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# Markovian motion of beads in the Galton-Board

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The Galton's board is a periodic lattice made with fixed nails at its nodes, spherical grains travel through them due to gravity. We show the convenience of this system to present the main concepts of Markovian-stochastic trajectories during the motion of only one particle. In a special case, the Galton board was modified, a set of nails (20%) were removed randomly. Under such a change the main characteristics of the random motion are mantained.

*Keywords:* Stochastic processes; Galton board; Langevin equation.

La tabla de Galton es una red periódica hecha con clavos fijos en sus nodos, partículas esféricas viajan a través de ellos debido a la gravedad. Se muestra la utilidad de este sistema para presentar los principales conceptos de trayectorias estocásticas de Markov durante el movimiento de una sola partícula. En un caso especial, la tabla de Galton fue modificada, un conjunto de clavos (20%) se eliminaron al azar. Bajo este cambio las características principales del movimiento al azar prácticamente no se modifican.

*Descriptores:* Procesos estocásticos; tabla de Galton; ecuación de Langevin.

PACS: 05.10.Gg; 05.40.Ca; 05.40.Jc

## 1. Introduction

In simple physical systems we can measure macroscopic quantities such as density and energy, this allows for well-defined values. Actually, this is only an approximation because matter is not really continuous, since it consists of discrete particles. When we look particles and their interactions we need to take into account the existence of fluctuations (also called noise). A special type of noise is the so called Brownian motion. This type of motion generates random or stochastic trajectories of a small particle immersed in a liquid. It is well known that each molecule of the liquid shocks against this small particle giving a motion where the time-averaged position of the particle is zero, $\langle r(t)\rangle = 0$, ($\langle\bullet\rangle$ denotes the average over along time and $r(t)$ is the instantaneous position of the particle) and the second moment is $\langle r(t)r(\tau)\rangle = A\delta(t-\tau)$), it means that motion is Markovian, *i.e.*, a memoryless motion.

A very similar motion, but in a mechanical system occurs when a bead falls down in the so called Galton´s board. In fact, in 1889 Francis Galton [1–3], used this system as a mechanical device to show both the law of error and the normal distribution. This device consists of a vertical board with interleaved rows of nails, beads are dropped from the top, bouncing left and right as they hit the nails. Below, beads are collected in bins where the height of bead columns approximates a bell curve.

Motivated by such a device, here we are interested in describing, in a detailed manner, the irregular trajectory or the trace of the bead crossing this board. When a particle travels the board, it has a probability of motion in a certain direction at certain speed. This latter gives a random motion or noise movement to the particle's path that can be viewed as an example of stochastic motion.

Diffusion problems can be approached and modeled in essentially two ways: a macroscopic description which is only concerned with mass conservation and phenomenological constitutive equations; and a microscopic description, which models this phenomena taking into account the stochastic nature of the interaction among diffusing particles, *i.e.* it models the behavior of individual solute particles bouncing around with the solution molecules and possibly interacting with the substrate. The elementary particles under the effect of different force fields of different nature perform complex motion, the trajectories of these particles reproduce the geometrical complex structure, ejemplo de este tipo de sistema is the naturally and artificially fractured reservoirs.

In this paper we are interested in emphasizing the role of describing the random or stochastic motion of a falling grain, due the gravity in a lattice of nails, in terms of stochastic differential equations, like the Langevin equation [4]. Here the dynamics of a particle is affected by an effective frictional force, $F_f = -\lambda V$, directed against the trajectory of the particle. Through this model, we should make plausible some of the main hypothesis of this type of description and we can study the time-dependent fluctuations, that is, how the deviations at different times are correlated with each other.

This paper is organized as follows. In Sec. 2, we present the experimental setup known as the Galton´s board, together with the measuring techniques. The bases of the theoretical treatment are presented in Sec. 3; after in Sec. 4 we applied this technique to understand the stochastic motion of beads

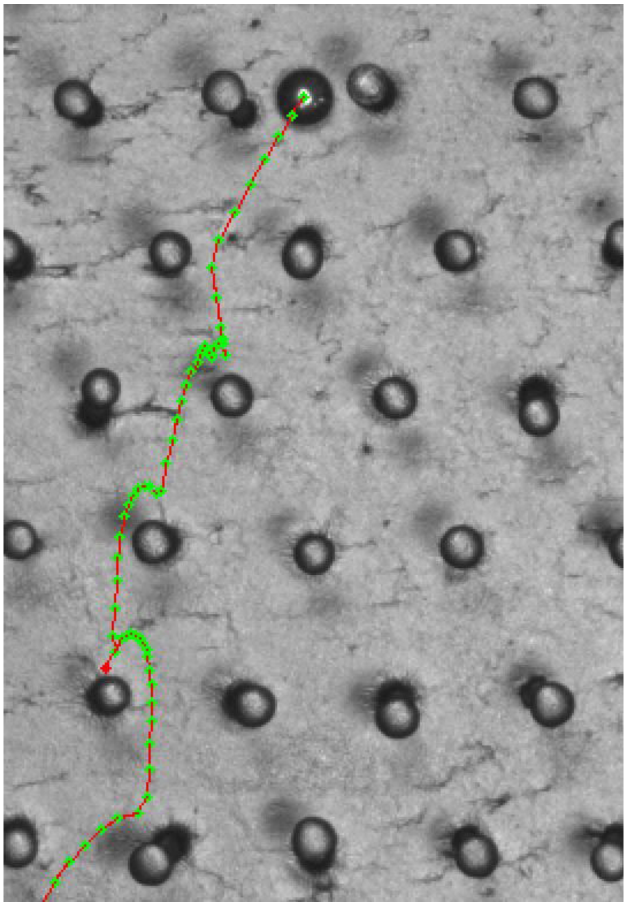

FIGURE 1. Galton´s board showing the track of a steel-bead.

