Point symmetries of the Euler–Lagrange equations

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We give an elementary derivation of the equations for the point symmetries of the Euler–Lagrange equations for a Lagrangian of a system with a finite number of degrees of freedom. We show that given a divergence symmetry of a Lagrangian, there exists an equivalent Lagrangian that is strictly invariant under that transformation. The corresponding description in the Hamiltonian formalism is also investigated.

Keywords: Lagrangians; symmetries; equivalent Lagrangians; constants of motion; Hamiltonian formalism.

Damos una derivación elemental de las ecuaciones para las simetrías puntuales de las ecuaciones de Euler–Lagrange para una lagrangiana de un sistema con un número finito de grados de libertad. Mostramos que dada una simetría hasta una divergencia de una lagrangiana, existe una lagrangiana equivalente que es estrictamente invariante bajo esa transformación. También se investiga la descripción correspondiente en el formalismo hamiltoniano.

Descriptores: Lagrangianas; simetrías; lagrangianas equivalentes; constantes de movimiento; formalismo hamiltoniano.

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1. Introduction

One of the advantages of the Lagrangian formalism in classical mechanics is that, roughly speaking, each continuous symmetry of the Lagrangian function of a system can be related to the existence of a constant of motion. However, usually, this relationship is not fully exploited, and is employed only in connection with simple geometrical transformations, such as translations, rotations, and time displacements.

In the case of a system with one degree of freedom, a constant of motion amounts to a first-order ordinary differential equation (ODE) so that, with the aid of a constant of motion, instead of having to solve a second-order ODE, one only has to solve a first-order ODE. When the number of degrees of freedom is greater than 1, any constant of motion also helps to simplify the equations of motion.

The (strict) *variational symmetries* of a Lagrangian $L(q_i, \dot{q}_i, t)$ ($i = 1, 2, \ldots, n$), are the point transformations, $q'_i = q'_i(q_1, \ldots, q_n, t)$, $t' = t'(q_1, \ldots, q_n, t)$, that leave the action integral

$$\int_{t_0}^{t_1} L \, \mathrm{d}t$$

invariant, that is,

$$L(q'_i, \frac{\mathrm{d}q'_i}{\mathrm{d}t'}, t') \, \frac{\mathrm{d}t'}{\mathrm{d}t} = L(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t), \tag{1}$$

with $\mathrm{d}t'/\mathrm{d}t = \partial t'/\partial t + \dot{q}_i \, \partial t'/\partial q_i$ (here and henceforth there is summation over repeated indices). The *one-parameter groups* of strict variational symmetries are determined by the first-order linear partial differential equation (PDE)

$$\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) + \frac{\partial L}{\partial t}\xi + L\,\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0, \tag{2}$$

where $\eta_i(q_j, t)$ and $\xi(q_i, t)$ are $n + 1$ unknown functions (see, *e.g.*, Refs. [1–6]) (note that, *e.g.*, $\mathrm{d}\eta/\mathrm{d}t = \partial\eta/\partial t + \dot{q}_i \, \partial\eta/\partial q_i$). A nontrivial solution of this equation yields the constant of motion

$$\varphi(q_i, \dot{q}_i, t) = \eta_i \frac{\partial L}{\partial \dot{q}_i} - \xi\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right). \tag{3}$$

A wider class of variational symmetries, also related to constants of motion, is formed by the one-parameter families of point transformations, $q'_i = q'_i(q_1, \ldots, q_n, t, s)$, $t' = t'(q_1, \ldots, q_n, t, s)$, such that

$$L(q'_i, \frac{\mathrm{d}q'_i}{\mathrm{d}t'}, t') \, \frac{\mathrm{d}t'}{\mathrm{d}t} = L(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t) + \frac{\mathrm{d}}{\mathrm{d}t}F(q_i, t, s), \tag{4}$$

for all values of the parameter s for which the transformation is defined, where F is some function of q_i, t, and s only. These transformations are sometimes called Noether symmetries [2], or divergence symmetries [3], but it seems more appropriate to call them Noether–Bessel-Hagen symmetries [7]. From Eq. (4) it follows that a set of functions $\xi(q_i, t)$, $\eta_i(q_j, t)$ generates a one-parameter group of variational symmetries of L if there exists a function $G(q_i, t)$ (defined up to an additive trivial constant) such that

$$\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) + \frac{\partial L}{\partial t}\xi + L\,\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{5}$$

[*cf.* Eq. (2)]. (The function G is equal to the partial derivative of F with respect to s, at $s = 0$, assuming that at $s = 0$ the transformation reduces to the identity.) In this case, in addition to ξ and the η_i, one has to find G. The constant of motion associated with a solution of Eq. (5) is

$$\varphi(q_i, \dot{q}_i, t) = \eta_i \frac{\partial L}{\partial \dot{q}_i} - \xi\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) - G. \tag{6}$$

Even though it is more complicated to solve Eq. (5) than Eq. (2), for some Lagrangians Eq. (5) leads to many more constants of motion than Eq. (2) (see, *e.g.*, the example given in Sec. 2.1.1 of Ref. [6] and the examples below).

