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Quantum damped nonstationary oscillator and Dynamical Casimir Effect

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I consider the model of quantum damped oscillator with time-dependent frequency and damping coefficients, which can be used to predict the results of current experiments aimed at the discovering the so-called Dynamical Casimir Effect. It is based on the Heisenberg–Langevin approach to the description of open quantum systems. The main emphasis is made on the comparison of the exactly solvable special cases with general approximate solutions found earlier. The examples considered demonstrate that simple approximate formulas are quite reliable for the analysis of realistic experimental situations.

Keywords: Dynamical Casimir effect; quantum damped oscillator; Heisenberg–Langevin equations; time-dependent frequency; parametric resonance.

Estudio el modelo de un oscilador cuántico amortiguado con frecuencia dependiente del tiempo y coeficientes de amortiguamiento, que puede ser utilizado para predecir resultados en experimentos que se están llevando a cabo y que conciernen al descubrimiento del llamado efecto Casimir dinámico. Este estudio se basa en el enfoque de Heisenberg–Langevin para la descripción de sistemas cuánticos abiertos. Se da especial énfasis a la comparación entre los casos especiales que son exactamente solubles con las soluciones aproximadas generales encontradas con anterioridad. Los ejemplos considerados demuestran que fórmulas simples aproximadas son bastante confiables para el análisis de situaciones experimentalmente factibles.

Descriptores: Efecto Casimir dinámico; oscilador cuántico amortiguado; ecuaciones de Heisenberg–Langevin; frecuencia dependiente del tiempo; resonancia paramétrica.

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1. Introduction

The term *Dynamical Casimir Effect* (DCE) [1,2] is used nowadays for the plethora of phenomena connected with the photon generation from vacuum due to fast changes of the geometry (in particular, the positions of some boundaries) or material properties of electrically neutral macroscopic or mesoscopic objects. The most recent reviews on the subject can be found in Ref. 3 and 4.

A rough qualitative explanation of such phenomena is the parametric amplification of quantum fluctuations of the electromagnetic (EM) field. Therefore the simplest model which can be used to describe at least some features of the DCE is that of the quantum harmonic oscillator(s) (representing the selected field mode(s) in a cavity) with time-dependent parameters (frequencies or coupling coefficients). It was considered in the frameworks of different approaches in many studies [5–8], where it was shown that a measurable number of quanta could be indeed created from the initial vacuum state of the EM field, provided one could move the cavity walls with a high frequency (twice the frequency of the field mode) and big amplitude.

However, exciting high amplitude oscillations of real boundaries turned out to be an extremely difficult task. For this reason, proposals based on the *imitation* of their motion attracted more and more attention with the course of time [1,9,10]. The key idea of the experiment named “MIR” [11,12] is to imitate the motion of a boundary, using an effective “plasma mirror” formed by *real electron–hole pairs* in a thin film near the surface of a semiconductor slab, il-

luminated by a periodical sequence of short laser pulses. If the interval between pulses exceeds the recombination time of carriers in the semiconductor, a highly conducting layer will periodically appear and disappear on the surface of the slab. This can be interpreted as periodical displacements of the boundary. The abbreviation MIR can be understood as “Mirror-Induced Radiation” or “Motion-Induced Radiation,” in accordance with the papers [13,14] where these names were introduced.

The main advantage of the semiconductor mirror is a great increase of the amplitude of variations of the instantaneous cavity eigenfrequency, compared with the case of real vibrating surface. This amplitude is determined mainly by the thickness of the semiconductor slab. Using the slabs of a few millimeters thickness one can easily obtain the frequency variation amplitude $\Delta\omega \sim 10^7 \text{ s}^{-1}$ or even bigger in the GHz range of the cavity resonance frequencies. Then the total excitation time can be reduced to less than $1 \mu\text{s}$ and the required cavity quality factor can be lowered to the easily achievable values of the order of 10^4 .

However, using the semiconductor mirror in the DCE experiments one has to overcome several serious difficulties, resulting from the fact that laser pulses create pairs of *real* carriers which change mainly the *imaginary part* $\epsilon_2 \equiv 4\pi\sigma/\omega$ of the complex dielectric permittivity $\epsilon = \epsilon_1 + i\epsilon_2$. Here σ is the conductivity in the CGS system of units. As a consequence, the “instantaneous” time-dependent resonance frequency becomes complex-valued function $\Omega(t) = \omega(t) - i\gamma(t)$. This means the appearance of inevitable intrinsic losses inside the

semiconductor slab, which can significantly deteriorate the effect of parametric amplification. The model which takes into account these losses was suggested in Ref. 15 to 17.

