Disturbance Rejection Using SPR0 Substitutions

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Abstract. The disturbance rejection, defined as the problem of designing control laws that ensure, where possible, exogenous disturbances that do not affect the output of the perturbed system, has been resolved by means of algebraic and geometric techniques. This is a steady linear case by means of the static feedback state. Modifications of the Smith form through the SPR0 substitutions are presented which guarantees infinite zeros of a linear single input, single output (SISO) system.

Keywords. Algebraic technique, geometric technique, SPR0 perturbation, SISO.

1 Introduction

The disturbance rejection problem given the structural conditions is an interesting study concerning the robustness of such conditions in the presence of uncertainty in a mathematical system model.

In this paper a special case of uncertainty characterized in terms of strictly positive real substitutions with a relative zero degree is contemplated. Such substitutions correspond to non-linear uncorrelated uncertainty of system parameters under a single input, single output (SISO) system.

Under the action of a control law

\[ u(t) = F \cdot x(t) + G \cdot v(t) \]

with G invertible, the resulting system \( (A + B \cdot F, B \cdot G, C) \) is such that for an input \( v(t) = h \cdot e^{\omega t} \), with \( h \) and \( \omega \) finite constants, the initial conditions and the values of \( h \) can always be found such that the trajectory of the system with feedback \( x(t) \) is equal to \( x(0) \cdot e^{\omega t} \) for all \( t \geq 0 \), and the output is zero for all \( t \geq 0 \), \( \omega = 1 \), and \( \omega = 2 \). Then \( \{1, 2\} \) is the set of the “finite invariant zeros” of the system \( (A, B, C) \).

1.1 Zeros and Poles of a Stationary Linear System

A number \( \lambda \) (real or complex) is said to be a pole of a proper rational transfer function \( \hat{g}(s) \) if

\[ |\hat{g}(\lambda)| = \infty \].

A zero of \( \hat{g}(s) \) is such that

\[ |\hat{g}(\lambda)| = 0 \].

Polynomial roots of equations \( P(s) = 0 \) and \( Q(s) = 0 \) are called zeros and poles, respectively.

1.2 Notion of Zeros at Infinity

Let \( T(s) \in \mathbb{R} \), then if \( \lim_{s \to \infty} T(s) = \alpha \in \mathbb{R} \), \( t(s) \) is said to be proper; if \( \lim_{s \to \infty} T(s) = \infty \in \mathbb{R} \), then \( t(s) \) is said to be not proper. If \( \alpha = 0 \), \( t(s) \) is strictly proper [6].
1.3 Basic Subspaces

**Definition 1.** Let \((A, B, C)\) be a stationary linear system defined in (1):

\[
\begin{aligned}
\sum \begin{bmatrix} x(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t) \end{bmatrix},
\end{aligned}
\tag{1}
\]

where vector \(x(\bullet) \in \mathbb{R}^n\) denotes the state, vector \(u(\bullet) \in \mathbb{U} \subseteq \mathbb{R}^m\) denotes the input, and vector \(y(\bullet) \in \mathbb{Y} \approx \mathbb{R}^p\) denotes the output, with \(n, m, p\) positive integers elements. \(A: \mathbb{X} \rightarrow \mathbb{X}, B: \mathbb{U} \rightarrow \mathbb{X},\) and \(C: \mathbb{X} \rightarrow \mathbb{Y}\) are linear mappings represented by real constant matrices chosen from \(\mathbb{X}, \mathbb{U}, \mathbb{Y},\) and \(\mathbb{D}\.\) Then we have the following:

i. \(V(K)\) is the largest subspace \((A, B)\)-invariant containing a subspace \(K \subseteq \mathbb{X}\).

ii. \(S^*(B)\) is the smaller subspace \((C, A)\)-invariant containing a subspace \(B \subseteq \mathbb{X}\).

iii. \(R^*(K)\) is the largest subspace \((A, B)\)-controllability-invariant included in a subspace \(K \subseteq \mathbb{X}\).

iv. \(V(K)_{stab}\) denotes the largest internally stabilized subspace \((A, B)\)-invariant containing \(K \subseteq \mathbb{X}\).