The method usually employed to solve Eqs. (2) and (5) relies on the fact that ξ and η_i are functions of q_i and t only and, in many cases, the left-hand sides of Eqs. (2) and (5) are polynomials in the $\dot{q}_i$, with coefficients that depend on q_i and t. Since Eqs. (2) and (5) must hold for all values of q_i, $\dot{q}_i$, and t, without imposing the equations of motion, by equating the coefficients of the products of the $\dot{q}_i$ on each side of the equation, one obtains a system of equations that only involve the variables q_i and t (see, *e.g.*, Refs. [5, 6, 8]). (It may be noticed that the function G *cannot* depend on the $\dot{q}_i$ because one would have terms proportional to $\ddot{q}_i$ on the right-hand side of Eq. (5), while the left-hand side of this equation is a function of q_i, $\dot{q}_i$, and t, only. *Cf.*, Eq. (9.2.10) of Ref. [9].)

In the case of Eq. (5), one obtains in this manner some expressions for the partial derivatives, $\partial G/\partial t$ and $\partial G/\partial q_i$, of the unknown function G in terms of L, ξ, η_i, and their first partial derivatives. From the equality of the mixed second partial derivatives of G with respect to q_i and t, one finds $n(n+1)/2$ equations, that do not contain G. Once ξ and η_i are determined from the set of equations thus obtained, the function G can be finally calculated (see, *e.g.*, Ref. [6]).

As we shall show below, these calculations can be simplified if, instead of starting from Eq. (5), one looks for the symmetries of the Euler–Lagrange equations corresponding to the given Lagrangian (see also Refs. [8, 10]), because these are n PDEs for ξ and η_i only (Eqs. (26) below). In Sec. 2 we prove that given a variational symmetry of a Lagrangian, there exist Lagrangians equivalent to it for which the transformation is a strict variational symmetry. We derive the transformation rules for the Euler–Lagrange equations under point transformations, which lead to the equations for the generators of the one-parameter groups of symmetries of the Euler–Lagrange equations. In Sec. 3 we show that, if the Lagrangian is regular (*i.e.*, $\det(\partial^2 L/\partial\dot{q}_i\partial\dot{q}_j) \neq 0$), the variational symmetries are canonical transformations.

2. Symmetries of the Lagrangians and of the Euler–Lagrange equations

As is well known, two Lagrangians, $L_1(q_i, \dot{q}_i, t)$ and $L_2(q_i, \dot{q}_i, t)$, lead to the same Euler–Lagrange equations, that is

$$\frac{\partial L_1}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_1}{\partial \dot{q}_i} = \frac{\partial L_2}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_2}{\partial \dot{q}_i},$$

if and only if there exists a function $F(q_i, t)$ such that

$$L_2 = L_1 + \frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i}. \tag{7}$$

In such a case, it is said that L_1 and L_2 are *equivalent* (see, *e.g.*, Ref. [8]). (Note that this indeed defines an equivalence relation. In the literature, this equivalence is also called gauge equivalence and the function F is called gauge function.) Thus, a variational symmetry of L is a point transformation that leaves L invariant, or leads to a Lagrangian equivalent to L [see Eq. (4)].

A straightforward computation shows that if the functions ξ, η_i generate a one-parameter group of variational symmetries of L_1 (*i.e.*, Eq. (5) holds for some function G_1), then ξ, η_i also generate a one-parameter group of variational symmetries of L_2 with

$$G_2 = G_1 + \left(\xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial q_i}\right) F, \tag{8}$$

up to an additive trivial constant. In other words, each solution, ξ, η_i of Eq. (5) represents a variational symmetry of a whole class of Lagrangians or, equivalently, a symmetry of a set of Euler–Lagrange equations, which are common to all the Lagrangians of a class.

Making use of Eq. (8) we can readily show that if a point transformation is a variational symmetry of a given Lagrangian L_1, then we can always find another Lagrangian, L_2, equivalent to L_1, for which the point transformation is a *strict* variational symmetry, that is, $G_2 = 0$. This conclusion follows from the fact that, for any function G_1, it is always possible to find a function F such that

$$G_1 + \left(\xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial q_i}\right) F = 0. \tag{9}$$

In fact, the solution is determined up to an additive function arbitrary function of n variables.

2.1. Transformation of the Euler–Lagrange equations

In order to find the equations for the symmetries of the Euler–Lagrange equations, we shall study the effect of a point transformation on the Euler–Lagrange equations.

In the case of a coordinate transformation of the form

$$q'_i = q'_i(q_1, \ldots, q_n, t), \tag{10}$$

where the new coordinates may depend explicitly on t, but the time itself is not changed, the inverse relations, $q_i = q_i(q'_1, \ldots, q'_n, t)$, must exist, and making use repeatedly of the chain rule one finds that

$$\frac{\partial L}{\partial q'_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}'_i} = \frac{\partial q_j}{\partial q'_i}\left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}\right). \tag{11}$$

