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INTERFACIAL FORCES ON A BUBBLE GROWING WITH ECCENTRIC CELL MODEL

FUERZAS INTERFACIALES EN LA EXPANSIÓN DE UNA BURBUJA CON UN MODELO DE CELDA EXCÉNTRICA

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Abstract

This article presents the development of a two-phase flow model in which the variation of the bubble radius due to changes in pressure was taken into account. The development considers expansion effects in the interaction forces between a dilute dispersion of gas bubbles and a continuous liquid phase. The closure relationships, associated with the spatial deviation around averaging variables, were formulated as functions of known variables. In order to solve the closure problem, as an approach of the heterogeneous structure of the two-phase flow, a geometric model given by an eccentric unit cell was applied. The obtained closure relationships include terms that represent combined effects from translation and pulsation due to displacement and size variation of the bubbles.

Keywords: potential flow, two-phase flow, Navier-Stokes equations, closure relationships.

Resumen

En este artículo se presenta el desarrollo de un modelo de flujo en dos fases que considera la variación del radio de burbuja debido a cambios en la presión. El desarrollo se realizo considerando efectos de expansión en las fuerzas de interacción que se presentan entre una dispersión diluida de burbujas de gas y una fase continua de líquido. Se formularon relaciones de cerradura, asociadas a las desviaciones espaciales alrededor de las variables promedio, como función de variables conocidas. Con el fin de resolver el problema de cerradura, como una aproximación de la estructura heterogénea del flujo en dos fases, se aplicó un modelo geométrico dado por una celda unitaria excéntrica. Las relaciones de cerradura obtenidas incluyen términos que representan efectos combinados de traslación y pulsación debido al desplazamiento y variación del radio de las burbujas.

Palabras clave: flujo potencial, flujo en dos fases, ecuaciones de Navier-Stokes, relaciones de cerradura.

1. Introduction

In this paper a two-phase flow model was developed to consider the phenomenological effects, due to size variations of the gas phase during large pressure changes of a system. Phenomena where the bubble size variations might be crucial, is the flow regime relaxation instability, which is caused by the pressure drop characteristics of the different flow regimes. As it has been pointed by Lahey and Podowski (1989), slug flow exhibits less pressure drop at the same gas and liquid flow rates than the bubble flow regime. Therefore, if a system is operating in the bubbly flow regime near the flow regime boundary, a small negative perturbation in liquid flow rate may cause a transition to slug flow and the channel pressure drop will tend to reduce, inducing more inlet liquid flow. This, may cause the system to revert back to bubble flow.

In previous works, in order to take into account the bubble radius variations due to expansion effects, Cheng *et al.* (1985) and Lahey (1992) assumed that a single bubble (each bubble being isolated from the others) is surrounded by an infinite liquid medium, and excited by sinusoidal pressure oscillations. The response of a stationary bubble is assumed to be spherically symmetric. The continuum assumption is valid for pressure excitations having wavelengths much greater than the bubble radius. Using these ideas, the closure relationships were developed by Espinosa-Paredes *et al.* (2004) based on the concentric cell approach, for a pulsating and translating bubbles.

In contrast with previous works, in the present paper the closure relationships were developed using

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an eccentric cell model considering interaction forces between liquid and bubbles due to radius variations by expansion effects. Expansion effects are very important in BWR and other industries related with steam generator.

2. Theoretical development

The problem under consideration is a dilute dispersion of non-interacting identical gas bubbles in a continuous liquid phase, moving through an isothermal, incompressible Newtonian fluid (illustrated in Fig. 1). This system is contained in a vertical duct with a much bigger diameter than the size of the individual bubbles. Interfacial mass transfer does not take place but interfacial momentum transfer is possible. The surface tension is considered constant. This study will be far away from the solid walls of the duct, so wall effects can be neglected.