The aim of this paper is to apply the model to the case of the initial high-amplitude *classical field* inside the cavity. This problem was not considered earlier, but it is important from the experimental point of view, since the first step of the experiments is to find the conditions under which the regime of parametric amplification of a classical signal can be reached. Only after these conditions will be found, people can move to the truly quantum regime of excitation.

The plan of the paper is as follows. The general features of the model, its justification and predictions are considered in Sec. 2. Some exactly solvable special cases which clarify the meaning and validity of general approximate formulas used to calculate the amplification coefficients are considered in Sec. 3. The last section contains conclusions.

2. The model and its consequences

2.1. Main assumptions

The theoretical model developed in Ref. 15 to 17 is based on the assumption that the main features of the phenomena under study can be understood in the frameworks of a simple model of a quantum harmonic oscillator with a time dependent frequency, which is determined by the instantaneous geometry of the cavity and the instantaneous values of different material parameters characterizing the cavity. This oscillator describes the selected mode of the electromagnetic (EM) field in the cavity. The influence of other field modes is neglected, because it is expected that only the selected one (which is supposed to be in resonance with external perturbations) can be excited after many periods of oscillations in the process of parametric amplification. The main problem is how to take into account the effects of dissipation whose strength rapidly varies with time.

At first glance, it could seem attractive to try to find some Hamiltonian, which would result in the given classical equations of motion, and then to solve the Schrödinger equation with this Hamiltonian. The simplest possibility is to use a Hamiltonian of the form

$$\hat{H}(t) = \frac{1}{2} [\mu(t)\hat{p}^2 + \nu(t)\hat{x}^2 + \rho(t) (\hat{x}\hat{p} + \hat{p}\hat{x})], \quad (1)$$

generalizing the so called Caldirola–Kanai model [18,19] (which corresponds to $\mu(t) = \exp(-2\gamma t) = \nu^{-1}(t)$ and $\rho(t) = 0$ in the case of $\gamma = \text{const}$). However, this approach suffers from many drawbacks. The Caldirola–Kanai Hamiltonian depends on time explicitly even when $\gamma = \text{const}$, so it correspond sooner to the system with time-dependent mass than to the genius dissipative system. Moreover, the problem of finding the Hamiltonian for the given equations of motion has no unique solution, and practically all such Hamiltonians have some defects from the point of view of physics [20,21]. Therefore, although the Schrödinger equation with

Hamiltonian (1) can be solved exactly, and the solutions have sometimes nice properties [22,23], this model hardly can be used for the description of the DCE in real nonideal cavities. A detailed analysis of different models of damped quantum oscillator can be found in Ref. 24.

Actually, the main physical defect of the Caldirola–Kanai approach is that it implies that the quantum state of the system remains *pure* during the evolution, whereas the dissipation is always connected with the loss of quantum purity. Thus one has to describe the system in terms of the density matrix or its equivalent forms, such the Wigner function,

I suppose that this can be done, at least approximately, in the frameworks of the following Heisenberg–Langevin equations for the operators $\hat{x}(t)$ and $\hat{p}(t)$, describing the two quadrature components of the field mode:

$$d\hat{x}/dt = \hat{p} - \gamma_x(t)\hat{x} + \hat{F}_x(t), \quad (2)$$

$$d\hat{p}/dt = -\gamma_p(t)\hat{p} - \omega^2(t)\hat{x} + \hat{F}_p(t). \quad (3)$$

Two noncommuting noise operators $\hat{F}_x(t)$ and $\hat{F}_p(t)$ (with zero mean values) are necessary to preserve the canonical commutator $[\hat{x}(t), \hat{p}(t)] = i$ (the variables are supposed to be normalized in such a way that formally $\hbar = 1$). The damping coefficients $\gamma_x(t)$ and $\gamma_p(t)$ should be deduced from some microscopic model, which could take into account the coupling of the field mode with electron–hole pairs inside the semiconductor slab, the coupling of electrons and holes with phonons or other quasiparticles, responsible for the damping mechanisms, and the time dependence of the number of carriers, which disappear after a short recombination time. Unfortunately, no such detailed study has been performed until now. A simplified model was considered recently in Ref. 25, where the real dissipative system was replaced by a set of harmonic oscillators (*bosonic* reservoir) and the real interactions were replaced by an effective quadratic bilinear bosonic Hamiltonian of the most general form with time-dependent coefficients. It was shown that in the limit case of very short interaction time (much smaller than the period of free oscillations T_0) the coefficients $\gamma_x(t)$ and $\gamma_p(t)$ must coincide: $\gamma_x(t) = \gamma_p(t) = \gamma_s(t)$. On the other hand, the same result is well-known in the opposite limit case when the time of interaction with an environment is much bigger than the period of oscillations in the selected mode [26]. Such an interaction describes the losses in the cavity walls responsible for the finite quality factor $Q = \omega_0/(2\gamma_c)$ of the cavity. Therefore I consider here the model with $\gamma_x(t) = \gamma_p(t) = \gamma(t) = \gamma_c + \gamma_s(t)$, where $\gamma_c = \text{const}$ takes into account the losses in the cavity walls while the time-dependent function $\gamma_s(t)$ describes additional losses inside the laser-excited semiconductor slab.