(Equation (11) explicitly demonstrates the covariance of the Euler–Lagrange equations under the coordinate transformations (10), which means that different choices of the "generalized coordinates" lead to equivalent equations of motion.) Here it is assumed that the function L appearing in both sides of Eq. (11) is the same function, expressed in terms of two different coordinate systems, but, when t is also transformed, the Lagrangian $L(q_i, \dot{q}_i, t)$ must be replaced by a new Lagrangian L' according to

$$L'(q'_i, \dot{q}'_i, t') = L(q_i, \dot{q}_i, t)\frac{\mathrm{d}t}{\mathrm{d}t'}, \tag{12}$$

with $\dot{q}'_i \equiv \mathrm{d}q'_i/\mathrm{d}t'$, so that the action integral remains invariant

$$\int_{t_0}^{t_1} L\mathrm{d}t = \int_{t'_0}^{t'_1} L'\mathrm{d}t'.$$

In order to find a relation analogous to Eq. (11), applicable to an arbitrary point transformation

$$q'_i = q'_i(q_1,\ldots,q_n,t), \qquad t' = t'(q_1,\ldots,q_n,t), \tag{13}$$

instead of attempting a direct computation, it is convenient to define $q_0 \equiv t$ (note that this is not related to Relativity, it is just a useful notation), so that Eqs. (13) are equivalent to the single equation

$$q'_\alpha = q'_\alpha(q_1,\ldots,q_n,q_0), \qquad \alpha = 0,1,\ldots,n, \tag{14}$$

and we introduce an auxiliary variable u in terms of which the coordinates q_i and t will be expressed. Then, since t is now a function of u, according to the elementary rules for a change of variable in an integral,

$$\int_{t_0}^{t_1} L(q_i, \mathrm{d}q_i/\mathrm{d}t, t)\,\mathrm{d}t = \int_{u_0}^{u_1} L(q_i, \frac{\mathrm{d}q_i/\mathrm{d}u}{\mathrm{d}t/\mathrm{d}u}, t)\,\frac{\mathrm{d}t}{\mathrm{d}u}\,\mathrm{d}u.$$

Hence, the use of the variable u must be accompanied by the use of the Lagrangian

$$\widetilde{L}(q_\alpha, \mathrm{d}q_\alpha/\mathrm{d}u) \equiv L(q_i, \frac{\mathrm{d}q_i/\mathrm{d}u}{\mathrm{d}t/\mathrm{d}u}, t)\,\frac{\mathrm{d}t}{\mathrm{d}u}.$$

In fact, a straightforward computation (using again the chain rule) shows that, for $i = 1,2,\ldots,n$,

$$\frac{\partial \widetilde{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial \widetilde{L}}{\partial(\mathrm{d}q_i/\mathrm{d}u)} = \frac{\mathrm{d}t}{\mathrm{d}u}\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right), \tag{15}$$

proving that the original Euler–Lagrange equations are indeed reproduced with $\widetilde{L}$, and

$$\begin{aligned}\frac{\partial \widetilde{L}}{\partial q_0} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial \widetilde{L}}{\partial(\mathrm{d}q_0/\mathrm{d}u)} &= \frac{\mathrm{d}t}{\mathrm{d}u}\frac{\partial L}{\partial t} \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}u}\left(L - \frac{\mathrm{d}q_i}{\mathrm{d}u}\frac{1}{\mathrm{d}t/\mathrm{d}u}\frac{\partial L}{\partial \dot{q}_i}\right) \\ &= \frac{\mathrm{d}t}{\mathrm{d}u}\left[\frac{\partial L}{\partial t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right)\right] \\ &= -\frac{\mathrm{d}t}{\mathrm{d}u}\,\dot{q}_i\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right),\end{aligned} \tag{16}$$

which is trivially equal to zero when the other n Euler–Lagrange equations for $\widetilde{L}$ are satisfied. (That is, we only get n equations of motion from $\widetilde{L}$, as in the case of L.)

Applying now the relation (11) to the auxiliary Lagrangian $\widetilde{L}$, we find that under a point transformation (13), for $i = 1,2,\ldots,n$,

$$\frac{\partial \widetilde{L}}{\partial q'_i} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial \widetilde{L}}{\partial(\mathrm{d}q'_i/\mathrm{d}u)} = \frac{\partial q_\alpha}{\partial q'_i}\left(\frac{\partial \widetilde{L}}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial \widetilde{L}}{\partial(\mathrm{d}q_\alpha/\mathrm{d}u)}\right)$$

(the lower case Greek indices run over $0,1,2,\ldots,n$). With the aid of Eqs. (15) and (16) we see that this last relation amounts to

$$\begin{aligned}\frac{\mathrm{d}t'}{\mathrm{d}u}\left(\frac{\partial L'}{\partial q'_i} - \frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial L'}{\partial \dot{q}'_i}\right) &= \frac{\partial q_j}{\partial q'_i}\frac{\mathrm{d}t}{\mathrm{d}u}\left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}\right) \\ &\quad - \frac{\partial t}{\partial q'_i}\frac{\mathrm{d}t}{\mathrm{d}u}\,\dot{q}_j\left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}\right)\end{aligned}$$

which means (*e.g.*, taking $u = t'$) that

$$\begin{aligned}&\frac{\partial L'}{\partial q'_i} - \frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial L'}{\partial \dot{q}'_i} \\ &\qquad = \left(\frac{\partial q_j}{\partial q'_i}\frac{\mathrm{d}t}{\mathrm{d}t'} - \frac{\partial t}{\partial q'_i}\frac{\mathrm{d}q_j}{\mathrm{d}t'}\right)\left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}\right).\end{aligned} \tag{17}$$

This relation reduces to Eq. (11) in the case where $t' = t$, and demonstrates the covariance of the Euler–Lagrange equations under the point transformations (13).