In a previous study on two-phase flow (Cazarez-Candia, 2001), it was shown that the volume average equations, which can be obtained from the local Reynolds equations, are given by:

$$\underbrace{\frac{\partial\left(\varepsilon_k\left\langle\rho_k\right\rangle^k\right)}{\partial t}}_{accumulation}+\underbrace{\nabla\bullet\left(\varepsilon_k\left\langle\rho_k\right\rangle^k\left\langle\mathbf{v}_k\right\rangle^k\right)}_{convection}=0 \tag{1}$$

$$\begin{aligned}&\underbrace{\frac{\partial}{\partial t}\left(\varepsilon_k\left\langle\rho_k\right\rangle^k\left\langle\mathbf{v}_k\right\rangle^k\right)}_{accumulation}+\underbrace{\varepsilon_k\nabla\left\langle p_k\right\rangle^k}_{pressure}\\&+\underbrace{\nabla\bullet\left(\varepsilon_k\left\langle\rho_k\right\rangle^k\left\langle\mathbf{v}_k\right\rangle^k\left\langle\mathbf{v}_k\right\rangle^k\right)}_{convection}+\underbrace{\varepsilon_k\left\langle\rho_k\right\rangle^k\mathbf{g}}_{gravity}\\&=\underbrace{\left\langle\Delta p_{km}\right\rangle\nabla\varepsilon_k}_{\substack{difference\ of\ the\\ averages\ of\ pressure}}\ \underbrace{-\nabla\bullet\left(\varepsilon_k\left\langle\rho_k\right\rangle^k\left\langle\tilde{\mathbf{v}}_k\tilde{\mathbf{v}}_k\right\rangle^k\right)}_{dispersion}\\&+\underbrace{\mathbf{M}_{km}}_{\substack{interfacial\\ force}}\end{aligned} \tag{2}$$

where k and m ($k \neq m$) denotes either gas ($k = g$) or liquid ($k = l$) phases; ρ_k, $\mathbf{v}_k$, p_k, are the local variables in the k-phase representing density, velocity and pressure vectors, respectively, $\mathbf{g}$ is the gravity acceleration vector, and ε_k is the void fraction in the k-phase and $\langle\ \rangle^k$ indicates the intrinsic phase average. The equation of state is given by $\left\langle\rho_k\right\rangle^k=\left\langle\rho_k\right\rangle^k\left\langle p_k\right\rangle^k$. The aim of this work is to develop appropriate closure relationships, that include expansion effects, with an eccentric cell model for the terms $\left\langle\Delta p_{km}\right\rangle$, $\left\langle\tilde{\mathbf{v}}_k\tilde{\mathbf{v}}_k\right\rangle^k$ and $\mathbf{M}_{km}$ as functions of the averaged variables. The interfacial force per unit volume applied on the k-phase is defined by:

$$\mathbf{M}_{km}=-\frac{1}{\mathscr{V}}\int_{A_{km}}\mathbf{n}_{km}\,\tilde{p}_{km}\,dA+\frac{1}{\mathscr{V}}\int_{A_{km}}\mathbf{n}_{km}\cdot\mathbf{T}_{km}\,dA \tag{3}$$

where $\mathbf{n}_{km}$ is the unit normal vector at the interfacial pointing out of the k-phase and the second integral is the interfacial drag force, which was defined by Lahey (1991) as $\varepsilon_g\mathbf{F}_D$; where $\mathbf{F}_D$ is the interfacial drag force per unit volume. The first integral in Eq. (3) is given by Espinosa-Paredes (1998):

$$\begin{aligned}\frac{1}{\mathscr{V}}\int_{A_{lg}}\tilde{p}_{lg}\mathbf{n}_{lg}\,dA=&-\rho_l\nabla\cdot\left(\varepsilon_l\langle\tilde{\mathbf{v}}_l\tilde{\mathbf{v}}_l\rangle^l\right)\\&+\frac{1}{2}\rho_l\varepsilon_l\nabla\left[\varepsilon_l^{-1}\nabla\cdot\left(\varepsilon_l\langle\tilde{\phi}\tilde{\mathbf{v}}_l\rangle^l\right)\right]-\langle\Delta p_{lg}\rangle\nabla\varepsilon_g\\&-\rho_l\varepsilon_l\left[\frac{\partial\mathbf{m}}{\partial t}+\frac{1}{2}\nabla\left(\langle\mathbf{v}_l\rangle^l+\langle\mathbf{v}_g\rangle^g\right)\cdot\mathbf{m}\right]\end{aligned} \tag{4}$$

where

$$\mathbf{m}=\frac{1}{\varepsilon_l\mathscr{V}}\int_{A_{lg}}\tilde{\phi}\mathbf{n}_{lg}\,dA \tag{5}$$