2.2. Parametric amplification of the classical signal

Let us suppose that the field mode was initially in the *classical* state. In such a case, quantum fluctuations can be considered as negligible. This means that one can put formally

$\hat{F}_x(t) = \hat{F}_p(t) = 0$ for analyzing the conditions of the parametric excitation. Then the linear operator Eqs. (2) and (3) can be replaced by the similar equations for the quantum-mechanical average values by removing the symbols of operators (carets). Using the notation $E(t) \equiv \langle \hat{x}(t) \rangle$ for one of the quadrature components of the classical RF-signal [radio-frequency], one can find the exact solution of Eqs. (2) and (3) with the initial condition

$$E_{in}(t) = E_0 \cos(\omega_0 t - \vartheta)$$

in the form

$$E(t) = E_0 e^{-\Gamma(t)} \operatorname{Re}(\varepsilon(t) e^{i\vartheta}), \quad (4)$$

where

$$\Gamma(t) = \int_0^t \gamma(\tau) d\tau \quad (5)$$

and the complex function $\varepsilon(t)$ satisfies the following differential equation and initial condition:

$$\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0, \quad \varepsilon(t < 0) = \exp(-i\omega_0 t). \quad (6)$$

Note that function $\omega(t)$ in Eq. (6) is the same as in (3), so that it does not depend on the friction coefficient $\gamma(t)$. This is the consequence of the equality $\gamma_x(t) = \gamma_p(t) = \gamma(t)$.

In the case of the MIR experiment function $\omega(t)$ has the form of *periodic* pulses with the periodicity T (so that the k th pulse begins at $t_k = (k-1)T$, $t_1 = 0$), separated by intervals of time with $\omega = \omega_0$. Therefore the function $\varepsilon(t)$ can be written as

$$\varepsilon_k(t) = a_k e^{-i\omega_0 t} + b_k e^{i\omega_0 t} \quad (7)$$

in the intervals between the end of the k th pulse and beginning of the $(k+1)$ th pulse. The sets of the nearest constant coefficients (a_{k-1}, b_{k-1}) and (a_k, b_k) are related by means of a linear transformation

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = M_k \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}. \quad (8)$$

Each matrix M_k has the form

$$M_k = \begin{vmatrix} f_k & g_k^* \\ g_k & f_k^* \end{vmatrix}, \quad (9)$$

where complex coefficients f_k and g_k satisfy the identity

$$|f_k|^2 - |g_k|^2 = \det M_k \equiv 1. \quad (10)$$

If matrix M_1 with the coefficients $f_1 \equiv f$ and $g_1 \equiv g$ corresponds to the pulse starting at the moment $t_1 = 0$, then matrix M_k related to the identical pulse starting at the moment t_k can be written as

$$M_k = \Phi_k^\dagger M_1 \Phi_k = \begin{vmatrix} f & g^* e^{-2it_k} \\ g e^{2it_k} & f^* \end{vmatrix} \quad (11)$$

where matrix

$$\Phi_k \equiv \begin{vmatrix} \exp(-i\omega_0 t_k) & 0 \\ 0 & \exp(i\omega_0 t_k) \end{vmatrix}. \quad (12)$$

describes the influence of the time shift between pulses.

After n identical pulses with the periodicity T one obtains

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \mathcal{M}_n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad (13)$$

with

$$\mathcal{M}_n = M_n M_{n-1} \cdots M_1 = \Phi^{\dagger n} (\Phi M_1)^n \quad (14)$$

and

$$\Phi \equiv \begin{vmatrix} \exp(-i\omega_0 T) & 0 \\ 0 & \exp(i\omega_0 T) \end{vmatrix}.$$

Since $\det(\Phi M_1) = 1$, one can use the known formula for the powers of any two-dimensional unimodular matrix S [27],

$$S^n = U_{n-1}(z)S - U_{n-2}(z)I, \quad z \equiv \frac{1}{2} \operatorname{Tr} S, \quad (15)$$

where I means the unit matrix and $U_n(z)$ is the Tchebyshev polynomial of the second kind. In the case involved,

$$z = \frac{1}{2} \operatorname{Tr}(\Phi M_1) = \operatorname{Re}[f \exp(-i\omega_0 T)]. \quad (16)$$

