_j \left( l_2 = j + \frac{1}{2} \right)_j \right),$$

the solution for each term is given by

$$2c_j M^j_j + d_j S_j(S_j + 1).$$

(23)

Each $j$ sector is diagonalized independently because the $K$ operators for different $j$ commute and also the total color spin operators between two different sectors.

5. Conclusions

In this contribution we reported on the current status of an algebraic approach to the low energy QCD. In a former contribution [6] we restricted analysis to the $s$-orbital level only. In this contribution we have extended the description to an arbitrary large combination of orbital levels. We analyzed the structure of the kinetic and potential interaction, obtaining interesting results which may be of relevance for finding an analytic solution.

At present, however, the Hamiltonian needs to be diagonalized numerically. Unfortunately, the spaces to be considered turn out to be quite large, though, manageable. Nevertheless, we are still looking for more attractive methods to treat the problem. A few have been discussed briefly and are currently under investigation.

In Sec. 4.3 we have obtained an analytical solution, involving several orbitals. This shows to us that there is possibly a promising way to treat QCD at low energy which does not need the time consuming treatment of Lattice Gauge calculations and is even more transparent, giving hope not only to treat non-perturbatively the QCD but also to extract more information in a relatively easy manner.

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