Here $\tilde{\phi}$ represents the spatial deviations of the velocity potential, which is defined by Gray (1975) as:

$$\tilde{\phi}=\phi-\langle\phi\rangle^l \tag{6}$$

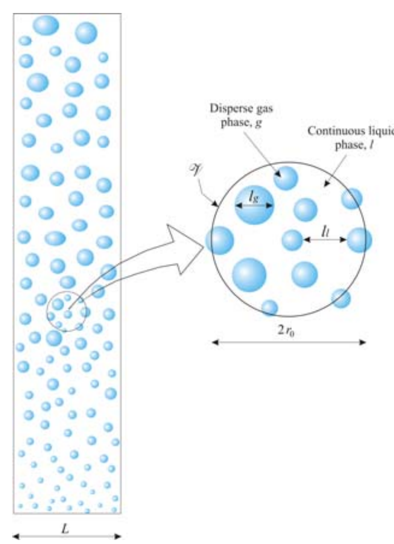

Fig. 1. Schematic diagram of the bubble flow system and averaging volume.

It should be pointed out that in Eqs. (4) and (5), spatial deviations of the velocity and potential for the liquid phase, are present. Then, it is necessary to establish a functional relationship for the spatial deviations with the average variables. In order to obtain these relationships, a geometric model known as unit cell model, was considered to approximate the structure of the two-phase flow, which has remained agglutinated within the integral in the averaging process.

2.1. Eccentric unit cell

Fig. 2 shows an eccentric unit cell, which is proposed in this work to solve the closure problem in terms of spatial deviation variables, which is briefly described in this section.

The radius of the spherical bubble in an eccentric cell with respect to the centroid is not constant as in the case of a concentric cell. In order to determine the eccentric radius r, an expression was developed as a function of the position of the spherical bubble with respect to the centroid of the cell:

$$r(\theta,\varphi) = -\left[g(\varphi)\sin\theta + l\cos\theta\right] + \sqrt{\left[g(\varphi)\sin\theta + l\cos\theta\right]^2 + b^2 - \gamma^2} \quad (7)$$

where b is the cell radius relative to the concentric coordinates, γ is the distance from the centroid of the cell to the centre of the bubble and $g(\varphi) = h\cos\varphi + k\sin\varphi$ (h, k, l represent a point relative to the centroid of the cell). It can be observed from this equation that the limit $\gamma \rightarrow 0$, the radius of the concentric cell is recovered, .i.e., b.

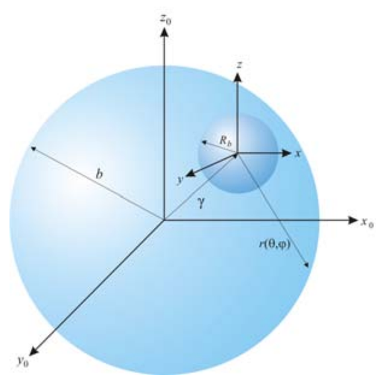

Fig. 2. Eccentric cell model. Bubble radius R_b, concentric cell radius b, distance from the centroid of the cell to the centre of the bubble γ, eccentric radius respect to the centre of the bubble $r(\theta,\varphi)$, coordinates of the concentric (x_0,y_0,z_0) and eccentric (x,y,z) cell.

3. Closure relationships

In order to obtain the averaging velocity potential, an eccentric cell model was applied. The process under consideration is illustrated in Fig. 2, where the bubble has a radius R_b and the cell is a sphere of radius b, thus $(R_b/b)^3$ is equal to the volume fraction in the cell.

Under the above considerations the velocity potential for flow around a pulsating and translating sphere is given by (Lahey, 1992):

$$\phi(r,\theta) = \underbrace{\frac{1}{2}\frac{R_b^3}{r^2}U\cos\theta}_{translating} - \underbrace{\frac{R_b^2\dot{R}_b e^{ik(r-R_b)}}{r(1-ikR_b)}}_{stationary\ pulsating\ bubble} \quad (8)$$

where r, and θ are the spherical co-ordinates, R_b is the sphere radio, k is the wave number ($k = \omega / C_l = 1/\lambda$; ω is the angular frequency, C_l, is the speed of sound in the liquid phase, λ is the wavelength), U is the sphere velocity, and $\dot{R}_b$ represents the radial velocity given by:

$$\dot{R}_b = \frac{d_g R_b}{dt} = \frac{\partial R_b}{\partial t} + \langle \mathbf{v}_g \rangle^g \cdot \nabla R_b \quad (9)$$