Consequently, the amplification of the initial signal can happen if only $|z| > 1$. For small variations of the frequency $\omega(t)$ the complex coefficient f is close to unity, whereas the factor $\exp(-i\omega_0 T)$ is close to -1 if $T \approx T_0/2$. Therefore it is convenient to introduce the parametrization

$$\frac{1}{2} \operatorname{Tr}(\Phi M_1) = \operatorname{Re}[f \exp(-i\omega_0 T)] = -\cosh(\nu). \quad (17)$$

Using (13)-(17) with the initial conditions $a_0 = 1$ and $b_0 = 0$ one can obtain the following expressions for a_n and b_n :

$$a_n = (-1)^{n-1} \left[f \frac{\sinh(n\nu)}{\sinh(\nu)} e^{i\omega_0 T(n-1)} + \frac{\sinh[(n-1)\nu]}{\sinh(\nu)} e^{i\omega_0 T n} \right], \quad (18)$$

$$b_n = (-1)^{n-1} g \frac{\sinh(n\nu)}{\sinh(\nu)} e^{-i\omega_0 T(n-1)}. \quad (19)$$

Writing $f = |f| \exp(i\varphi)$ and $g = |g| \exp(i\phi)$ one has

$$\cosh(\nu) = |f| \cos(\delta), \quad \delta = \omega_0 (T - T_{res}), \quad (20)$$

where

$$T_{res} = \frac{1}{2} T_0 (1 + \varphi/\pi). \quad (21)$$

is the resonance periodicity of pulses, for which the maximal parametric amplification effect can be reached. In terms of the frequencies one can write (neglecting corrections of the second order with respect to the small parameter φ)

$$w_{res} = 2\omega_0 (1 - \varphi/\pi), \quad \delta = \pi \frac{w_{res} - w}{2\omega_0}. \quad (22)$$

Hereafter $w = \omega/(2\pi)$ is the frequency expressed in Hz. Note that the resonance frequency must be different from the value $2w_0$.

For small variations of the frequency the coefficients $|\varphi|$ and $|g|$ are also small. Therefore one can use the approximate formula $|f| = \sqrt{1+|g|^2} \approx 1 + |g|^2/2$. Moreover, since the reasonable values of the detuning from the strict resonance are also small ($|\delta| \ll 1$), one can replace $\sin(\delta)$ by δ . Consequently, in view of (20) a good approximation for the parameter ν is

$$\nu = \sqrt{|g|^2 - \delta^2}. \quad (23)$$

The time dependent frequency can be written in the case involved as $\omega(t) \equiv \omega_0[1 + \chi(t)]$ with $|\chi(t)| \ll 1$. Then the following approximate formulas can be obtained for the parameters g and φ (see [28] for details):

$$g \approx i\omega_0 \int_0^{t_f} \chi(t)e^{-2i\omega_0 t} dt, \quad (24)$$

$$\varphi = -\omega_0 \int_0^{t_f} \chi(t) dt. \quad (25)$$

It is assumed that each pulse starts at the instant $t = 0$ and finishes at $t = t_f$. The immediate consequence of formulas (24) and (25) is the inequality $|g| \leq |\varphi|$, which must hold for any function $\chi(t)$ which does not change its sign (as it happens in the MIR experiment).

Using Eqs. (4), (7), (18) and (19) one can arrive at the following formula for the field component after n pulses (*i.e.*, for $t > nT$):

$$E_n(t) = D_n(t) \left\{ \cosh(n\nu) \cos[\psi_n(t)] - \frac{\delta}{\nu} \sinh(n\nu) \sin[\psi_n(t)] + \frac{|g|}{\nu} \sinh(n\nu) \cos[\psi_n(t) + \Psi] \right\}, \quad (26)$$

where

$$D_n(t) \equiv E_0 \exp[-\Gamma_n(t)], \quad (27)$$

$$\psi_n(t) = \omega_0 t - n\omega_0(T - T_0/2) - \vartheta, \quad (28)$$

$$\Psi = 2\vartheta + \phi + \omega_0(T - T_0/2). \quad (29)$$

The damping factor can be written as

$$\Gamma_n(t) = n(\Lambda + \gamma_c T) + \gamma_c(t - nT), \quad (30)$$

where

$$\Lambda = \int_0^{t_f} \gamma_s(\tau) d\tau. \quad (31)$$

Therefore the field component performs rapid oscillations at frequency ω_0 with the slowly varying (as function of the number of pulses n) amplitude

$$A_n = D_n(t) \left\{ 1 + \frac{2|g|^2}{\nu^2} \sinh^2(n\nu) \times \left[1 + \frac{\delta}{|g|} \sin(\Psi) \right] + \frac{|g|}{\nu} \sinh(2n\nu) \cos(\Psi) \right\}^{1/2}. \quad (32)$$