**Remark 1.** A subspace \(V\) is \((A, B)\)-invariant if \(AV \subseteq V + B\).

**Remark 2.** A subspace \(S\) is \((C, A)\)-invariant if \(A(C \cap S) \subseteq S\).

**Remark 3.** A subspace \(R\) is \((A, B)\)-controllability-invariant if \(AR \subseteq R + B\) and furthermore \(\sigma(|R|A + BF|R|) = \Lambda\), where \(\Lambda\) is a symmetric set in the complex plane \(\mathbb{C}\) and \(\text{card}(\Lambda) = \dim(R)\). \(F: \mathbb{X} \rightarrow \mathbb{U}\) is any linear application such that \((A + BF) \subseteq R\). \(R|A + BF|R\) denotes the double restriction of \(R\) to \(A + BF\).

**Remark 4.** A subspace \(V_g\) is \((A, B)\)-invariant \(\mathbb{C}_g\) and internally stabilized if a linear mapping \(F: \mathbb{X} \rightarrow \mathbb{U}\) can be found such that \((A + BF) V_g(K) \subseteq V_g(K)\) and \(\sigma(V_g|A + BF|V_g) \subseteq \mathbb{C}_g\). \(R|A + BF|R\) denotes the double restriction of \(R\) to \(A + BF\).

Subspaces \(V^*(K), S^*(B)\) can be computed using the following algorithms [1]:

**Algorithm ISA**

\[
\begin{aligned}
V^0(K) &= X \\
V^\mu(K) &= K \cap A^{-1}\{V^\mu(K) + \text{Im } B\},
\end{aligned}
\tag{2}
\]

for \(\mu \geq 1\),

**Algorithm CISA**

\[
\begin{aligned}
S^0_K(Z) &= 0 \\
S^\mu_K(Z) &= Z + A\left(K \cap S^{\mu+1}_K\right),
\end{aligned}
\tag{3}
\]

for \(\mu \geq 1\), and on the other hand,

\[
R^*(K) = V^*(K) \cap S^*(B).
\tag{4}
\]

Concerning \(V(K)_{stab}\), this is defined as follows.

**Definition 2.** Let \(\beta(\lambda)\) be the minimal polynomial of \(A_0|V(K)/R(K)\), where \(A_0 = A + BF_0\). The factor \(\beta(\lambda) = \beta_0(\lambda) \cdot \beta_0(\lambda)\), where the zeros of \(\beta_0(\lambda)\) \(\beta_0(\lambda)\) in \(\mathbb{C}\) belongs to \(\mathbb{C}^- \cup \mathbb{C}^+\) and are defined as

\[
\begin{aligned}
X_-^* &= \left(\frac{V^*(K)}{R^*(K)}\right) \cap \text{Ker } \beta_0(A_0), \\
X_+^* &= \dim \left(\frac{V^*(K)}{R^*(K)}\right) \cap \text{Ker } \beta_0(A_0),
\end{aligned}
\tag{5}
\]

and

\[
V^*_{stab}(K) = P^{-1} \cdot X_+^*.
\tag{6}
\]

**Remark 5.** The subspace \(V(k)_{stab}\) is the largest member of the family \(B\) defined as
\[ B := \left\{ V : V \in S(K), \exists F \in F(V) \right\} \] \[ \sigma \left[ (A + B \cdot F) \mid V \right] \subseteq C^- \] \[ (7) \]

### 1.4 Cumulus to Infinity

The total sum of the infinite system \((A, B, C)\) is a positive integer number denoted as \(C_{\infty}(ABC)\) and called the Infinite Cumulus System \((A, B, C)\). In geometric terms [18]:

\[ C_\infty(A, B, C) := \sum n_i = \sum p_i = \dim \left( \frac{V^*(K) + S'(B)}{V_{stab}^*(K)} \right). \]

(8)

### 1.5 Unstable Cumulus

The total sum of the multiplicity of orders of unstable roots (i.e., roots located at \(-\infty\)) of the polynomials in \(\Sigma\) \((A, B, C)\) is called unstable cumulus and is given in [8] as

\[ C^+(A, B, C) := \dim \left( \frac{V^*(K)}{V_{stab}^*(K)} \right). \]

(9)