2.2. Symmetries of the Euler–Lagrange equations

We shall say that the point transformation (13) is a *symmetry of the Euler–Lagrange equations* corresponding to the Lagrangian $L(q_i,\dot{q}_i,t)$ if Eq. (17) holds with $L' = L$. According to the definitions given in Sec. 1, if a point transformation (13) is a variational symmetry of a Lagrangian L, the Lagrangian L' appearing in Eq. (17) is equal to L or is equivalent to L; in either case, we can replace L' by L on the left-hand side of Eq. (17) and therefore, any variational symmetry of L is also a symmetry of its Euler–Lagrange equations.

In what follows it will be convenient to use the abbreviation [8]

$$E_i \equiv \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}. \tag{18}$$

By contrast with the Lagrangian L, the functions E_i depend on q_j, $\dot{q}_j$, $\ddot{q}_j$, and t. Thus, the point transformation (13) is a symmetry of the Euler–Lagrange equations if

$$\begin{aligned}&E_i(q'_k,\dot{q}'_k,\ddot{q}'_k,t') \\ &\qquad = \left(\frac{\partial q_j}{\partial q'_i}\frac{\mathrm{d}t}{\mathrm{d}t'} - \frac{\partial t}{\partial q'_i}\frac{\mathrm{d}q_j}{\mathrm{d}t'}\right)E_j(q_k,\dot{q}_k,\ddot{q}_k,t),\end{aligned} \tag{19}$$

or, equivalently, interchanging the roles of q_α and q'_α,

$$\begin{aligned}&E_i(q_k,\dot{q}_k,\ddot{q}_k,t) \\ &\qquad = \left(\frac{\partial q'_j}{\partial q_i}\frac{\mathrm{d}t'}{\mathrm{d}t} - \frac{\partial t'}{\partial q_i}\frac{\mathrm{d}q'_j}{\mathrm{d}t}\right)E_j(q'_k,\dot{q}'_k,\ddot{q}'_k,t').\end{aligned} \tag{20}$$

2.2.1. Example

The Euler–Lagrange equations corresponding to the Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy, \tag{21}$$

where m and g are constants, and $(q_1, q_2) = (x, y)$, possess a one-parameter group of symmetries given by

$$\begin{aligned} x' &= x\mathrm{e}^{s/2}, \\ y' &= y\mathrm{e}^{s/2} - \frac{1}{2}gt^2(\mathrm{e}^{2s} - \mathrm{e}^{s/2}), \quad t' = t\mathrm{e}^{s}. \end{aligned} \tag{22}$$

In fact, for this family of point transformations (treating s as an independent parameter),

$$\begin{aligned} \frac{\mathrm{d}x'}{\mathrm{d}t'} &= \frac{\mathrm{e}^{s/2}\mathrm{d}x}{\mathrm{e}^{s}\mathrm{d}t} = \mathrm{e}^{-s/2}\dot{x}, \\ \frac{\mathrm{d}y'}{\mathrm{d}t'} &= \frac{\mathrm{e}^{s/2}\mathrm{d}y - gt(\mathrm{e}^{2s} - \mathrm{e}^{s/2})\,\mathrm{d}t}{\mathrm{e}^{s}\mathrm{d}t} \\ &= \mathrm{e}^{-s/2}\dot{y} - gt(\mathrm{e}^{s} - \mathrm{e}^{-s/2}), \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{\mathrm{d}^2x'}{\mathrm{d}t'^2} &= \frac{\mathrm{d}(\mathrm{d}x'/\mathrm{d}t')}{\mathrm{d}t'} = \frac{\mathrm{e}^{-s/2}\mathrm{d}\dot{x}}{\mathrm{e}^{s}\mathrm{d}t} = \mathrm{e}^{-3s/2}\ddot{x}, \\ \frac{\mathrm{d}^2y'}{\mathrm{d}t'^2} &= \frac{\mathrm{d}(\mathrm{d}y'/\mathrm{d}t')}{\mathrm{d}t'} = \frac{\mathrm{e}^{-s/2}\mathrm{d}\dot{y} - g(\mathrm{e}^{s} - \mathrm{e}^{-s/2})\,\mathrm{d}t}{\mathrm{e}^{s}\mathrm{d}t} \\ &= \mathrm{e}^{-3s/2}\ddot{y} - g(1 - \mathrm{e}^{-3s/2}). \end{aligned}$$

On the other hand [see Eq. (18)],

$$E_1 = -m\ddot{x}, \qquad E_2 = -mg - m\ddot{y}, \tag{23}$$

hence, for instance, the right-hand side of Eq. (20) with $i = 2$ gives

$$\begin{aligned} &\left(\frac{\partial x'}{\partial y}\frac{\mathrm{d}t'}{\mathrm{d}t} - \frac{\partial t'}{\partial y}\frac{\mathrm{d}x'}{\mathrm{d}t}\right)(-m\ddot{x}') \\ &\quad + \left(\frac{\partial y'}{\partial y}\frac{\mathrm{d}t'}{\mathrm{d}t} - \frac{\partial t'}{\partial y}\frac{\mathrm{d}y'}{\mathrm{d}t}\right)(-mg - m\ddot{y}') \\ &\quad = \mathrm{e}^{s/2}\mathrm{e}^{s}\big\{-mg - m\big[\mathrm{e}^{-3s/2}\ddot{y} - g(1 - \mathrm{e}^{-3s/2})\big]\big\} \\ &\quad = -mg - m\ddot{y}, \end{aligned}$$

which coincides with E_2, and, in a similar manner, one finds that the equation with $i = 1$ also holds.