Depending on the phase Ψ this amplitude can assume values between

$$A_n^{(\pm)} = D_n(t) \left\{ 1 + \frac{2|g|^2}{\nu^2} \sinh^2(n\nu) \pm \frac{|g|}{\nu} \sinh(2n\nu) \sqrt{1 + \frac{\delta^2}{\nu^2} \tanh^2(n\nu)} \right\}^{1/2}.$$

In the case of strict resonance ($\delta = 0$ and $\nu = |g|$) one obtains

$$A_n^{(\pm)} = E_0 \exp(-\Gamma_n \pm |g|n). \quad (33)$$

Therefore the necessary condition of the parametric amplification is the fulfillment of the inequality

$$F \equiv |g| - \Lambda > 0. \quad (34)$$

Besides, for the nonzero detuning δ the amplification of the initial signal can be observed if only $|\delta| < |g|$, *i.e.*, if

$$|w - w_{res}| < \delta w_{res} = \frac{2}{\pi} |g| w_0. \quad (35)$$

Putting $w = 2w_0$ in Eq. (22) one obtains $|\delta| = |\varphi|$. Since $|\varphi| \geq |g|$ in the case involved, this means that in the MIR experiment the amplification at exactly the double frequency $2w_0$ is impossible: the frequency of laser pulses must be shifted from $2w_0$.

If $|\delta| > |g|$ (totally out of the resonance), then the hyperbolic functions in Eqs. (26) and (32) should be replaced by the trigonometric ones. In this case the amplitude of oscillations A_n decays with time (or the number of pulses n), but this decay also shows some oscillating structure (as function of n). Only for a very big detuning, $|\delta| \gg |g|$, the oscillations in the amplitude can be neglected, and the decay amplitude becomes strictly monotonous:

$$E_n(t) = E_0 e^{-\Gamma_n(t)} \cos[\psi_n(t) - n|\delta|].$$

3. Some special cases

One of the most important general results of the preceding section is contained in Eq. (22), which shows that the frequency of laser pulses in the MIR experiment must be shifted from the double frequency of the field mode, otherwise the parametric amplification cannot be achieved. This result seems strange at the first glance, since usually people believe that the best condition for the parametric resonance

is the frequency modulation exactly at the double frequency. This is explained by the fact that the standard case of the parametric resonance considered in textbooks (see, e.g., [29-31]) corresponds to the *harmonic* and *symmetrical* modulation of the eigenfrequency, whereas the function $\chi(t)$ in the realistic case of the MIR experiment is asymmetrical and strongly unharmonic. To see better the origin of differences, it seems reasonable to consider a few special cases admitting explicit exact or approximate solutions.

3.1. Harmonic asymmetrical variations of the frequency

Consider the function $\omega(t)$ of the form

$$\omega(t) = \omega_0 \{1 - \kappa + \kappa \cos[(2\omega_0 + \eta)t]\}, \quad |\kappa| \ll 1. \quad (36)$$

It preserves the sign of $\chi(t)$, as it happens in the MIR experiment. In accordance with the method of slowly varying amplitudes [29-31] one can seek for the solution to Eq. (6) in the form

$$e(t) = a(t) \exp[-i(\omega_0 + \delta)t] + b(t) \exp[+i(\omega_0 + \delta)t]$$

with *slowly varying* functions $a(t)$ and $b(t)$. The choice $2\delta = \eta$ results in the simple set of equations with constant coefficients

$$\begin{aligned} \dot{a} &= \frac{i}{2} [(\eta + 2\omega_0\kappa)a - \omega_0\kappa b], \\ \dot{b} &= -\frac{i}{2} [(\eta + 2\omega_0\kappa)b - \omega_0\kappa a]. \end{aligned}$$

Looking for solutions in the form $a = a_0 e^{\lambda t}$ and $b = b_0 e^{\lambda t}$ one gets

$$\lambda = \frac{1}{2} \sqrt{(\omega_0\kappa)^2 - (\eta + 2\omega_0\kappa)^2}.$$

Consequently, the maximal amplification can be achieved for the value of the frequency shift $\eta = -2\omega_0\kappa$, which is negative if $\kappa > 0$, i.e., if $\chi(t) < 0$. This result coincides exactly with formulas (22) and (25), provided the integration in (25) is performed over the period of the frequency variation: $t_f = 2\pi/(2\omega_0 + \eta) \approx \pi/\omega_0$. Note that the resonance frequency of variations of parameters $w_{\text{res}} = 2\omega_0(1 - \kappa)$ coincides exactly with twice the *average frequency* $\langle w(t) \rangle = \omega_0(1 - \kappa)$.

