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Dynamics of particles in corrugated waveguides classical description

A.J. Martínez-Mendoza, J.A. Méndez-Bermúdez, and G.A. Luna-Acosta

Instituto de Física, Universidad Autónoma de Puebla,
Apartado Postal J-48, Puebla 72570, Mexico.

N. Atenco-Analco
Escuela de Ciencias, Universidad Autónoma “Benito Juárez” de Oaxaca,
Av. Universidad S/N, Oaxaca 68120, Mexico.

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We construct a classical map to describe the dynamics of point particles moving inside a corrugated waveguide. By the use of Chirikov’s overlapping resonance criterion we are able to identify the onset of global chaos as a function of the geometrical parameters of the waveguide. Then, (i) in the regime of global chaos we derive an heuristic expression for the diffusion coefficient of the angle; and (ii) in the regime of mixed chaos we analyze the scaling of the angle dispersion.

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1. Introduction

Dynamical billiards have been a paradigm in the study of dynamical systems and quantum chaos since they capture all the complexity of Hamiltonian systems [1, 2]. The dynamics of billiards, which is entirely defined by their shape, can range from integrable to completely chaotic. Examples of integrable billiards are rectangles and ellipses while the stadium and the Sinai billiard are the most popular billiards developing full chaos [2]. Moreover, most billiards with smooth convex boundaries have a phase space consisting of KAM-islands merged into chaotic components, a situation known as mixed chaos [3].

There are several examples of billiards whose dynamics transits from integrable to chaotic as a function of their geometrical parameters. Among them, we can mention the quadrupole billiard [4], the limaçon billiard [5], and the cosine billiard [6]. In particular, the cosine billiard has found a prominent place in the quantum chaos literature [7] since it can be studied in its close [8], infinitely periodic [6, 9], and finite periodic [10] versions; having applications to quantum dots, two-dimensional crystals, and electron or electromagnetic waveguides, respectively. Moreover, a disordered-like realization of the cosine billiard has been introduced in Ref. 11 to study the influence of corrugation on the wave properties of guided electrons. However, the classical (or ray) properties of disorder-like corrugated waveguides has not been analyzed yet. So, in the present paper we undertake this task.

The organization of this paper is as follows. In the next Section we define the model of disordered-like corrugated waveguide and construct a classical map to describe the dynamics of point particles moving inside it. In Sec. 3 by the use of Chirikov’s overlapping resonance criterion we identify the onset of global chaos as a function of the geometrical parameters of the waveguide. Then, (i) in the regime of global chaos we derive an heuristic expression for the diffusion coefficient of the angle (Sec. 4); and (ii) in the regime of mixed chaos we analyze the scaling of the angle dispersion (Sec. 5). Finally, we draw our conclusions in Sec. 6.

2. Model and map

The model we use in our analysis is the corrugated waveguide shown in Fig. 1. It has two hard walls: one flat at \( y = 0 \) and a corrugated one given by the function \( y = L_y W \xi(x) \). \( L_y \) is the average width of the waveguide, \( W \) the corrugation amplitude, and

\[
\xi(x) = \sum_{n=1}^{N} A_n \cos(nx).
\]  

(1)

Here, \( A_n \) are random numbers distributed uniformly in the interval \([-A, A]\) and \( N \) drives the modulated boundary of the waveguide from smooth \((N \sim 1)\) to rough \((N \sim 100)\) [11]. \( A \) is chosen such that every realization of the waveguide fullfills \(|\xi(x)| \leq 1\); with this prescription we numerically found that \( \langle A \rangle \propto N^{-0.6} \). Examples of waveguide geometries may be found at Ref. 11.
We study the dynamics of the point particle colliding specularly with the walls of the corrugated waveguide of Fig. 1 via the Poincaré map $M$:

\[
(\theta_{n+1}, x_{n+1}) = M(\theta_n, x_n).
\]

Here, $\theta_n$ is the angle the particle’s trajectory makes with the $x$-axis just before the $n$th bounce with the modulated boundary at $x_n$. From Fig. 1 we see that given the initial condition $(\theta_0, x_0)$,

\[
\phi_0 = \arctan[y'(x_0)] = \arctan[W\xi'(x_0)].
\]

Our numerical algorithm can determine exactly the values of $x$ and $\theta$ for each subsequent collision and hence obtain the exact Poincare map $M$. However, in order to obtain analytical results we construct an approximate Poincare map as follows. Since the collisions are specular, we have $\beta_1 = 2\phi_0 - \theta_0$ and $\beta_2 = -\theta_1$. So,

\[
\theta_1 = \theta_0 - 2\phi_0 = \theta_0 - 2\arctan[W\xi'(x_0)].
\]

Assuming that there are no multiple collisions with the upper wall, then $x_0^* = x_0 + [L_y + W\xi(x_0)]\cot(\theta_1)$ and $x_1 = x_0^* + [L_y + W\xi(x_1)]\cot(\theta_1)$. Making the approximation $W\xi(x_1) = W\xi(x_0)$, expected to be valid for $W/L_y \ll 1$, and generalizing for all collisions, we obtain the map

\[
M : \begin{cases} 
\theta_{n+1} = \theta_n - 2\arctan[W\xi'(x_n)] \\
x_{n+1} = x_n + 2[L_y + W\xi(x_n)]\cot(\theta_{n+1})
\end{cases}
\]