It can be readily verified that the point transformations (22) are not strict variational symmetries of the Lagrangian (21). In fact, they satisfy Eq. (4) with

$$F = (\mathrm{e}^{3s/2} - 1)mgt\left[-y + \frac{1}{6}gt^2(2\mathrm{e}^{3s/2} - 1)\right].$$

The one-parameter groups of symmetries of the Euler–Lagrange equations are more easily obtained by finding firstly their generators. Proceeding as in Ref. [6], we now consider a one-parameter family of symmetries of the Euler–Lagrange equations

$$q_i' = q_i'(q_1, \ldots, q_n, t, s), \qquad t' = t'(q_1, \ldots, q_n, t, s), \tag{24}$$

and we shall assume that $q_i'(q_1, \ldots, q_n, t, 0) = q_i$ and $t'(q_1, \ldots, q_n, t, 0) = t$. Taking the partial derivative of both sides of Eq. (20) with respect to s, at $s = 0$, using the chain rule, we obtain

$$\begin{aligned} 0 = \delta_{ij}\Bigg(&\frac{\partial E_j}{\partial q_k}\,\frac{\partial q_k'}{\partial s}\bigg|_{s=0} + \frac{\partial E_j}{\partial \dot{q}_k}\,\frac{\partial \dot{q}_k'}{\partial s}\bigg|_{s=0} + \frac{\partial E_j}{\partial \ddot{q}_k}\,\frac{\partial \ddot{q}_k'}{\partial s}\bigg|_{s=0} \\ &+ \frac{\partial E_j}{\partial t}\,\frac{\partial t'}{\partial s}\bigg|_{s=0}\Bigg) + \Bigg(\frac{\partial}{\partial s}\frac{\partial q_j'}{\partial q_i}\bigg|_{s=0} + \delta_{ij}\,\frac{\partial}{\partial s}\,\frac{\mathrm{d}t'}{\mathrm{d}t}\bigg|_{s=0}\Bigg)E_j, \end{aligned}$$

or, equivalently, with the aid of the standard definitions

$$\begin{aligned} \eta_i(q_j, t) &\equiv \frac{\partial q_i'(q_j, t, s)}{\partial s}\bigg|_{s=0}, \\ \xi(q_i, t) &\equiv \frac{\partial t'(q_i, t, s)}{\partial s}\bigg|_{s=0}, \end{aligned} \tag{25}$$

we have (see, *e.g.*, Refs. [4, 6])

$$\begin{aligned} \frac{\partial E_i}{\partial q_k}\eta_k &+ \frac{\partial E_i}{\partial \dot{q}_k}\left(\frac{\mathrm{d}\eta_k}{\mathrm{d}t} - \dot{q}_k\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) \\ &+ \frac{\partial E_i}{\partial \ddot{q}_k}\left(\frac{\mathrm{d}^2\eta_k}{\mathrm{d}t^2} - 2\ddot{q}_k\frac{\mathrm{d}\xi}{\mathrm{d}t} - \dot{q}_k\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\right) \\ &+ \frac{\partial E_i}{\partial t}\xi + E_j\frac{\partial \eta_j}{\partial q_i} + E_i\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0. \end{aligned} \tag{26}$$

(Equation (26) is derived in Ref. [8], Sec. 8.3, making use of the language of fibred manifolds and jet prolongations.)

As in the case of Eqs. (2) and (5), Eqs. (26) are PDEs for the $n + 1$ functions η_i and ξ. Any linear combination (with constant coefficients) of solutions of Eqs. (26) is also a solution of these equations, and the fact that Eqs. (26) have to hold for all values of $\dot{q}_k$ and $\ddot{q}_k$ (without imposing the equations of motion), leads to several conditions that in some cases are readily solved (see the example below). The main differences between Eqs. (26) and Eqs. (2) and (5) are that Eqs. (26) constitute a system of n PDEs, not a single equation when $n > 1$. Equations (26) determine the variational symmetries of a Lagrangian and all other Lagrangians equivalent to it, and it does not contain the unknown function G. However, in order to find the constant of motion associated with a given solution ξ, η_i of Eqs. (26), we have to make use of Eq. (5) (to obtain G) and then of Eq. (6). (Alternatively, the *differential* of this constant of motion is equal to the contraction of the vector field

$$\xi\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i} + \left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_i}$$

with the differential of the Cartan 1-form, defined below [Eq. (37)].)