When a single bubble is surrounded by an infinite liquid medium, and excited by sinusoidal pressure excitations having wavelengths λ, which are much greater than the bubble radius, R_b, the terms of order $k R_b$, or higher, can be neglected in Eq. (8), resulting:

$$\phi(r,\theta) = \frac{1}{2}\frac{R_b^3}{r^2}U\cos\theta - \frac{R_b^2\dot{R}_b}{r} \quad (10)$$

3.1 Spatial deviations of the velocity potential

Substituting Eq. (10) into Eq. (5), we obtain:

$$\mathbf{m} = \frac{1}{\varepsilon_l \mathscr{V}}\int_{A_{lg}} \phi\,\mathbf{n}_{lg}\,dA + \frac{\langle\phi\rangle^l}{\varepsilon_l}\nabla\varepsilon_l \quad (11)$$

where $\langle\phi\rangle^l$ is constant with respect to the integral:

$$\frac{\langle\phi\rangle^l}{\varepsilon_l}\nabla\varepsilon_l = -\frac{\langle\phi\rangle^l}{\varepsilon_l}\left(\frac{1}{\mathscr{V}}\int_{A_{lg}} \mathbf{n}_{lg}\,dA\right) \quad (12)$$

The averaging velocity potential in spherical co-ordinates is given by:

$$\langle\phi\rangle^l = \frac{1}{V_l}\int_0^{2\pi}\int_0^{\pi}\int_{R_b}^{r(\theta,\varphi)} \phi(r,\theta)r^2 dr\sin\theta\, d\theta\, d\varphi \quad (13)$$

where $r^2 dr\sin\theta\, d\theta\, d\varphi$ is the differential element of volume dV, $V_l = 4/3\pi\left(b^3 - R_b^3\right)$ and $r(\theta,\varphi)$ is the eccentric radius given by Eq. (7), which depends on the bubble position within the cell (Fig. 2).

Substituting Eq. (10) into Eq. (13) and integrating, we obtained the following result

$$\langle\phi\rangle^l = \frac{1}{2}\frac{\varepsilon_g}{\varepsilon_l}\left[c_1 U - \left(c_2 R_b^{-1} + 3R_b\right)\dot{R}_b\right] \quad (14)$$

where

$$c_1 = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi} r(\theta,\varphi)\cos\theta\sin\theta\, d\theta\, d\varphi \quad (15)$$

$$c_2 = \left(\frac{4}{3}\pi\right)^{-1} \int_0^{2\pi} \int_0^{\pi} r^2(\theta,\varphi)\sin\theta d\theta d\varphi \qquad (16)$$

When the eccentricity γ is defined, c_1 and c_2 are constants. It is important to point out that, when the unit cell model is concentric: $\langle \phi \rangle^l = 0$. Substituting the Eqs. (8) and (14) into Eq. (6), we obtain the spatial deviations of the velocity potential which are given by

$$\tilde{\phi} = \frac{1}{2}\frac{R_b^3}{r^2(\theta,\varphi)}U\cos\theta - \frac{R_b^2}{r(\theta,\varphi)}\dot{R}_b - \frac{1}{2}\frac{\varepsilon_g}{\varepsilon_l}\left[c_1 U - \left(c_2 R_b^{-1} + 3R_b\right)\dot{R}_b\right] \qquad (17)$$

Then, the integral on the interfacial area of the spatial deviations of the velocity potential (Eq. (5)) is given by

$$\frac{\rho_l}{\mathscr{V}}\int_{A_{lg}} \tilde{\phi}\mathbf{n}_{lg} dA = \rho_l C_{IMC}(\varepsilon_g)\{U\mathbf{e}_z - \varepsilon_l^{-1}[c_1 U - (c_2 R_b^{-1} + 3R_b)\dot{R}_b]\nabla\varepsilon_g\} \qquad (18)$$

where the virtual mass coefficient to concentric cell is given by $C_{IMC}(\varepsilon_g) = \varepsilon_g/2$; the differential element of area is $dA = R_b^2 \sin\theta d\theta d\varphi$ and the unit vector is given by $\mathbf{n}_{lg} = -\mathbf{e}_r = \mathbf{e}_x \sin\theta\cos\varphi + \mathbf{e}_y \sin\theta \sin\varphi$.