(3)

Note that the approximate map $M$ is also area preserving

\[
|\partial(\theta_{n+1}, x_{n+1})/\partial(\theta_n, x_n)| = 1.
\]

To see the effect of ignoring multiple collisions in our approximate map Eq. (3), in Figs. 2 and 3 we plot $\theta_{n+1} - \theta_n$ vs. $x_n$.
using the exact map Poincaré map (circles) and the approximate map (solid curve). Points falling away from the curves represent multiple collisions with the modulated boundary. In both figures we use waveguides with \( N = 1, 25, 50, \) and 100; but while in Fig. 2 we set \( W/L_y = 0.001 \), in Fig. 3 we use \( W/L_y = 0.1 \). It is clear from these figures that for \( W/L_y \ll 1 \), as in Fig. 2, the approximate map \( M \) reproduces very well the exact dynamics.

We remark that maps similar to Eq. (3) were also derived in Refs. 6 and 12 for corrugated waveguides but only for the particular case of \( N = 1 \).

3. Onset of global chaos

The map \( M \) can be further simplified by imposing \( W \ll L_y \) and \( W \xi'(x) \ll 1 \). This allows us to write

\[
\theta_{n+1} \approx \theta_n - 2W \xi'(x_n)
\]

\[
x_{n+1} \approx x_n + 2L_y \cot(\theta_{n+1}).
\]

Following [3] we linearize the map of Eq. (4) around the period-one fixed point \( \theta_{n+1} = \theta_n = \theta^* \). This requires \( \cot(\theta^*) = 0 \). For an angle close to \( \theta^* \) we can write

\[
\theta_n = \theta^* + \Delta \theta_n \quad \text{obtaining}
\]

\[
\theta_{n+1} = \theta^* + \Delta \theta_{n+1} = \theta^* + \Delta \theta_n - 2W \xi'(x_n).
\]

Then,

\[
\Delta \theta_{n+1} = \Delta \theta_n - 2W \xi'(x_n).
\]

Now, for \( x \) we have

\[
x_{n+1} = x_n + 2L_y \cot(\theta^* + \Delta \theta_{n+1})
\]

\[
= x_n + 2L_y \cot(\theta^*) + 2L_y \cot'(\theta^*) \Delta \theta_{n+1}.
\]

But since \( \cot(\theta^*) = 0 \) and \( \cot'(\theta^*) = -1 \),

\[
x_{n+1} = x_n - 2L_y \Delta \theta_{n+1},
\]

Finally, by substituting the new angle \( \delta \theta_n \equiv -2d \Delta \theta_n \) in (5) and (6) we get the linearized map

\[
\delta \theta_{n+1} = \delta \theta_n + K f(x_n)
\]

\[
x_{n+1} = x_n + \delta \theta_{n+1},
\]

where \( K = 4WL_y \max[\xi'(x)] \), \( f(x) = \xi'(x)/\max[\xi'(x)] \), and \( \max[\xi'(x)] \) is the maximum value of \( \xi'(x) \); so that \( |f(x)| \leq 1 \) (7) is exactly the Standard map, where \( \delta \theta \) and \( x \)
Figure 4. $\langle \text{max}[\xi'(x)] \rangle$ as a function of $N$ (open symbols). The continuous line is $0.62N$, the best linear fit to the numerical data. The average of $\text{max}[\xi'(x)]$ was taken over 100 profile realizations, i.e. 100 different sequences of random numbers $A_n$ were used in Eq. (1).

Chirikov’s overlapping resonance criterion predicts that transition to global chaos occurs for $K > 1$ [3]. Global chaos means that chaotic regions are interconnected over the whole phase space (stability islands may still exist but are sufficiently small that the chaotic region extends throughout the vast majority of phase space). For our waveguide, this reads as

$$WL_y N > K_{\text{chaos}} \approx 0.4.$$  \hfill (8)

In Fig. 5 we present Poincaré maps from the exact dynamics of particles moving inside waveguides with $N = 1, 25, 50$ and 100. In all cases we set $WL_y N \approx 0.8 \approx 2K_{\text{chaos}}$. Only a single trajectory was used to construct these maps. We chose the modulated boundary of the waveguide as surface of section: each time a ray impinges on that wall, we plot the position $x$ and $\cos(\alpha)$, where $\alpha$ is the angle the ray makes with the tangent of the boundary at $x$.

We have observed that Eq. (8) works well as the condition for global chaos for our corrugated waveguides when $N > 10$, see for example Figs. 5(b-d). However, for $N \sim 1$ we concluded that $K_{\text{chaos}}$ is approximately 2. See for example Figs. 5(a), where the Poincaré map in the case $N = 1$ still shows stability islands for $WL_y N \approx 0.8$.