2.2.2. *Example. Particle in a uniform gravitational field*

We will determine the generators of the point symmetries of the Euler–Lagrange equations (23), corresponding to the standard Lagrangian for a particle of mass m in a uniform gravitational field (21). Substituting Eqs. (23) into Eqs. (26) we obtain [with $(x, y) = (q_1, q_2)$]

$$\begin{aligned} &-m\left(\frac{\mathrm{d}^2\eta_1}{\mathrm{d}t^2} - 2\ddot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \dot{x}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\right) - m\ddot{x}\frac{\partial\eta_1}{\partial x} \\ &- (mg + m\ddot{y})\frac{\partial\eta_2}{\partial x} - m\ddot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0, \\ &-m\left(\frac{\mathrm{d}^2\eta_2}{\mathrm{d}t^2} - 2\ddot{y}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \dot{y}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\right) - m\ddot{x}\frac{\partial\eta_1}{\partial y} \\ &- (mg + m\ddot{y})\frac{\partial\eta_2}{\partial y} - (mg + m\ddot{y})\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0. \end{aligned}$$

Canceling the common factor $-m$ and writing more explicitly the derivatives $\mathrm{d}^2\eta_k/\mathrm{d}t^2$, we have

$$\begin{aligned} &\frac{\partial^2\eta_1}{\partial t^2} + 2\dot{q}_j\frac{\partial^2\eta_1}{\partial q_j\partial t} + \ddot{q}_j\frac{\partial\eta_1}{\partial q_j} + \dot{q}_j\dot{q}_k\frac{\partial^2\eta_1}{\partial q_j\partial q_k} - 2\ddot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t} \\ &\quad - \dot{x}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \ddot{x}\frac{\partial\eta_1}{\partial x} + (g + \ddot{y})\frac{\partial\eta_2}{\partial x} + \ddot{x}\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0, \\ &\frac{\partial^2\eta_2}{\partial t^2} + 2\dot{q}_j\frac{\partial^2\eta_2}{\partial q_j\partial t} + \ddot{q}_j\frac{\partial\eta_2}{\partial q_j} + \dot{q}_j\dot{q}_k\frac{\partial^2\eta_2}{\partial q_j\partial q_k} - 2\ddot{y}\frac{\mathrm{d}\xi}{\mathrm{d}t} - \dot{y}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} \\ &\quad + \ddot{x}\frac{\partial\eta_1}{\partial y} + (g + \ddot{y})\frac{\partial\eta_2}{\partial y} + (g + \ddot{y})\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0. \end{aligned} \tag{27}$$

Making use of the fact that the coefficients of $\ddot{x}$ and $\ddot{y}$ must be equal to zero we find that

$$2\frac{\partial\eta_1}{\partial x} - \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0, \quad \frac{\partial\eta_1}{\partial y} + \frac{\partial\eta_2}{\partial x} = 0, \quad 2\frac{\partial\eta_2}{\partial y} - \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0.$$

Since $\mathrm{d}\xi/\mathrm{d}t = \partial\xi/\partial t + \dot{q}_j\partial\xi/\partial q_j$, from the preceding equations it follows that $\partial\xi/\partial q_j = 0$, hence

$$\begin{aligned} &\xi = A(t), \qquad \eta_1 = \frac{x}{2}\frac{\mathrm{d}A}{\mathrm{d}t} + B_1(y, t), \\ &\eta_2 = \frac{y}{2}\frac{\mathrm{d}A}{\mathrm{d}t} + B_2(x, t), \end{aligned} \tag{28}$$

where $A(t)$, $B_1(y, t)$, and $B_2(x, t)$ are some functions, with

$$\frac{\partial B_1}{\partial y} + \frac{\partial B_2}{\partial x} = 0. \tag{29}$$

By considering the coefficients of the quadratic terms in the velocities in Eqs. (27), one finds that η_1 and η_2 must be polynomials of first degree in x and y: $B_1(y, t) = D_1(t)y + D_2(t)$, $B_2(x, t) = -D_1(t)x + D_3(t)$, where D_1, D_2, and D_3 are functions of a single variable, and we have taken into account the condition (29). The vanishing of the linear terms in the velocities in Eqs. (27) implies that D_1 is some constant. Finally, from the terms that do not contain $\ddot{q}_k$ or $\dot{q}_k$ one obtains the conditions

$$\frac{x}{2}\frac{\mathrm{d}^3A}{\mathrm{d}t^3} + \frac{\mathrm{d}^2D_2}{\mathrm{d}t^2} - gD_1 = 0, \quad \frac{y}{2}\frac{\mathrm{d}^3A}{\mathrm{d}t^3} + \frac{\mathrm{d}^2D_3}{\mathrm{d}t^2} + \frac{3g}{2}\frac{\mathrm{d}A}{\mathrm{d}t} = 0,$$

which imply that A is a polynomial of degree not greater than 2, and the solution of Eqs. (27) is given by

$$\begin{aligned} &\xi = c_3 + c_7 t + c_8 t^2, \\ &\eta_1 = c_1 + c_4 t + c_6\left(\frac{1}{2}gt^2 + y\right) + c_7\frac{x}{2} + c_8 xt, \\ &\eta_2 = c_2 + c_5 t - c_6 x \\ &\quad + c_7\left(\frac{y}{2} - \frac{3}{4}gt^2\right) + c_8\left(yt - \frac{1}{2}gt^3\right), \end{aligned} \tag{30}$$

where $c_1, \ldots, c_8$ are arbitrary real constants, which agrees with the solution obtained from Eq. (5) [6]. Indeed, substituting these functions ξ and η_i into Eq. (5) one readily finds that, up to a trivial constant, the corresponding function G is given by

$$\begin{aligned} &-c_2 mgt + c_4 mx + c_5 m\left(y - \frac{1}{2}gt^2\right) \\ &\quad + c_6 mgxt + c_7 m\left(-\frac{3}{2}gyt + \frac{1}{4}g^2t^3\right) \\ &\quad + c_8 m\left(-\frac{3}{2}gt^2 y + \frac{1}{2}(x^2 + y^2) + \frac{1}{8}g^2t^4\right). \end{aligned}$$

Then, making use of Eq. (6), one can calculate the associated constant of motion.