It can be observed in Eq. (18) that the liquid density is included to analyze the physical interpretation. This equation represents the virtual mass effect, which includes in addition to the classical term (translation velocity), a new term: the stationary pulsates of the bubble. The gradient of the void fraction is due to eccentric cell model approach. When $\dot{R}_b = 0$, this result is simplified to $\rho_l C_{IMC}(\varepsilon_g) U(\mathbf{e}_z - \varepsilon_l^{-1} c_1 \nabla\varepsilon_g)$, which corresponds to an eccentric cell model without expansion effect. Now, when $\dot{R}_b = 0$, without expansion effects and for concentric cell model, the classical virtual mass term (Wallis, 1989): $\rho_l C_{IMC}(\varepsilon_g) U\mathbf{e}_z$ is obtained. According with Eq. (18), the Eq. (5) can be rewritten as:

$$\mathbf{m} = \varepsilon_l^{-1}[C_{IMT}(\varepsilon_g)\mathbf{U} + C_{IMP}(\varepsilon_g)\dot{R}_b\mathbf{e}_z] \qquad (19)$$

where $\mathbf{U} = U\,\mathbf{e}_z$ and the coefficients are given by

$$C_{IMT}(\varepsilon_g) = C_{IMC}(\varepsilon_g)\left(1 - \frac{c_1}{\varepsilon_l}\frac{\partial\varepsilon_g}{\partial z}\right) \qquad (20)$$

$$C_{IMP}(\varepsilon_g) = C_{IMC}(\varepsilon_g)\varepsilon_l^{-1}(c_2 R_b^{-1} + 3R_b)\frac{\partial\varepsilon_g}{\partial z} \qquad (21)$$

Results in literature have been expressed in the form $C_{IM} = C_{IMC}(1 - \alpha\varepsilon_g)$ (Zuber, 1964; van Wijngaarden, 1976; Geurst, 1985; Biesheuvel and Spoelstra, 1989), where α is a constant that depends on the details of the pair wise interaction of the bubbles and $C_{IMC} = 1/2\rho_l\varepsilon_g$. In the absence of interactions, the virtual mass of each sphere is $1/2\rho_l\varepsilon_g$.

3.2 Spatial deviations of the velocity

The spatial deviations of the velocity are defined by

$$\tilde{\mathbf{v}}_l = \mathbf{v}_l - \langle \mathbf{v}_l \rangle^l \qquad (22)$$

where

$$\mathbf{v}_l = -\nabla\phi = \mathbf{e}_r \frac{\partial\phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad (23)$$

where $\mathbf{e}_r$ was defined previously and $\mathbf{e}_\theta = \mathbf{e}_x \cos\theta \cos\varphi + \mathbf{e}_y \cos\theta \sin\varphi - \mathbf{e}_z \sin\theta$. Then,

$$\mathbf{v}_l = -\frac{UR_b^3}{2r^3}\left(3\mathbf{e}_r\cos\theta - \mathbf{e}_z\right) + \frac{R_b^2\dot{R}_b}{r^2}\mathbf{e}_r \qquad (24)$$

The intrinsic average of the velocity is given by

$$\langle \mathbf{v}_l \rangle^l = \frac{1}{V_l}\int_0^{2\pi}\int_0^{\pi}\int_{R_b}^{r(\theta,\varphi)} \mathbf{v}_l\, r^2 dr \sin\theta d\theta d\varphi \qquad (25)$$

Substituting the Eq. (24) in the Eq. (25) and integrating it the following result is obtained

$$\langle \mathbf{v}_l \rangle^l = -\frac{\varepsilon_g}{\varepsilon_l}\left(\frac{1}{2}U\mathbf{a} - R_b^{-1}\dot{R}_b\mathbf{b}\right) \qquad (26)$$

where

$$\mathbf{a} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi} \ln\left[r(\theta,\varphi)\right]\left(3\mathbf{e}_r\cos\theta - \mathbf{e}_z\right)\sin\theta d\theta d\varphi \qquad (27)$$

$$\mathbf{b} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi} r(\theta,\varphi)\mathbf{e}_r \sin\theta d\theta d\varphi \qquad (28)$$

Substituting Eqs. (24) and (25), into Eq. (22) the final form of the spatial deviations of the velocity is obtained:

$$\tilde{\mathbf{v}}_l = -\frac{UR_b^3}{2r^3}\left(3\mathbf{e}_r\cos\theta - \mathbf{e}_z\right) + \frac{R_b^2\dot{R}_b}{r^2}\mathbf{e}_r + \frac{\varepsilon_g}{\varepsilon_l}\left(\frac{1}{2}U\mathbf{a} - R_b^{-1}\dot{R}_b\mathbf{b}\right) \qquad (29)$$

The third term on the right hand side of this equation is identified as eccentric effects, since it involves values of **a** and **b** that are evaluated on a point that is not located at the centroid of the cell.