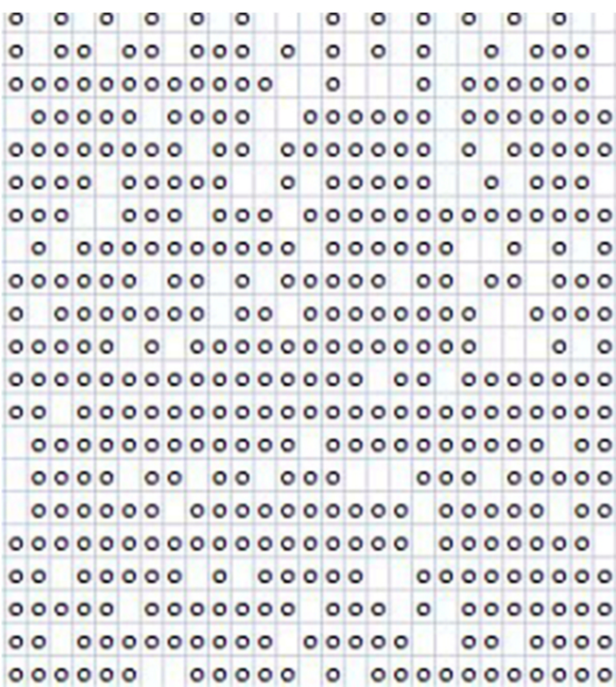

FIGURE 2. Modified Galton’s board where has been removed a 20% of the nails.

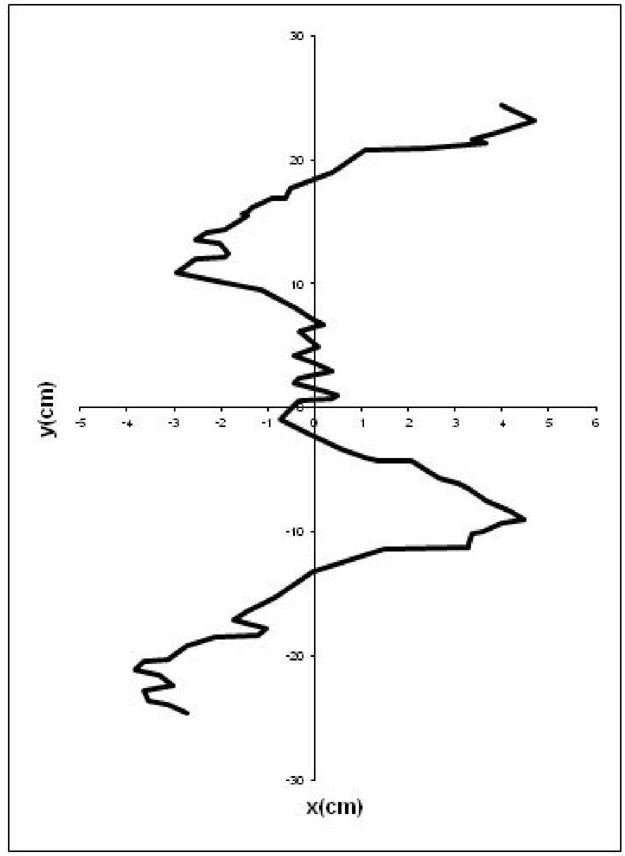


FIGURE 3. Typical actual trajectory motion of steel-bead on the Galton´s board.

in the experiments. Finally, a brief summary as some possible extensions of this work are given in Sec. 5.

## 2. Experimental Setup

### 2.1. The apparatus

In this section we describe the experimental set-up. The experiment was built as a board of nails, see Fig. 1, where also is shown the trajectory of steel bead. Here steel-nails were fixed to a wood-board on regular form, resulting into a triangular lattice, like Lorentz lattice of fixed scatterers [5]. The horizontal distance between nails is $D = 10.0 \pm 0.05$ mm and the oblique distance is $D' = 11.5 \pm 0.05$ mm. The nails have a diameter mean size of $\delta = 1.96 \pm 0.05$ mm. The size of the board was $58.0$ cm length by $45.0$ cm height.

Motions of two types of grains, spherical steel-bead and gel-beads of mean size $d = 4$ mm, falling down due to gravity, from rest were video recording with a camera at 30 frames per second. Each frame has a resolution of $480 \times 640$ pixels. In our experiment we have that $1$ pixel $= 1$ mm. The average number of pixels that the bead travels between frame is approximately 4 along the $x-$axis and 6 along the $y-$axis. In order to avoid grains leaving the board, we have inclined it to an angle $\theta < 90°$, respect to the horizontal.

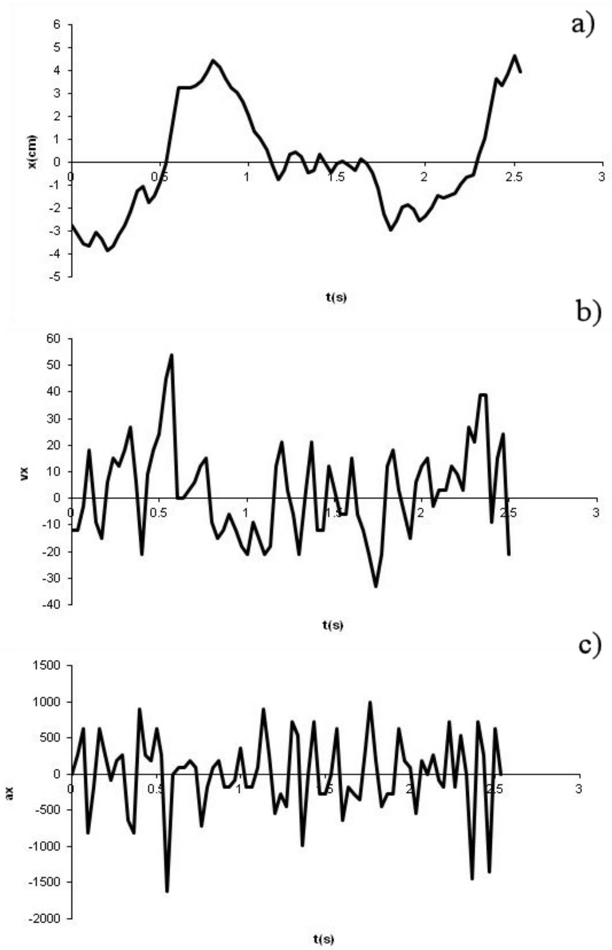


FIGURE 4. Typical behavior of $x$-component for steel-bead, (a) trajectory, (b) velocity and (c) acceleration.