### 1.6 Total Cumulus

Total cumulus is the total sum of \((A, B, C)\) and the sum of their infinite cumulus and unstable cumulus. This is a positive integer denoted as \(C^*_\infty(A, B, C)\) and defined as

\[ C^*_\infty(A, B, C) := C_\infty(A, B, C) + C^+(A, B, C) \]

\[ = \dim \left( \frac{V^*(K) + S'(B)}{V_{stab}^*(K)} \right). \]

(10)

#### Example 1.

Consider a continuous stationary linear system \((A, B, C)\) given as

\[ \sum = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases}, \]

\[ (11) \]

where

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } [-1 & 1]. \]

(12)

Expression (9) is minimal (i.e., controllable and observable) and the corresponding transfer function is given as

\[ T(s) = \frac{s-1}{(s+1)^2}. \]

(13)

The relative degree system (11) is equal to 1, i.e., expression (10) has a zero at infinity of the order one. In addition, (10) has an instable zero of finite multiplicity equal to one in \(s = 1\). Therefore,

\[ C_\infty(T(s)) = 1, \]

\[ C^+(T(s)) = 1, \]

\[ C^*_\infty(T(s)) = C_\infty(T(s)) + C^+(T(s)) = 2. \]

(14)

### 2 SPR0 Substitutions

The \((A, B, C)\) stationary system is a linear time invariant system (LTI) with minimal representation (13), i.e., it is controllable and observable in the time domain:

\[ \sum = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t) \end{cases}, \]

\[ (15) \]

whose transfer function is defined as

\[ Y(s) = \left( D + C \cdot (sI - A)^{-1} \cdot B \right) \cdot U(s), \]

(16)

where \(X(s) \in \mathbb{R}^n; U(s), Y(s) \in \mathbb{R}^m\).

#### Definition 3.

Let \(RH\) be the Euclidean domain of real functions, proper and stable rational transfer. It has the following form:

\[ \left\| P(s) \right\|_\infty := \sup_{\omega} \left| P(j\omega) \right| \]

(17)
It is a subspace of $H_\infty$, which include open right half-plane and bounded analytic functions with the same standard space. The real number $\| P (s) \|_\infty$ is the $H_\infty$ norm of $P (s)$.

Let $P(s)= Np(s) / Dp(s)$ be a proper rational function, where $Np(s)$ and $Dp(s)$ are real polynomials. $P(s)$ has a relative degree zero, with $\deg(N_p(s)) = r$, $\deg(D_p(s)) = l$, and $r = l$:

$$\deg(N_p(s)) = \deg(D_p(s)).$$

Now we introduce a formal definition of a Strictly Positive Real Function of relative degree zero:

**Definition 4.** In agreement with [1] and [2], let $P(s)$ be a rational function with a relative degree equal to zero. It is SPR0 if and only if

1. $P(s)$ is analytic in $\text{Re}[s] \geq 0$,
2. $\text{Re}[P(j\omega)] > 0$ for all $\omega \in \mathbb{R}$.

The set of functions SPR0 is denoted by SPR0.

**Definition 5.** A rational function is in SPR0* if it is described by a limit of sequence functions that is in SPR0.

**Note 1.** A rational function $s \in \text{SPR0}^*$ can be interpreted as the limit of SPR0 sequence of functions:

$$\lim_{s \to 0} \frac{s + a}{as + 1} \to s,$$

where $a > 0$ and $a^2 \neq 1$.

**Example 2.** Let $Z(s)$ be described as

$$Z(s) = \frac{s^2 + 4s + 1}{s^2 + s + 2}.$$
4 Defining the Disturbance Rejection Problem

Consider a disturbed stationary linear system \((A, B, C, E)\) [10] described as

\[
\begin{cases}
(sI - A)x(s) = B \cdot u(s) + E \cdot d(s) \\
y(s) = C \cdot x(s)
\end{cases},
\]

where \(x(s)\), \(u(s)\), and \(y(s)\) denote the Laplace transforms of the state vectors \(x(\cdot) \in X \approx \mathbb{R}^n\), input \(u(\cdot) \in U \approx \mathbb{R}^m\), and output \(y(\cdot) \in Y \approx \mathbb{R}^p\), respectively, \(d(s)\) is the Laplace transform of the disturbance vector \(d(\cdot) \in D \approx \mathbb{R}^q\). \(A: X \rightarrow X\), \(B: U \rightarrow X\), and \(C: X \rightarrow Y\) are linear mappings (represented by constant real shades for freely chosen \(X, U, Y\), and \(D\)).