3.3 Average of products of deviation

In Eq. (4) it can be observed, that the first and second terms of the right side contain product averages of variables between spatial deviations, which were determined in this section.

The intrinsic average of the spatial deviations of the velocity potential with the spatial deviations of the velocity can be expressed in the following form

$$\langle\tilde{\phi}\tilde{\mathbf{v}}_l\rangle^l = \langle\phi\mathbf{v}_l\rangle^l - \langle\phi\rangle^l\langle\mathbf{v}_l\rangle^l \quad (30)$$

where $\langle\phi\rangle^l$ and $\langle\mathbf{v}_l\rangle^l$ are given by Eqs. (14) and (26), respectively, but it is necessary to calculate the first term of this equation. The calculation leading to the $\langle\mathbf{v}_l\rangle^l$ can be repeated for $\langle\phi\mathbf{v}_l\rangle^l$. Then:

$$\langle\phi\mathbf{v}_l\rangle^l = \frac{\varepsilon_g}{\varepsilon_l}\left[\frac{1}{8}R_b^3\mathbf{c}U^2 - \frac{1}{2}R_b^2(\mathbf{e}+7\mathbf{e}_z)U\dot{R}_b - R_b\mathbf{d}(\dot{R}_b)^2\right] \quad (31)$$

where

$$\mathbf{c} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\frac{(3\mathbf{e}_r\cos\theta - \mathbf{e}_z)\cos\theta\sin\theta}{r^2(\theta,\varphi)}d\theta d\varphi \quad (32)$$

$$\mathbf{d} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\ln[r(\theta,\varphi)]\mathbf{e}_r\sin\theta d\theta d\varphi \quad (33)$$

$$\mathbf{e} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\frac{(4\mathbf{e}_r\cos\theta - \mathbf{e}_z)\sin\theta}{r(\theta,\varphi)}d\theta d\varphi \quad (34)$$

Substituting Eqs. (14), (26) and (31) into Eq. (30) we obtained:

$$\langle\tilde{\phi}\tilde{\mathbf{v}}_l\rangle^l = \frac{\varepsilon_g}{\varepsilon_l}\left[\frac{1}{4}\mathbf{f}U^2 - \frac{1}{2}(7\mathbf{e}_zR_b^2 + \mathbf{h})U\dot{R}_b + \mathbf{q}\dot{R}_b^2\right] \quad (35)$$

where

$$\mathbf{f} = \frac{1}{2}R_b^3\mathbf{c} + \frac{\varepsilon_g}{\varepsilon_l}c_1\mathbf{a} \quad (36)$$

$$\mathbf{h} = \frac{1}{2}\left(\frac{\varepsilon_g}{\varepsilon_l}c_2R_b^{-1} + 3R_b\right)\mathbf{a} + \frac{\varepsilon_g}{\varepsilon_l}c_1\mathbf{b}R_b^{-1} + \mathbf{e}R_b^2 \quad (37)$$

$$\mathbf{q} = \left(\frac{1}{2}\frac{\varepsilon_g}{\varepsilon_l}c_2R_b^{-2} + 3\right)\mathbf{b} - R_b\mathbf{d} \quad (38)$$

When the unit cell model is concentric, this result can be simplified to $\langle\tilde{\phi}\tilde{\mathbf{v}}_l\rangle^l = -7\mathbf{e}_z\varepsilon_l^{-1}\varepsilon_gR_b^2U\dot{R}_b/2$. Now, the intrinsic average of the dyad of the spatial deviations can be decomposed in two terms:

$$\langle\tilde{\mathbf{v}}_l\tilde{\mathbf{v}}_l\rangle^l = \langle\mathbf{v}_l\mathbf{v}_l\rangle^l - \langle\mathbf{v}_l\rangle^l\langle\mathbf{v}_l\rangle^l \quad (39)$$

where $\langle\mathbf{v}_l\rangle^l$ is given by Eq. (26). The procedure leading to $\langle\phi\mathbf{v}_l\rangle^l$ can be repeated for $\langle\mathbf{v}_l\mathbf{v}_l\rangle^l$. Then,