The fact that c_1 and c_3 do not appear in G means that if only the constants c_1 and c_3 are different from zero, the transformations generated are strict variational symmetries of L, which corresponds to the fact that x and t do not appear in L. (When only c_1 and c_3 are different from zero, the vector field $\xi\,\partial/\partial t + \eta_i\,\partial/\partial q_i$ reduces to $c_1\partial/\partial x + c_3\partial/\partial t$; $\partial/\partial x$ generates translations along the x-axis, and $\partial/\partial t$ generates time displacements.) The one-parameter group of point transformations (22) is generated by the vector field $\xi\,\partial/\partial t + \eta_i\,\partial/\partial q_i$ with $c_7 = 1$ and all the other constants c_k equal to zero.

The vector field $\partial/\partial y$ (obtained setting $c_2 = 1$, and all the other constants c_k equal to zero) generates translations along the y-axis, which are not strict variational symmetries of the Lagrangian (21), since in this case $G = -mgt$. With the aid of (8) we can readily obtain a Lagrangian, equivalent to (21), for which $\partial/\partial y$ generates strict variational symmetries. In view of Eq. (9), we need a function F such that

$$G + \left(\xi\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial q_i}\right)F = -mgt + \frac{\partial F}{\partial y} = 0.$$

Choosing $F = mgty$, from Eq. (7) we have

$$\begin{aligned} L_2 &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy + \frac{\mathrm{d}}{\mathrm{d}t}(mgty) \\ &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgt\dot{y}. \end{aligned}$$

Now y is ignorable, but L_2 depends explicitly on t.

2.2.3. Example. A singular Lagrangian

As a second example, we shall consider the singular Lagrangian

$$L = \dot{q}_1\dot{q}_3 - q_2\dot{q}_3 + q_1 q_3, \tag{31}$$

already studied in Ref. [10] and the references cited therein. In this case

$$E_1 = q_3 - \ddot{q}_3, \qquad E_2 = -\dot{q}_3, \qquad E_3 = q_1 - \ddot{q}_1 + \dot{q}_2.$$

The solution to Eqs. (26) for these functions E_i, given in Ref. [10], is

$$\xi = c_1, \qquad \eta_1 = -c_2 q_1 + c_3 \mathrm{e}^t + c_4 \mathrm{e}^{-t},$$
$$\eta_2 = -c_2 q_2 + b(q_3), \qquad \eta_3 = c_2 q_3, \tag{32}$$

where $c_1, \ldots, c_4$ are arbitrary real constants, and b is an *arbitrary real-valued function* of one variable. (We omit an unnecessary additive constant, denoted by C_3 in Ref. [10].) In fact, substituting Eqs. (31) and (32) into Eq. (5) we find that the left-hand side of Eq. (5) amounts to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[c_3 \mathrm{e}^t q_3 - c_4 \mathrm{e}^{-t} q_3 - \int^{q_3} b(u)\mathrm{d}u\right]. \tag{33}$$

Hence, only c_1 and c_2 are related to strict variational symmetries of the Lagrangian (31); c_1 is related with the obvious invariance of L under time displacements, while c_2 is related to a scaling symmetry ($q'_1 = q_1 \mathrm{e}^{-c_2 s}$, $q'_2 = q_2 \mathrm{e}^{-c_2 s}$, $q'_3 = q_3 \mathrm{e}^{c_2 s}$, $t' = t$).

Even though the Lagrangian (31) is singular, the general expressions given in the preceding sections are also applicable in this case. For instance, making use of Eqs. (6), (32), and (33) we find the constant of motion

$$\varphi = c_1(q_1 q_3 - \dot{q}_1\dot{q}_3) + c_2(q_3\dot{q}_1 - q_1\dot{q}_3 - q_2 q_3)$$
$$+ c_3 \mathrm{e}^t(\dot{q}_3 - q_3) + c_4 \mathrm{e}^{-t}(\dot{q}_3 + q_3) + \int^{q_3} b(u)\mathrm{d}u,$$

which differs from the expressions reported in Ref. [10]. (The expressions presented in Ref. [10] are the ones given by Eq. (3), which are valid only in the case of a strict variational symmetry.)

If we look for a Lagrangian equivalent to (31), for which $b(q_3)\,\partial/\partial q_2$ represents a strict variational symmetry, making use of Eq. (9), with $G = -\int^{q_3} b(u)\mathrm{d}u$ [see Eq. (33)], we need a function F such that

$$-\int^{q_3} b(u)\mathrm{d}u + b(q_3)\frac{\partial F}{\partial q_2} = 0$$

(note that this PDE is simpler than the system of second-order PDEs considered in Ref. [10]). Thus, we can choose, *e.g.*,

$$F = \frac{q_2}{b(q_3)} \int^{q_3} b(u)\mathrm{d}u,$$

which differs from the result found in Ref. [10] (the error in Ref. [10] was produced by the implicit assumption that the new Lagrangian is also independent of $\dot{q}_2$).