As a special case of the Galton board, it was modified as follows: randomly and uniformly (on rows) nails were removed by a 20%. To do it was used a random number generator [6]. As a result of it, was obtained a no homogeneus board (with uniform distribution of "holes"). In Fig. 2 we show a typical Galton board with 20% of nails removed.

### 2.2. Measurements

We mainly report experiments made with an angle $\theta = 74°$, in order to limit the friction force between the wood-board and beads on the particle motion. As aforementioned at time $t = 0$, one grain was falling down from the top, and its noisy path was recorded and analyzed frame by frame on a computer monitor. A set of twenty experiments, for each grain, was recorded in this study. The position versus time needed to determine the speeds and accelerations used to calculate the instantaneous velocities and accelerations along the transversal direction are respectively

$$v_{x_i} = \frac{x_i - x_{i-1}}{t_i - t_{i-1}}, \qquad a_{x_i} = \frac{v_{x_i} - v_{x_{i-1}}}{t_i - t_{i-1}} \tag{1}$$

where $x_i$ indicate the transversal component at time $t_i$, the same is valid for the downward longitudinal component, $y_i$.

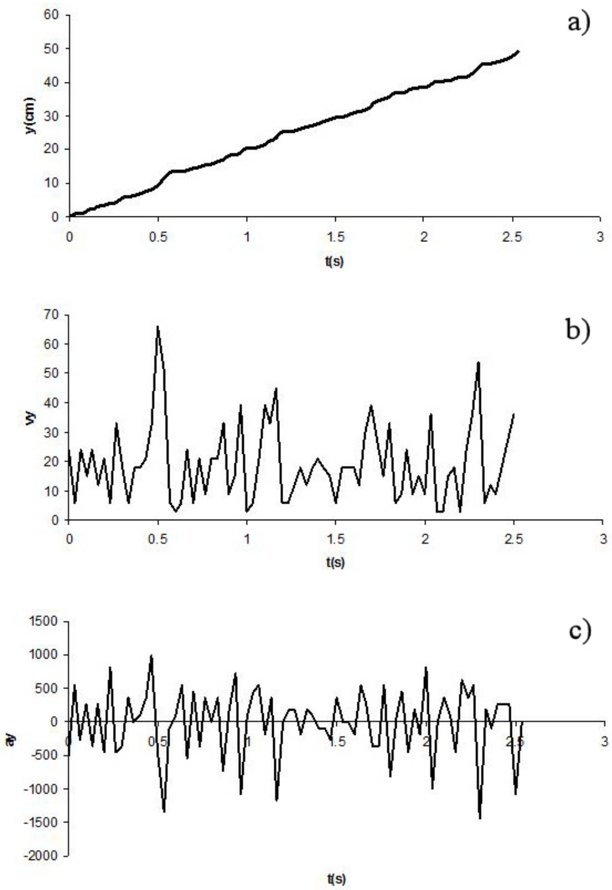


FIGURE 5. Typical behavior of $y$-component for steel-bead, a) trajectory, b) velocity and (c) acceleration.

A typical trajectory of the steel-bead in the board is given in the Fig. 3. The gravity has a component in the $y$ direction. Plots for the $x$ component of the trajectory and their speed and acceleration are shown in the Fig. 4, and the same is shown for the $y$ component (see Fig. 5). Notice that the behavior of each component is both quantitatively and qualitatively different from each other.

In all measurements, despite the abrupt behavior of the $x$ component, the time-averaged position, $\langle x \rangle$, turns into a zero mean value, *i.e.*, $\langle r(t) \rangle = 0$. The same is maintained for $\langle v_x \rangle$ and $\langle a_x \rangle$. On the other hand, the mean value of the $y$ component can always be approximated by straight lines, *i.e.*, $\langle y \rangle = ct$ (see the Fig. 4(a)). These facts will be used in the next section in order to model the fluctuating motion.

For the gel case, the experiments were performed with a gel-bead of mean size $d = 4$ mm. Plot of the trajectory of the gel-bead as a time function is given in Fig. 6. Their speed and acceleration as a function of time, along with the transverse $x$-component, are shown in the Fig. 7. Similar plots, but for the longitudinal component, $y$, are given in Fig. 8.

In the case where the nails were removed, the dynamics is very similar for both cases, steel-bead and gel bead case. This is shown through the trajectories in Fig. 9.