Figure 5. Poincaré plots for the exact dynamics of particles moving inside waveguides with (a) $N = 1, W/2\pi = 0.02$, (b) $N = 25, W/2\pi = 0.0008$, (c) $N = 50, W/2\pi = 0.0004$, and (d) $N = 100, W/2\pi = 0.0002$. $L_y = 2\pi$. In all cases $WL_y N \approx 0.8 \approx 2K_{\text{chaos}}$. 

By the substitution of map (3) in Eq. (7), the diffusion coefficient reads

\[ D = 4 \langle \arctan^2 |W\xi'(x)| \rangle_x, \]

where the average is taken over one period of \( \xi'(x) \). So,

\[ D = \frac{2}{\pi} \int_{0}^{2\pi} \arctan^2 |W\xi'(x)| dx \approx \frac{2W^2}{\pi} \int_{0}^{2\pi} |\xi'(x)|^2 dx. \]

The last approximation is valid when \( W\xi'(x) \ll 1 \).

Now, with the help of Eq. (1) we get

\[ D \approx 2W^2 \left( \sum_{n=1}^{N} (nA_n)^2 \right). \]

Since we numerically found that

\[ \sum_{n=1}^{N} (nA_n)^2 \approx 0.218N^{1.8}, \]

as can be seen in Fig. 6, we can finally write and expression for \( D \) as a function of \( W \) and \( N \):

\[ D(W, N) \approx 0.436W^2N^{1.8}. \] (10)

In Fig. 7 we plot \( D \) calculated from the dynamics generated by the map \( M \), Eq. (3), together with the expression for \( D(W, N) \) given in Eq. (10). It is clear from this figure that Eq. (10) gives a very good estimate of \( D \).

Notice that \( D \), as defined in Eq. (9), does not depend on \( L_y \). However, to calculate \( D \) in Fig. 6 we used \( WL_y N > K_{\text{chaos}} \) assuming \( L_y = 2\pi \).

5. Scaling of angle’s dispersion

In a recent series of papers, see a review in Ref. 14, it has been shown that generic two-dimensional area preserving maps exhibiting an integrability-to-chaos transition can be classified according to their scaling properties. In particular, the scaling properties of the cosine waveguide [12] (i.e. our corrugated waveguide with \( N = 1 \)) were found to be the same as those of Chirikov’s standard map [15]. It is the purpose of this Section to study the the scaling of the angle dispersion as the roughness in the waveguide is increased (i.e., as \( N \) increases).

We define the angle dispersion as

\[ \Omega(n, W) = \frac{1}{H} \sum_{j=1}^{H} \sqrt{\langle \theta_j^2(n, W) \rangle - \langle \theta_j(n, W) \rangle^2}, \] (11)

where

\[ \langle \theta_j(n, W) \rangle = \frac{1}{n} \sum_{j=1}^{n} \theta_j, \] (12)
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Figure 8. (a) $\Omega(n, W)$ as a function of $n$ for $N = 1$ and some values of $W$. Scaled $\Omega(n, W)$ for (b) $N = 1$, (c) $N = 50$, and (d) $N = 100$.

$H$ is the number of trajectories (or initial conditions) used to calculate $\Omega(n, W)$, and $n$ is the $n$th iteration of map (3). Below concentrate in the regime of small perturbations: $W \ll L_y$.

We do not want to reproduce here the comprehensive derivation of the scaling of $\Omega(n, W)$ for $N = 1$ made in Ref. 12, which in fact is applicable to our corrugated waveguide with arbitrary $N$. Instead we concentrate on the results and refer the reader to Ref. [12] for details.

In Fig. 8(a) we plot $\Omega(n, W)$ as a function of $n$ for $N = 1$ and some values of $W$ and in Fig. 8(b) the scaled data according to Refs. 12 and 15. Then in Figs. 8(c) and 8(d) we show the scaled $\Omega(n, W)$ for $N = 50$ and $N = 100$, respectively. With this result for $N \gg 1$ we confirm the hypothesis made in Ref. 15 on the universality of the scaling of Chirikov’s-like maps. The only effect produced by the increase of waveguide corrugation is a delay in the saturation of $\Omega(n, W)$, as seen in Fig. 8.

6. Conclusions

In this paper we study the classical, or ray, dynamics of point particles moving inside corrugated waveguides by means of a two-dimensional area preserving map.

The results presented above can be summarized in the following three points:

(i) By the use of Chirikov’s overlapping resonance criterion, applied to the linearized map, we are able to identify the onset of global chaos as a function of the geometrical parameters of the waveguide (mean width, corrugation amplitude and corrugation complexity).

(ii) In the regime of global chaos we derive an heuristic expression for the diffusion coefficient of the angle that depends on the corrugation amplitude and corrugation complexity only.

(iii) Our calculations are in agreement with the universality of the scaling of the angle dispersion, predicted for the Standard map.

We believe that our results might be relevant in the study of the dynamical and transport properties of the quantized version of our corrugated waveguide, which will be the subject of a future publication.

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