The Problem of Interference Rejection (PRB) is defined as follows.

**Definition 5.** Let a stationary linear system \((A, B, C, E)\) be perturbed. Find (if possible) a linear mapping \(F: X \rightarrow U\) such that

\[
\begin{cases}
1. & C \cdot (sI_n - (A + B \cdot F))^{-1} \cdot E \equiv 0, \\
2. & \sigma(A + B \cdot F) \subset \mathbb{C}^-
\end{cases}
\]

**Note 2.** The above definition states that the feedback \(u(s) = Fx(s)\) rejects the disturbance \(d(s)\) in the retrofitted system:

\[
\begin{cases}
(sI_n - (A + B \cdot F)) \cdot x(s) = E \cdot d(s) \\
y(s) = C \cdot x(s)
\end{cases}
\]

ensuring internal stability.

5 Geometric Solution

**Theorem 1.** Let \((A, B, C)\) be a fictitious system and \((A, [BE], C)\) be a combined system:

\[
\begin{cases}
\dot{x}(t) := A \cdot x(t) + [B \ E] \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}, \\
y(s) = C \cdot x(t)
\end{cases}
\]

i.e. \((A, B, C)\) corresponds to the fictitious system \((A, [BE], C)\) when \(d(t) = 0\), while \((A, [BE], C)\) results when \(d(t)\) is notionally regarded as an input control. Furthermore, from (23) we have

\[
y(s) = T(s) \cdot u(s) + T_d(s) \cdot d(s)
\]

with

\[
T(s) := C \cdot (sI - A)^{-1} \cdot B \quad \text{and} \quad T_d(s) := C \cdot (sI - A)^{-1} \cdot E.
\]

Now consider the control law

\[
u(t) = Fx(t),
\]

where \(F: X \rightarrow U\). From the system in (28) we obtain

\[
\begin{cases}
x(s) = (sI - A)^{-1} Bu(s) + (sI - A)^{-1} Ed(s) \\
u(s) = Fx(t)
\end{cases}
\]

and

\[
u(s) = C(s) \cdot d(s),
\]

where

\[
C(s) := \left(I - F(sI - A)^{-1} B\right)^{-1} F(sI - A)^{-1} E.
\]

**Note 3.** Compensator resulting control law in (28) with \(C(s)\) defined in (29) is strictly proper.

Then the Disturbance Rejection with Internal Stability (PRPEI) problem is defined in algebraic terms as follows.

**Definition 6.** The problem is to find an application \(F\) represented by a constant matrix selected for the involved base such that
\[ T_s(s) = T(s) - C(s) + T_e(s) = C \cdot (sI_n - (A + B \cdot F))^{-1} \cdot E = 0, \quad (32) \]

\[ \sigma(A + B \cdot F) \subseteq \mathbb{C}^+. \quad (33) \]

The geometric solution to the problem is as follows.

**Remark 5.** In agreement with [4], the Disturbance Rejection problem has a solution if and only if there is the largest sub-space \((A, B)\)-invariant contained in \(\text{ker}C\).

### 6 Structural Solution

Here we present a structural solution to the Disturbance Rejection problem without measuring the disturbance \([9]\). It is necessary to introduce two fictitious systems built from a perturbed system. The first modified system is

\[
\begin{cases}
\dot{x}(t) := \tilde{A} \cdot x(t) + \tilde{B} \cdot x(t) + \tilde{E} \cdot d(s) \\
y(s) = \tilde{C} \cdot x(t)
\end{cases}, \quad (34)
\]

where

\[
\tilde{A} = \begin{bmatrix} A & I_n \\ 0 & 0 \end{bmatrix}; \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} C \\ 0 \end{bmatrix}; \tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad (35)
\]

which are associated fictitious systems \((\tilde{A}, \tilde{B}, \tilde{C})\) and \((\tilde{A}, \tilde{B}, \tilde{C})\), whose respective transfer functions are

\[
\tilde{T}(s) = \tilde{C} \cdot (sI - \tilde{A})^{-1} \cdot \tilde{B} \quad (36)
\]

and

\[
\begin{bmatrix} \tilde{T}(s) \\ \tilde{T}_d(s) \end{bmatrix} = \tilde{C} \cdot (sI - \tilde{A})^{-1} \cdot \begin{bmatrix} \tilde{B} \\ \tilde{E} \end{bmatrix}. \quad (37)
\]

The following theorem provides necessary and sufficient conditions for solving PRPEI.