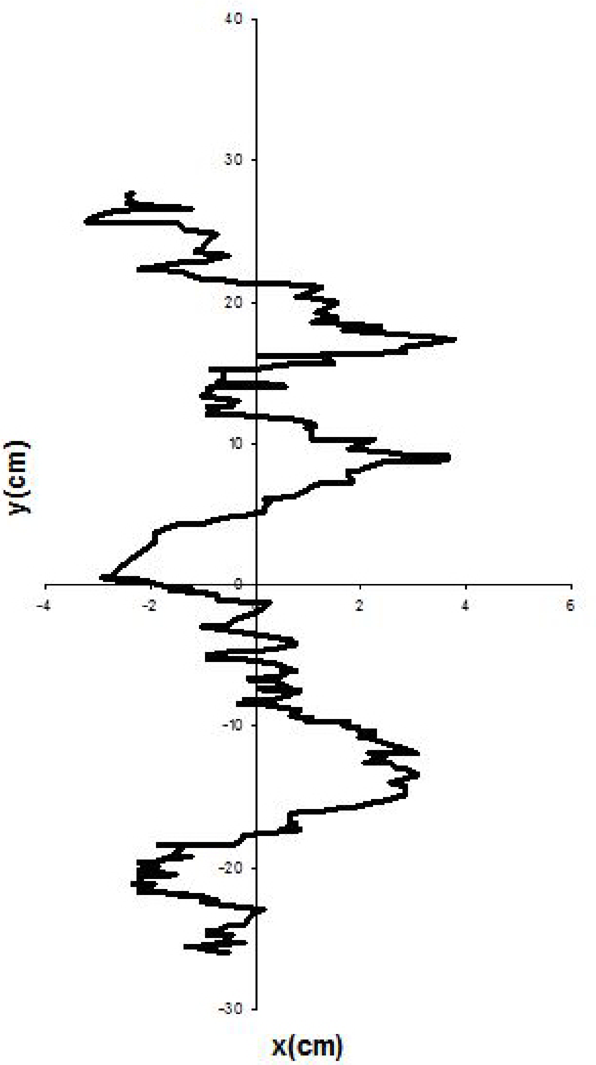


FIGURE 6. Typical actual trajectory motion of the gel-bead on the Galton´s board.

The steel-bead an gel-bead have a restitution coefficient $e = 0.72$ and $e = 0.77$ respectively, this was found experimentally over a steel plate $7.32$ mm, using the equation

$$e = \frac{v_R}{v_i} = \frac{h_R}{h_i}, \tag{2}$$

where $h_R$ is the rebound height and $h_i$ is the initial height [7], as it can see, the restitution coefficients are near the same, but the experiment in the Galton's board it can see that those coefficients change. In this equation it can see the cause of change:

$$e = \sqrt{\frac{W_{\text{kin}.R}}{W_{\text{kin}}}} = \sqrt{1 - \frac{W_{\text{abs}}}{W_{\text{kin}}}} = \frac{\mid v_R \mid}{v}. \tag{3}$$

In the previous equation restitution coefficient is the square root of the ratio of elastic energy $W_{\text{kin}.R}$, released during the impact, *i.e,* the kinetic energy of the initial impact minus the absorbed energy, $W_{\text{kin}} - W_{\text{abs}}$. This later expression

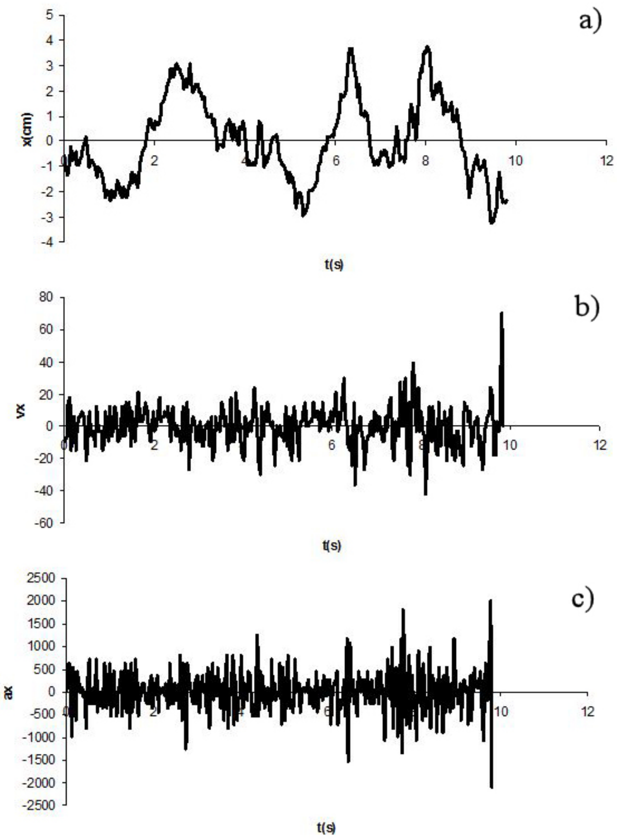


FIGURE 7. Typical behavior of $x$-component for gel-bead, (a) trajectory, (b) velocity and (c) acceleration.

also can be written in terms of the impact and rebound velocities, $v$ and $v_R$, respectively [8].

Trajectories in the steel-bead and gel-bead are different among them, essentially because in the case of gel-beads rebounds and oblique shocks are much more frequent and loss a small quantity of kinetic energy, the distances attained in this case are larger in both components.

## 3. Theoretical Approach

### 3.1. The Langevin´s Treatment

It is convenient to remember some important facts of the Brownian motion. One of the first phenomenological descriptions of Brownian motion was made in 1908 by French physicist Paul Langevin. He established the following arguments: if a large particle (compared with atomic dimensions) is introduced into a fluid, then, according to hydrodynamics, it experience an opposite force that depends on speed. This opposing force is due to the viscosity of the fluid. The greater the speed with which the body moves inside the fluid, the greater the opposing force or viscous friction that is created.