Formula (24) gives the value $|g| = \pi\kappa/2 = |\varphi|/2$, which results, due to (34), in the same maximal increment of the amplitude as obtained above:

$$e^{|g|n} = e^{|g|wt} = \exp\left(\frac{\pi}{2}\kappa \cdot 2 \cdot \frac{\omega_0}{2\pi} \cdot t\right) = e^{\kappa\omega_0 t/2}.$$

Taking $w = 2\omega_0$ one gets $|w - w_{\text{res}}| = 2\omega_0|\kappa|$. Since this value twice bigger than $\delta w_{\text{res}} = |\kappa|\omega_0$, no amplification can happen if $w = 2\omega_0$.

3.2. Rectangular pulses

The second example of the frequency variations is as follows: $\chi = \text{const} \neq 0$ for $0 < t < \tau$ and $\chi = 0$ for $\tau < t < T$,

repeating with the period T . The straightforward calculations can be performed *exactly* for any values of parameters. Simplifying them in the special case of $|\chi| \ll 1$ one can obtain the following formulas for the elements of matrix M_1 defined in (9) (only the leading terms with respect to $|\chi|$ are preserved):

$$g = i\chi e^{-i\omega_0\tau} \sin(\tau\omega_0), \quad (37)$$

$$f = e^{-i\omega_0\chi\tau} \left[1 + \frac{\chi^2}{2} \sin^2(\tau\omega_0) \right]. \quad (38)$$

Consequently,

$$\varphi = -\omega_0\chi\tau. \quad (39)$$

Formulas (40) and (42) *coincide exactly* (for $|\chi| \ll 1$) with those arising from the integrals (24) and (25). Also, one can see that $|g| = |\chi \sin(\tau\omega_0)| \leq |\omega_0\chi\tau| = |\varphi|$ and $|g| = |\varphi|$ for short pulses, when $\tau\omega_0 \ll 1$.

3.3. A realistic model for short laser pulses

For the cavities used in the MIR experiment the functions $\chi(t)$ and $\gamma_s(t)$ can be approximated as follows [32]:

$$\chi(t) \approx \frac{\zeta_m A^2(t)}{A^2(t) + 1}, \quad \gamma_s(t) \approx \frac{\omega_0 |\zeta_m| A(t)}{A^2(t) + 1}. \quad (40)$$

Here ζ_m is the maximal frequency shift between the illuminated and non-illuminated cavities (it is roughly proportional to the thickness of the dielectric slab containing the semiconductor film on its surface) and $A(t) = A_0 \exp(-t/T_r) f(t)$, where T_r is the recombination time of photo-excited carriers in the semiconductor. The function $f(t)$ takes into account the finite duration of the laser pulse, properties of the semiconductor surface, etc. The dimensionless coefficient A_0 can be expressed as (in the CGS units)

$$A_0 = 2|eb|\varepsilon_b\kappa W K_c / (cE_g S),$$

where b is the mobility of carriers, e electron charge, c velocity of light, W the energy of a single laser pulse *absorbed in the semiconductor*, E_g the energy gap of the semiconductor, S the area of the illuminated semiconductor surface, κ the quantum efficiency, ε_b the dielectric constant of the non-illuminated semiconductor and K_c is the dimensionless coefficient determined by the geometry of the cavity (for very thin dielectric slabs it is also roughly proportional to the slab thickness).

Noticing that function $\chi(t)$ does not change its sign in the case of the MIR experiment one can conclude that the sign of φ is opposite to the sign of $\chi(t)$. This means [see Eq. (22)] that the sign of the difference $w_{\text{res}} - 2\omega_0$ *coincides* with the sign of coefficient ζ_m .