As a consequence of the fact that L is singular, its variational symmetries contain an arbitrary function, and the corresponding constants of motion are not useful because, by virtue of the equations of motion, $q_3 = 0$.

3. Hamiltonian formulation

As is well known, for a given system with Hamiltonian function $H(q_i, p_i, t)$, where q_i, p_i are (local) coordinates in the phase space, the coordinate transformation $Q_i = Q_i(q_j, p_j, t)$, $P_i = P_i(q_j, p_j, t)$ is canonical if there exists a function F such that

$$P_i \mathrm{d}Q_i - K\mathrm{d}t - (p_i \mathrm{d}q_i - H\mathrm{d}t) = \mathrm{d}F.$$

The function K is the Hamiltonian that determines the time evolution of the new coordinates Q_i, P_i (see, *e.g.*, Refs. [11, 12]). One can readily verify that if $T = T(q_j, p_j, t)$ is a new variable replacing t, then the existence of a function F such that

$$P_i \mathrm{d}Q_i - K\mathrm{d}T - (p_i \mathrm{d}q_i - H\mathrm{d}t) = \mathrm{d}F \tag{34}$$

assures that the Hamilton equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}, \tag{35}$$

are equivalent to

$$\frac{\mathrm{d}Q_i}{\mathrm{d}T} = \frac{\partial K}{\partial P_i}, \qquad \frac{\mathrm{d}P_i}{\mathrm{d}T} = -\frac{\partial K}{\partial Q_i}. \tag{36}$$

(For instance, this follows from the fact that the Hamilton equations determine the extremals of the integral $\int(p_i \mathrm{d}q_i - H\mathrm{d}t)$.) However, *the Poisson brackets are preserved only if T is a function of t exclusively* (this happens in the case of the family of transformations (22) and in the examples given above, since ξ is a function of t only). As in the case of the usual canonical transformations, the converse is not true, the equivalence of the Hamilton equations (35) and (36) *does not* imply the existence of a function F such that Eq. (34) holds [12].

According to the usual definitions of the canonical momenta and the Hamiltonian,

$$p_i \mathrm{d}q_i - H\mathrm{d}t = \frac{\partial L}{\partial \dot{q}_i}\mathrm{d}q_i - \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right)\mathrm{d}t$$
$$= L\mathrm{d}t + \frac{\partial L}{\partial \dot{q}_i}\big(\mathrm{d}q_i - \dot{q}_i \mathrm{d}t\big).$$

The linear differential form

$$\theta_L \equiv L\mathrm{d}t + \frac{\partial L}{\partial \dot{q}_i}\big(\mathrm{d}q_i - \dot{q}_i \mathrm{d}t\big), \tag{37}$$

is known as the Cartan 1-form (see, *e.g.*, Ref. [8] and the references cited therein). Making use of this definition we see that, given two equivalent Lagrangians L_1 and $L_2 = L_1 + \mathrm{d}F/\mathrm{d}t$,

$$\theta_{L_2} = \left(L_1 + \frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i} \right) \mathrm{d}t + \left(\frac{\partial L_1}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i} \right) \big(\mathrm{d}q_i - \dot{q}_i \mathrm{d}t \big) = \theta_{L_1} + \mathrm{d}F$$

and the condition (4) is equivalent to

$$L(q'_i, \frac{\mathrm{d}q'_i}{\mathrm{d}t'}, t')\, \mathrm{d}t' + \frac{\partial L}{\partial \dot{q}'_i} \big(\mathrm{d}q'_i - \dot{q}'_i \mathrm{d}t' \big) = L(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t)\, \mathrm{d}t + \frac{\partial L}{\partial \dot{q}_i} \big(\mathrm{d}q_i - \dot{q}_i \mathrm{d}t \big) + \mathrm{d}F.$$

Thus, if the Lagrangian is regular, a variational symmetry corresponds to a canonical transformation [in the sense of Eq. (34)].

In the standard Hamiltonian formulation one only considers transformations of the coordinates of the phase space, maintaining t unchanged and for many purposes this is enough. For instance, any constant of motion can be associated with a one-parameter group of canonical transformations, with t unchanged; in the case of the Hamilton–Jacobi method, one looks for a transformation to new coordinates which are constant of motion, but if a function is a constant of motion, its derivative with respect to any "time" coordinate will be equal to zero. Nevertheless, it seems interesting to explore the applications of more general transformations. By contrast, in the Lagrangian formalism, it is very useful to consider point transformations in which t is also transformed, as we can see in the examples given above and in Refs. [6, 8].

4. Concluding remarks

As a by-product of the derivations in this paper, we have found the transformation rules of the Euler–Lagrange equations under point transformations [Eq. (17)]. As pointed out above, Eqs. (26) constitute a convenient way to obtain all the variational symmetries of a given Lagrangian, because they do not contain the function G, present in Eq. (5).

As we have shown, when one considers point transformations in the phase space, there are two nonequivalent ways of defining a canonical transformation; the transformations induced by the variational symmetries of a Lagrangian obviously preserve the form of the Hamilton equations, but the Poisson brackets may not be preserved.

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