Moreover, as described above it is known that introducing a small particle in a fluid, it experiences forces due to collisions suffering with fluid molecules. Given the large number of collisions occurring at every moment, this second force varies in a very random and violent manner. This means that

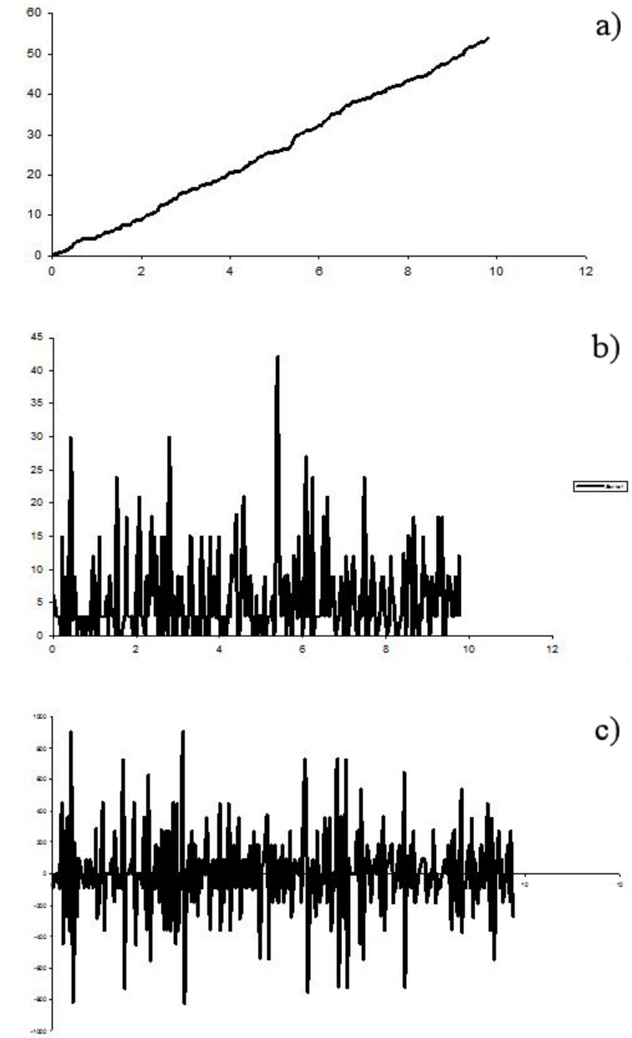


FIGURE 8. Typical behavior of $x$-component for gel-bead, (a) trajectory, (b) velocity and (c) acceleration.

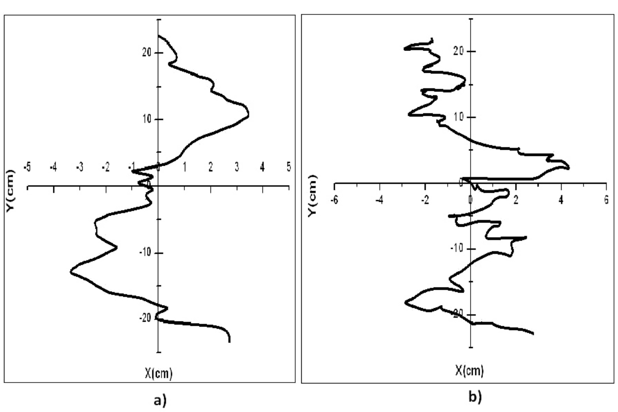


FIGURE 9. Typical actual trajectory motion of the steel bead a) and gel-bead b) on the Galton´s board with a 20% of the nails removed.

if, for example, make observations of the Brownian particle with a time scale of the order of seconds, the force due to collisions will change greatly, as have occurred in a second lot of collisions. On the other hand, in this same time scale, the first force of which we have spoken, the friction changes very little.

Below we report the analysis for the motion beads, for the steel-bead we can apply the theoretical treatment builded by Langevin and for gel-bead, we only identify the differences with the steel bead. For the Galton's board modified here are no major changes respect to results in the normal Galton's board.

### 3.1.1. *Motion of steel-bead*

An adequate approach to study the fluctuating motion of the particles and interactions having similar behavior is the Langevin´s treatment [5]. This inspired guess is able to short cut the general theory of fluctuating processes, turning out to be the only possibility for systems with a linear response. In this sense let $Q$ be physical quantity obeying a linear phenomenological law

$$\frac{dQ}{dt} = -\gamma Q, \tag{4}$$

where $\gamma$ is a constant and $Q$, for example, can be a component of the velocity of a heavy particle suspended in a gas or a liquid. In order to describe also the fluctuations, one writes for the instantaneous, detailed value $q$ of the same physical quantity, the Langevin equation

$$\frac{dq}{dt} = -\gamma q + f(t). \tag{5}$$

This equation is only meaningful if some information regarding the random force $f(t)$ is added. In general, when working with stochastic quantities, their description is in terms of their distribution. Two features that have the above distribution are: its mean and standard deviation. Since $f(t)$ is pictured as a very rapid and irregularly varying function of time, it can be only described by its stochastic properties. Specifically one assumes

$$\langle f(t) \rangle = 0, \tag{6}$$

here $\langle f \rangle$ denotes the average over a long time interval compared to the rapid variations in $f(t)$, but short compared to the phenomenological damping time $1/\gamma$. In these fluctuating forces, it would be reasonable to think that if we take an interval of, for example, a second, the force is exerted in one direction and in the opposite direction so that, on average, the force vanishes. Then at each instant the average stochastic force is zero.

Furthermore, we must keep in mind that this stochastic force changes with time, which means that not only must say something about their distribution at a given time, but also something about how to relate the values of the forces stochastic at various times. In addition one assumes

$$\langle f(t) f(\tau) \rangle = \Gamma \delta(t - \tau), \tag{7}$$

here $\Gamma$ is a constant independent of $t$ ($t > \tau$) and $q$. The delta-function is actually a sharply peaked but finite function, whose width is the autocorrelation time of $f(t)$. If we make observations on scales of the order of seconds, then the stochastic force value at a given time has nothing to do with the value it acquires in another moment that is separated by seconds. This is because in a second, the force varied greatly, so that the end of the interval, the force does not have a close relationship with the value it had at the beginning of the interval. It is possible say that stochastic or fluctuating forces are not correlated at various times. This assumption about $f(t)$ constitute the short cut replacing the general theory of fluctuating processes.