Simple analytical formulas for the coefficients φ and Λ can be obtained in the case of a good surface (having a small surface recombination rate) and short laser pulses (approximated by the delta-function of time), when $f(t) = 1$. Then the upper limits of integration in Eqs. (25) and (31) can be extended to infinity (since the recombination time must be

much smaller than the periodicity of laser pulses T). The results are as follows,

$$\Lambda = 2\pi\omega_0 T_r |\zeta_m| \tan^{-1}(A_0), \quad (41)$$

$$\varphi = -\pi\omega_0 T_r \zeta_m \ln(1 + A_0^2). \quad (42)$$

As was shown in Ref. 15, the amplification coefficient $F = |g| - \Lambda$ can be positive only for sufficiently big values of the parameter A_0 (at least bigger than 5). Therefore the most interesting case is $A_0 \gg 1$. Then Λ does not depend on A_0 (*i.e.*, on the laser pulse energy), whereas the dependence $\varphi(A_0)$ becomes rather weak:

$$\Lambda = \pi^2 \omega_0 T_r |\zeta_m|, \quad (43)$$

$$\varphi = -2\pi\omega_0 T_r \zeta_m \ln(A_0). \quad (44)$$

The integral (24) giving the coefficient g can also be calculated exactly for $f(t) = 1$ (provided the upper limit of integration is extended to infinity). The result can be expressed in terms of the Gauss hypergeometric function [17]. Simplifying the exact formula in the special case of $A_0 \gg 1$ and $T_r \ll T$ one can obtain the following simple expression:

$$g \approx i\zeta_m \sin(2\pi\omega_0 T_r \ln A_0) \exp(-2\pi i\omega_0 T_r \ln A_0). \quad (45)$$

Note that (44) and (45) coincide with formulas (39) and (37), derived for the rectangular pulse, if one defines the “effective duration” τ of the “equivalent rectangular pulse” as

$$\tau = T_r \ln(A_0). \quad (46)$$

The meaning of this effective duration is clear: this is the time necessary to diminish the function $A(t) = A_0 \exp(-t/T_r)$

from the initial value A_0 to the value $A(\tau) = 1$. The latter value is distinguished by two properties [see Eq. (40)]: when $A = 1$, then $|\chi(\tau)| = |\chi_{\max}|/2$ and the damping coefficient attains the maximal possible value $\gamma_{\max} = \omega_0 |\chi_{\max}|/2$.

Now let us see how formulas (44) and (45) can be derived from the exact solution of Eq. (6) with

$$\omega^2(t) = \omega_0^2 \left[1 + \frac{2\zeta_m A_0^2 \exp(-2\beta t)}{A_0^2 \exp(-2\beta t) + 1} \right]. \quad (47)$$

This time dependence is equivalent to (40) if $|\zeta_m| \ll 1$, $\beta = 1/T_r$ and $f(t) \equiv 1$, *i.e.*, for very short laser pulses (whose duration is much smaller than T_r). Function (47) belongs to the family of the so-called Epstein’s profiles [33].

Introducing the new variable $z = -A_0^2 \exp(-2\beta t)$ and looking for the solution in the form $\varepsilon(z) = z^d \psi(z)$, one can transform Eq. (6) to the form

$$z(1-z)\psi'' + (1+2d)(1-z)\psi' + 2d^2\zeta_m\psi = 0, \quad (48)$$

provided the coefficient d is chosen according to the relations

$$d^2 = -\alpha^2, \quad \alpha = \omega_0/(2\beta). \quad (49)$$

Comparing (48) with the canonical form of the hypergeometric equation

$$z(1-z)F'' + [c - (a+b+1)z]F' - abF = 0, \quad (50)$$

whose solution is the Gauss hypergeometric function $F(a, b; c; z)$, one can write the general solution as

$$\varepsilon(t) = f e^{-i\omega_0 t} F(-i\alpha\xi, i\alpha(2+\xi); 1+2i\alpha; -A_0^2 e^{-2\beta t}) + g e^{i\omega_0 t} F(i\alpha\xi, -i\alpha(2+\xi); 1-2i\alpha; -A_0^2 e^{-2\beta t}), \quad (51)$$

where

$$\xi = \sqrt{1 + 2\zeta_m} - 1. \quad (52)$$

The functions $e^{\mp i\omega_0 t}$ arise automatically in (51), because $z^d = (-A_0^2)^d \exp(\mp i\omega_0 t)$ if $d = \pm i\alpha$. Since $F(a, b; c; 0) = 1$, the constant coefficients f and g are exactly the same coefficients that form the transfer matrix M_1 for the one cycle of the process “laser excitation – recombination”, because $-A_0^2 e^{-2\beta t} \rightarrow 0$ and $\omega(t) \rightarrow \omega_0$ as $t \rightarrow \infty$. These

coefficients are determined by the initial conditions $\varepsilon(0) = 1$ and $\dot{\varepsilon}(0) = -i\omega_0$ (since we assume that $\varepsilon(t) = \exp(-i\omega_0 t)$ if $t < 0$). Remember that the amplification can happen only if $A_0 > 1$. For this reason, for small values of t it is necessary to transform the function (51) to an equivalent form, where the arguments of the hypergeometric functions are smaller than unity by their absolute values. This can be done with the aid of the known identity