$$\langle\tilde{\mathbf{v}}_l\tilde{\mathbf{v}}_l\rangle^l = \frac{\varepsilon_g}{\varepsilon_l}\left[(\mathbf{K}_c+\mathbf{K}_e)U^2 + \mathbf{K}_{Ub}U\dot{R}_b + (\mathbf{K}_{cb}+\mathbf{K}_{eb})\dot{R}_b^2\right] \quad (40)$$

where

$$\mathbf{K}_c = \frac{1}{5}\left(\frac{3}{4}\mathbf{e}_x\mathbf{e}_x + \frac{3}{4}\mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z\right) \quad (41)$$

$$\mathbf{K}_e = -\frac{1}{12}R_b^3\mathbf{A} - \frac{1}{4}\frac{\varepsilon_g}{\varepsilon_l}\mathbf{aa} \quad (42)$$

$$\mathbf{K}_{Ub} = \frac{1}{2}R_b^2\mathbf{B} + \frac{\varepsilon_g}{\varepsilon_l}R_b^{-1}\mathbf{ab} \quad (43)$$

$$\mathbf{K}_{cb} = \mathbf{I} \quad (44)$$

$$\mathbf{K}_{eb} = -R_b\mathbf{C} - \frac{\varepsilon_g}{\varepsilon_l}R_b^{-2}\mathbf{bb} \quad (45)$$

$$\mathbf{A} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\left[\frac{9\mathbf{e}_r\mathbf{e}_r\cos^2\theta}{r^3(\theta,\varphi)} - \frac{6\mathbf{e}_r\mathbf{e}_z\cos\theta + \mathbf{e}_z\mathbf{e}_z}{r^3(\theta,\varphi)}\right]\sin\theta d\theta d\varphi \quad (46)$$

$$\mathbf{B} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\left[\frac{3\mathbf{e}_r\mathbf{e}_r\cos\theta}{r^2(\theta,\varphi)} - \frac{\mathbf{e}_r\mathbf{e}_z}{r^2(\theta,\varphi)}\right]\sin\theta d\theta d\varphi \quad (47)$$

$$\mathbf{C} = \left(\frac{4}{3}\pi\right)^{-1}\int_0^{2\pi}\int_0^{\pi}\frac{\mathbf{e}_r\mathbf{e}_r}{r(\theta,\varphi)}\sin\theta d\theta d\varphi \quad (48)$$

The classic second-order tensor $\mathbf{K}_c$ was re-defined by Wallis (1989) for single rigid spheres in a fluid, which was obtained with the potential for the motion of a sphere with a radius R_b moving with speed U, in an infinite medium with spherical coordinates. In this equation we identified that $(\mathbf{K}_c+\mathbf{K}_e)U^2$ is due to translation effects on the sphere, while $\mathbf{K}_{Ub}U\dot{R}_b$ is due to combined effects of translation and pulsation, and $(\mathbf{K}_{cb}+\mathbf{K}_{eb})\dot{R}_b^2$ considers only pulsating effects.

3.4 Difference between the intrinsic and interface averages of the pressure

Considering spherical bubbles, the interfacial average pressure is given by

$$\langle p_l\rangle_{lg} = \frac{1}{4\pi R_b^2}\int_0^{2\pi}\int_0^{\pi}p_{lg}R_b^2\sin\theta d\theta d\varphi \quad (49)$$

where $4\pi R_b^2$ is the superficial area of the spherical bubble A $R_b^2\sin\theta d\theta d\varphi$ is the differential element of area dA and p_{lg} is the interfacial local pressure. To integrate the previous equation, it is necessary to calculate first p_{lg}. Then, the well known Bernoulli's equation (Currie, 1974), can be rewritten as:

$$\frac{p_l}{\rho_l} - \frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi = \frac{\langle p_l \rangle^l}{\rho_l} \quad (50)$$

where $f(t) = \langle p_l \rangle^l / \rho_l$ was used. Now substituting Eq. (10) and evaluating at the surface ($r = R_b$) R_b), the following result is obtained

$$\frac{p_{lg}}{\rho_l} = \frac{\langle p_l \rangle^l}{\rho_l} - R_b \ddot{R}_b - \frac{3}{2}\dot{R}_b^2 + \frac{1}{2}\left[R_b \frac{dU}{dt}\cos\theta + (3\cos^2\theta - 1)U^2 - \frac{1}{4}(3\cos^2\theta + 1)U^2 \right] \quad (51)$$