From Eqs. (6) and (7) it immediately follows that $\langle q\rangle$ satisfy the phenomenological law (4) and it may therefore be identified with the macroscopic quantity $Q$. In summary, using these arguments we conclude that the total force experienced by the Brownian particle is the sum of two forces: the systematic and stochastic. The stochastic force varies widely within the time scale that it changes the systematic force. However, if you add two quantities, one known, but the other stochastic nature, the sum will also be stochastic. Consequently, the total force experienced by the particle is random.

Once this identification is made, it may concluded that (5) describes correctly the phenomenology of the system.

The spectral density of a signal is a mathematical function that tells us how it is distributed the power or energy (as appropriate) of that signal on different frequencies that it is formed. It is often called simply the spectrum of the signal. Intuitively, the spectral density captures the frequency content of a stochastic process and helps identify periodicity. There exists a Fourier relationship between a time function and its spectrum, there also exists a Fourier relationship between the autocorrelation function and the power spectral density of a stochastic process. This important result in signal processing and communication theory is known as the *Weiner-Khintchine theorem*

$$S(\omega) = \int_{-\infty}^{\infty} \langle q(t)q(t+s)\rangle \exp[-i\omega s] ds = \Gamma, \qquad (8)$$

where $\omega = (2\pi/t)$ is the angular frequency given by Ramirez [9]. A random process with a spectrum of the correlation function which is flat and independent of the frequency $\omega$ is usually called a *white noise*. All frequencies are equally represented in such a process.

Of course, the white noise never can be performed in nature because a real random process has always a non-vanishing characteristic correlation time (memory) $\tau$. However, the correlation (7) serves as a very convenient mathematical idealization of a process, whose memory is short (as compared to all other characteristic times). More exactly, this is the limit of a short-correlated process for $\tau \to 0$.

The first moment $\langle f(t)\rangle$ in general is related to the average acceleration $\langle dv_i/dt\rangle$ ($i = x, y$). As can be seen in Figs. 4(c) and 5(c), was found that when removing nails evenly, up to 20%, changes in the plots of are smaller, also for the case of the gelatin-bead, this moment is zero because the mean value of the time series is exactly zero.

### *3.1.2. Motion of gelatin-bead*

We again apply the Langevin approach for the motion of the gel-bead. As in the previous section the experimental results given in Figs. 7(c) and 8(c) lead to conclude that the first moment also obeys the Eq. (6). The mean of the stochastic force is null, due the fact that the $x$-component of the trajectory is maintained around $x = 0$ and the mean velocity along $y$ is no accelerated. However the second moment is not constant, hence we have

$$\langle f(t)f(\tau)\rangle = g(t). \qquad (9)$$

According to the experimental data, we note that the spectral density increases as a function of power, hence the autocorrelation function has a non constan Fourier spectrum

$$S(\omega) = \omega^{\alpha}, \qquad (10)$$

where $\alpha \in R$. In the classification by spectral density is given color terminology, with different named types. In the next section we assigned the $\alpha$ value for the motion of the bead.

All the above analysis of the Langevin approach for the motion of both types of grains, applies to the $x$ and $y$ trajectories of each case. Langevin description will be useful in the problem to contextualize the experimental results of the dynamics of the particle in the lattice. We should note that the program to construct the equation of motion and the stochastic properties of force have not been used experimentally for the case of a single particle in a dense medium. In this paper does not intend to replace the medium by a discrete lattice, but rather to characterize the lattice effect on the trajectory of the bead.

## 4. The Langevin Approach to the Galton board

As we have found in Sec. 2, all grain paths have noise behavior, which will be modeled in this section. These are given to us for both the transversal and longitudinal components respectively,

$$X(t) = \langle x(t)\rangle = 0, \qquad (11)$$

$$Y(t) = \langle y(t)\rangle = V_0 t. \qquad (12)$$

Physically, the Eq. (11) gives the important result that, on average, the beads remain mainly at the center, therefore there is non a preferred direction of propagation. On the other hand, Eq. (12) indicates that the beads propagate towards the bottom of the board with a constant velocity $V_0$, although that gravity and continuous collisions with the nails, act on the bead and the apparent motion is very fluctuating .

We now analyze the particle dynamics; then we have two options for the analysis:

- to build the motion equation in terms of the Newton second law [10], or
- to build a less detailed motion equation containing gross effects on the bead.

In this work we chose this last option, because from a simple model we obtain a good description for the motion. The average motion can be understood if we suppose that the medium response on the grain´s path is linear. The linear response theory will let us to use the fact that $d\langle\omega\rangle/dt = \langle d\omega/dt\rangle$ to any quantity $\omega$. Specifically, the average velocity is then $d\langle x\rangle/dt = \langle dx/dt\rangle$. This leads to

$$V_x = \frac{dX}{dt} = 0, \tag{13}$$

$$V_y = \frac{dY}{dt} = V_0, \tag{14}$$

*i.e.*, the average propagation velocity, $V_y$, is a constant equal to $V_0$. For the steel-bead, plot 5(b), yields $V_0 = 19.34$ cm/s and for gel-bead, plot 8(b), yields $V_0 = 5.47$ cm/s both of mean size $d = 4$ mm. Finally, the components of the acceleration are

$$A_x = \frac{dV_x}{dt} = 0, \tag{15}$$

$$A_y = \frac{dV_y}{dt} = 0. \tag{16}$$

These two relationships indicate, as we mentioned before, that the motion is acceleration free.

Equations (15) and (16) apparently are similar expressions, but each one reflects distinct macroscopic and detailed behavior. We can use the linear Langevin´s approach as the model for the detailed motion equations see Sec. 3.1. Such approach assumes that the instantaneous time series $v_x(t)$ and $v_y(t)$ are due to fluctuating forces $h(t)$ and $f(t)$, respectively. So, the abrupt behavior of the $x$ component can be understood by using the Langevin equation

$$\frac{dv_x}{dt} = h(t), \tag{17}$$

here the average of $h(t)$ is null and it clearly describes the average null motion along the transversal motion (see Fig. 4(c)).