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1}), \quad (53)$$

so that the equivalent expression is as follows:

$$\varepsilon(t) = u e^{-i\omega_0(1+\xi)t} F(-i\alpha\xi, -i\alpha(2+\xi); 1-2i\alpha(1+\xi); -A_0^{-2} e^{2\beta t}) + v e^{i\omega_0(1+\xi)t} F(i\alpha\xi, i\alpha(2+\xi); 1+2i\alpha(1+\xi); -A_0^{-2} e^{2\beta t}), \quad (54)$$

$$u = fC_1 + gC_2^*, \quad v = fC_2 + gC_1^*,$$

$$C_1 = \frac{\Gamma(2i\alpha[1+\xi])\Gamma(1+2i\alpha)}{\Gamma(i\alpha[2+\xi])\Gamma(1+i\alpha[2+\xi])} \exp[2i\alpha\xi \ln(A_0)],$$

$$C_2 = \frac{\Gamma(-2i\alpha[1+\xi])\Gamma(1+2i\alpha)}{\Gamma(-i\alpha\xi)\Gamma(1-i\alpha\xi)} \exp[-2i\alpha(2+\xi) \ln(A_0)].$$

Until this point, all formulas were exact, and exact expressions can be also found for the coefficients f and g in terms of several different hypergeometric functions and their derivatives taken at the point $z_0 = -A_0^{-2}$. However, these expressions are extremely cumbersome. On the other hand, in reality one has to deal with the values $A_0 \gg 1$ and $|\zeta_m| \ll 1$, so that $\xi \approx \zeta_m$ and $|\xi| \ll 1$. Moreover, the parameter α is small, too: for $T_r = 10$ ps and $w_0 = 2.5$ GHz (these are typical values [34,35]) one has $\alpha \approx 0.08$. Remembering the expansion

$$F(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2c(c+1)}z^2 + \dots$$

and using the evaluation $|abz_0/c| \sim 2\alpha^2\xi/A_0^2 < 10^{-6}$ for each hypergeometric function in (54), one can conclude that the hypergeometric functions in (54) can be replaced by the unity with a high accuracy, if $\beta t \ll 1$. Then the initial conditions yield

$$u = \frac{1 + \xi/2}{1 + \xi} \approx 1 - \xi/2, \quad v = \frac{\xi}{2(1 + \xi)} \approx \xi/2. \quad (55)$$

The coefficients C_1 and C_2 can be simplified with the aid of the known formulas for the Gamma functions, such as $\Gamma(1+z) = z\Gamma(z)$ and

$$\Gamma(a+x) \approx \Gamma(a)e^{x\psi(a)}, \quad \Gamma(x) \approx 1/x,$$

which hold for $x \rightarrow 0$. Here $\psi(z) = (d/dz) \ln[\Gamma(z)]$. The result is as follows:

$$C_1 = \frac{\exp[2i\alpha\xi \ln(A_0)]}{1 + \xi/2} [1 + \mathcal{O}(\xi^2)],$$

$$C_2 = \frac{\xi \exp[-2i\alpha(2+\xi) \ln(A_0)]}{2(1+\xi)} [1 + \mathcal{O}(\xi^2)].$$

Following this way, finally one can obtain the following expressions for the coefficients f and g up to the corrections of the order of ξ^2 :

$$f = e^{-2i\alpha\xi \ln(A_0)}, \quad (56)$$

$$g = i\xi \sin[2\alpha \ln(A_0)] e^{-2i\alpha \ln(A_0)}. \quad (57)$$

These expressions are equivalent to (44) and (45), since $2\alpha = 2\pi w_0 T_r$ and $\xi = \zeta_m$ for $|\zeta_m| \ll 1$.

4. Conclusions

The main result of this paper is formula (32), which shows how the amplitude of the initial classical monochromatic signal should change after n laser pulses. Fitting experimental data to this dependence, one can determine the values of different parameters characterizing the process. Other important formulas are (34) and (35): they give the conditions under which the parametric amplification is possible. It is worth emphasizing that the frequency of laser pulses illuminating the semiconductor target must be shifted (although slightly) from the double unperturbed eigenfrequency of the selected field mode. Moreover, under the conditions of the MIR experiment (when the function describing the instantaneous deviation of the cavity eigenfrequency from its initial value preserves its sign) the pulses following with the exact double frequency never can lead to the parametric amplification. This result is contained in Eqs. (22) and (35), and it is confirmed and clarified in three explicit examples considered in Sec. 3. Besides, these explicit examples demonstrate the reliability of the approximate formulas (24) and (25) for the parameters g and φ , whose values are crucial for the experiment.

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