**Theorem 2.** In agreement with [5], PRPEI has a solution if and only if

\[
\text{rank} \left\{ \tilde{T}(s) \right\} = \text{rank} \left\{ \begin{bmatrix} \tilde{T}(s) & \tilde{T}_d(s) \end{bmatrix} \right\} \quad (38)
\]

and

\[
C_\infty^+ \left\{ \tilde{T}(s) \right\} = C_\infty^+ \left\{ \begin{bmatrix} \tilde{T}(s) & \tilde{T}_d(s) \end{bmatrix} \right\}. \quad (39)
\]

### 7 Numerical Example

Consider the following stationary linear single input, single output system described by

\[
\begin{cases}
\dot{x}(t) := A \cdot x(t) + B \cdot u(t) \\
y(s) = C \cdot x(t)
\end{cases} \quad (40)
\]

with

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [2 \ -3 \ 1]. \quad (41)
\]

Let us denote the inverse Laplace transform as \(\mathcal{Z}^{-1}\{\cdot\}\). Then the trajectory has

\[
x(t) = \mathcal{Z}^{-1}\{x(s)\} = \mathcal{Z}^{-1}\{(s \cdot I - A)^{-1} \cdot (x(0) + B \cdot u(s))\}. \quad (42)
\]

For an input of the form \(u(t) = g \cdot e^{\alpha t}, \ t \geq 0\) with \(g\) and \(\alpha\) finite constants, whose Laplace transform is

\[
u(t) = \frac{g}{s - \alpha}. \quad (43)
\]

Then it is transformed into
Now, we consider the case when $\alpha=1$ and $g=1$. It can be verified that for the vector of initial conditions $x(0) = \left( \begin{array}{c} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{array} \right)^T$ the system trajectory is

\[
x(t) = \mathcal{A}^t \left( \begin{array}{ccc} 1 & s^{-3} & s \\ 2 & s^2 & s \\ 2s & 3s^2 & s^3 \end{array} \right) \left( \begin{array}{c} x(0) \\ x_2(0) \\ x_3(0) + \frac{g}{s^{-\alpha}} \end{array} \right).
\]

(44)

and therefore the output is

\[
y(t) = C \cdot x(t) = \left( \begin{array}{c} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{array} \right) \cdot e^{t \cdot \mathcal{A}}.
\]

(45)

Similarly, for the case of $\alpha=2$, it can be verified that for $g=0$ and the vector of initial conditions $x(0) = (1 \ 2 \ 4)^T$, we have

\[
x(t) = \mathcal{A}^t \left( \begin{array}{ccc} 1 & s^{-3} & s \\ 2 & s^2 & s \\ 2s & 3s^2 & s^3 \end{array} \right) \left( \begin{array}{c} x(0) \\ x_2(0) \\ x_3(0) + \frac{g}{s^{-\alpha}} \end{array} \right).
\]

(46)

\[
y(t) = 0, \ \forall \ t \geq 0.
\]

(47)

It is can therefore be seen that for $\alpha=1$ and $\alpha=2$, there is a combination of the values of $g$ and $x(0)$ for which the trajectory of the system is equal to $x(0) \cdot e^{\alpha t}$ for all $t \geq 0$ and the output of the system is equal to zero for all $t \geq 0$, for an input of the form $u(t) = g \cdot e^{\alpha t}$. In fact, the values of $\alpha$ are the only ones for which this situation occurs.

8 Conclusions

In this work, the structural and geometric conditions that ensure the existence of at least one controller for a feedback static state which decouples the disturbance of a perturbed output system are presented. This guarantees that the $(A, B, C, D, E)$ system in closed loop has stability under the controller action.

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References


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