To study the $y$-component we can assume a force on the bead, like hard spheres, where the frictional force of the lattice is proportional to the average velocity in the $y$ direction, $V_0$, and other force (opposite to the motion) of stochastic nature, $f(t)$, proportional to the instantaneous velocity. In this case

$$\frac{dv_y}{dt} = f(t) + \lambda V_0 = -\lambda(v_y - V_0), \tag{18}$$

$\lambda$ the effective friction coefficient, is a constant independent of $v_y$ and $t$, for the steel-bead $\lambda = 31.42$ s$^{-1}$ and we have for gel-bead $\lambda = 40.71$ s$^{-1}$ for particles of mean size $d = 4$ mm, these values of $\lambda$ change for different $d$ and $\theta$ values. In both cases the change in value of the friction coefficient is minimal for the cases the modified Galton's board.

If we take into account equations derived from the experimental data, we can see that the stochastic properties of the random forces, of the equations (17) and (18) are respectively of the form

$$\langle h(t)\rangle = 0 \tag{19}$$

$$\langle f(t)\rangle = -\lambda V_0. \tag{20}$$

to all grain sizes here treated.

Another way to prove the non accelerated motion, can be investigated through the mean squared displacement $\langle y^2(t)\rangle$ (see Fig. 10), this quantity give us

$$\langle y^2(t)\rangle = \beta_n t^2, \tag{21}$$

where $\beta_n$ are constants depending on grain sizes. Expressions of the form (19) is valid for different diameters of grains used in our experiments and indicate the behavior of free particles. Note that when the grain grows his movement is slower in the the arrangement of nails.

There is the phenomenon of collision between the nails and beads, this causes that the particles exhibit dominantly elastic-plastic or plastic behavior during collisions. The kinetic energy absorption is the result of plastic deformation, adhesion and friction between nails and the table.

The restitution coefficient is an important parameter of a material, is used to describe the energy absorption and damping force. For an ideal elastic impact, the energy is absorbed during the impact and recovered on the rebound, so the relative velocity before impact is equal after impact. For the complete absorption of kinetic energy the restitution coefficient is zero. Finally in the case of elastic-plastic impact, the range of restitution coefficient is in a range from zero to one.

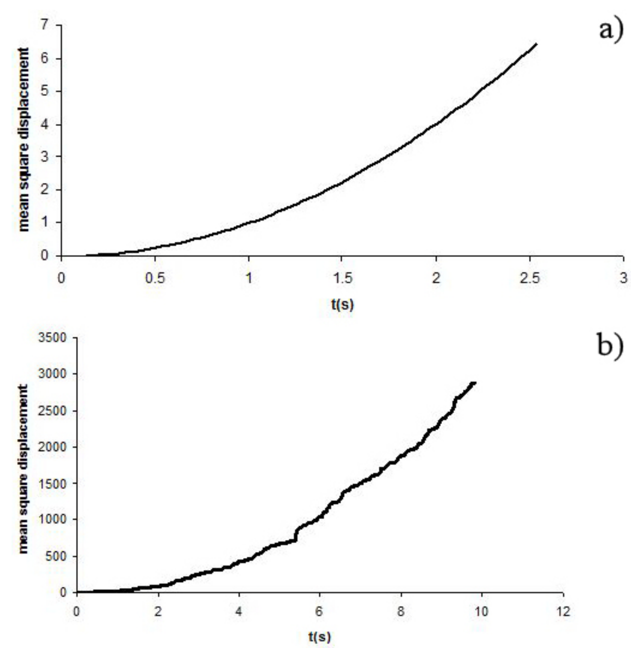


FIGURE 10. Mean square displacement $\langle y^2(t)\rangle$ as a function of time.

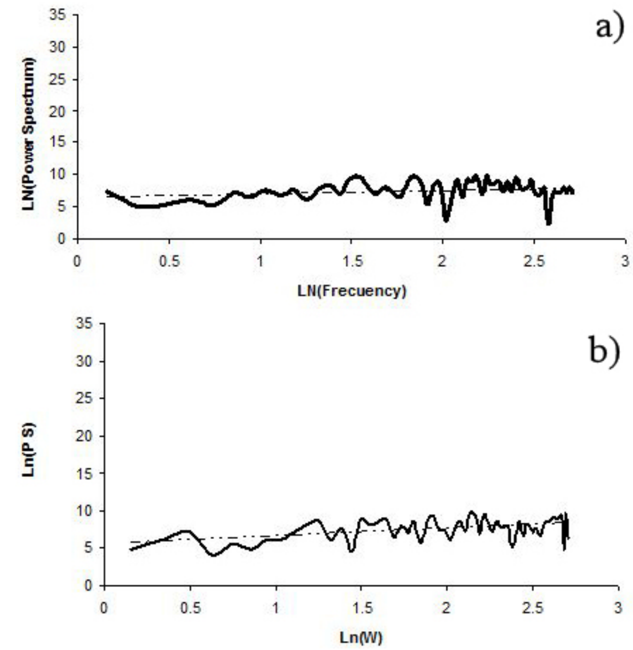


FIGURE 11. Fourier transform of steel-bead.

For this study, we used gelatin and steel beads, both materials are not purely elastic neither purely inelastic, that is why is important to present the restitution coefficients study. We determined the restitution coefficient [11] as the ratio of relative rebound velocity $V_R$ to that before the impact $V$, the normal and oblique impacts are described by normal and tangential restitution coefficients

$$e_n = |V_{R,n}| /V_n, e_t = |V_{R,t}| /V_t.$$

In both cases we determinated the average restitution coefficient because we obtain the instantaneous velocity in the $x$ and $y$ component. For gelatin bead the normal and tangential restitution coefficient are $e_n = 0.5750$ and $e_t = 0.7228$, for steel bead the restitution coefficient are $e_n = 0.6160$ and $e_t = 0.7232$.

In this paper we propose two stochastic differential equations (17) and (18) to describe the random motion; there are works where their principal aim is a numerical study of a bidimensional Galton board with determinist equations. Benito *et al.*, [12] have able to reproduce by means of computational simulations the geometrical features of a Galton board, they introduce the effect of bouncing without calculating any force and simulate disk of equal diameters but different elastic properties. In Ref. 13 present results of simulations of a model of the Galton board for various degrees of elasticity of the ball-to-nail collision, the fall of the ball is described with a set of two ordinary differential equations.

Finally [14] present a numerical simulation of generalized Langevin equation with arbitrary correlated noise for anomalous and ballistic diffusion applications.

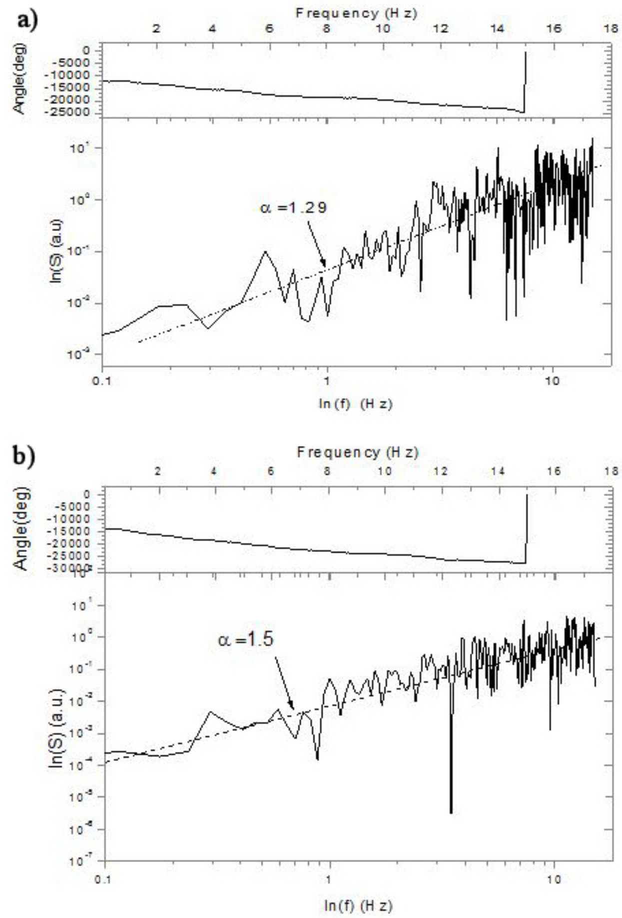


FIGURE 12. Fourier transform of gel-bead, (a) $x$-component , (b) $y$-component.

### 4.1. Power spectra

The noise of the paths can be characterized as mentioned before, from the power spectra that can be alternatively calculated by the expression

$$S(\omega) = |f(\omega)|^2/\omega, \tag{22}$$

$f(\omega)$ is the Fourier transform of $f(t)$. In Fig. 11 show calculations for all time series $h(t)$ and $f(t)$ give us constant values which confirm the existence of white noise for these components.

Using the Wiener-Khinchine theorem, *i.e.*, employing the inverse Fourier transform in (8), we easily obtain that the correlation of $h(t)$ and $f(t)$ gives delta correlations to each component

$$\langle h(t)h(\tau)\rangle = \Gamma_n\delta(t-\tau), \tag{23}$$

and

$$\langle f(t)f(\tau)\rangle - \langle f(t)\rangle^2 = \Gamma'_n\delta(t-\tau). \tag{24}$$

$\Gamma$ and $\Gamma'$ are constants.

### 4.2. Gel-bead

The noise of the paths can be characterized from the power spectra $S(\omega) = |f(\omega)|^2/\omega$, the calculations for noise of $h(t)$ and $f(t)$ do not give constant values as occurs in the case of white noise because there are not a unique characteristic frequency; instead in the present cases a color noise is raised. A color noise indicates that the power spectra fit a power law of the form $S \sim \omega^{\alpha}$, so we have color of noise, according to the (10) we need calculate $\alpha$. In the components $x$ and $y$ adjust with a logarithm function the adjusted functions are the power spectra for $h(t)$ and $f(t)$. The correlation in the graph (Fig. 12) fits to the following power functions.

For the force along the $x-$axis

$$S = 0.3272\omega^{1.29}, \tag{25}$$

and for the force along the $y-$axis

$$S = 0.1599\omega^{1.5}. \tag{26}$$

Perhaps the difference in the value of $\alpha$ among the $x$ and $y$ axis is due to bead motion that is forced by gravity mainly in the $y$ direction, *i.e.* there is a continuos energy addition.

## 5. Conclusions

The fluctuating motion of the particles (steel and gel beads) on the Galton´s board has a motion like a particle interacting with a hard spheres gas where the frictional term is proportional to the velocity. However, unlike this system where the particle has diffusive motion, in the Galton´s board the particle will propagate, in average, towards the board´s bottom like a free particle. Up to what we know, this is one of the very first time that the random motion of a single particle can be investigated, and the stochastic properties justified. On the other hand the Langevin´s method allows to search for dynamic coefficients, *i.e.*, the moments of the probability distribution with no need to study the probability distribution itself, as in the case of previous studies [13].

For the special case of the modified Galton board, the changes were minimal in comparison with the values of the parameter founded for the normal Galton´s board.

As a possible extension of this work we should note that the Galton´s board gives effective friction coefficient to both each bead size and each angle, however more complex friction coefficients can be investigated if the board is partially filled of nails in accordance with specific rules of filled and predefined concentrations for values greater than 20%. Calculations of diffusion coefficients could be of great importance to future work.

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