Here $p_{lg} = p_l|_{r=R_b}$. Then, substituting Eq. (51) into Eq. (49), and after integration yields

$$\langle \Delta p_{lg} \rangle = \langle p_l \rangle_{lg} - \langle p_l \rangle^l = -\frac{1}{4}\rho_l U^2 + \rho_l(\frac{3}{2}\dot{R}_b^2 + R_b \ddot{R}_b) \quad (52)$$

The closure of Eq. (2) now can be obtained by substitution of Eq. (52) for $\langle \Delta p_{km} \rangle$, Eq. (35) for $\langle \tilde{\phi}\tilde{\mathbf{v}}_l \rangle^l$ and Eq. (40) for $\langle \tilde{\mathbf{v}}_k \tilde{\mathbf{v}}_k \rangle^k$. The momentum transfer between the gas and liquid phases for each phase is given by Eq. (3) and can be written (Cheng *et al.*, 1985): $\mathbf{M}_{lg} = -\mathbf{M}_{gl}$.

The closed set of averaged equations must provide enough information for the estimation of the global phenomenological behavior in bubbly gas-liquid flows in terms of averaged variables.

Conclusions

The closure relationships using an eccentric cell model and the potential flow around a translating spherical bubble with radius variations was developed. The results show that the closure problem is a function of the eccentricity, which is defined by a parameter γ through Eq. (7).

The virtual mass effect given by Eq. (19) includes two coefficients: the first coefficient (Eq. 20) is due to translating effects, while the second coefficient (Eq. 21) considers radial effects, both coefficients includes the gradient of the void fraction, which change the structure of the non-linear differential averaged equation. The closure for $\langle \tilde{\phi}\tilde{\mathbf{v}}_l \rangle^l$ (Eq. 35) is a vector made for three terms: translating, pulsating and translating-pulsating effects. We found that the closure $\langle \tilde{\mathbf{v}}_l \tilde{\mathbf{v}}_l \rangle^l$ (Eq. 40) includes translating, pulsating and combined of translating and pulsating effects on the sphere.

An eccentric cell model allows approaching the asymmetric effects of a bubble moving in a continuous fluid. Most of the models consider that the bubble is a sphere due that the potential flow theory around a spherical object is relatively straightforward. Nevertheless, the sphericity of the bubble occurs when the internal and external forces are in equilibrium and in general, this happens for low void fractions and small bubbles. However, in most industrial processes and experimental studies, the bubbles are not spherical, and then the eccentric cell model allows approaching the non-spherical effects by means of the eccentricity parameters that were theoretically obtained in this article.

Nomenclature

$\mathbf{a}$	vector defined in Eq. (27)
A_{km}	interfacial area contained in $\mathscr{V}$
$\mathbf{A}$	defined in Eq. (46)
b	cell radius respect to concentric coordinates
$\mathbf{b}$	vector defined in Eq. (28)
$\mathbf{B}$	tensor defined in Eq. (47)
$\mathbf{c}$	vector defined in Eq. (32)
C_{IMC}	virtual mass coefficient to concentric cell (= ε_g / 2)
$\mathbf{C}$	tensor defined in Eq. (48)
$\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$	vectors defined in Eqs. (33), (34) and (36), respectively
$\mathbf{g}$	gravity acceleration vector
$\mathbf{h}$	vector defined in Eq. (37)
$\mathbf{I}$	unit tensor
$\mathbf{K}$	tensor
l_g	characteristic length of disperse phase
l_l	characteristic length of continuous phase
$\mathbf{m}$	vector defined in Eq. (5)
$\mathbf{M}_{km}$	interfacial force per unit volume applied on phase *k*
$\mathbf{n}_{km}$	unit vector at the interface point out of phase *k*
p_k	local pressure of phase *k*
$\tilde{p}_k$	spatial deviation of pressure of phase *k*
$\langle p_k \rangle_{km}$	interfacial average pressure
$\langle p_k \rangle^k$	intrinsic average pressure of phase *k*
r_0	characteristic length of average volume
R_b	bubble radius
$\dot{R}_b$	defined in Eq. (9)
$\mathbf{v}_k$	velocity vector of phase *k*
$\tilde{\mathbf{v}}_k$	spatial deviation of velocity vector of phase *k*
$\langle \mathbf{v}_k \rangle^k$	intrinsic average velocity vector of phase *k*
$\mathscr{V}$	averaging volume
V_k	volume of the *k*-phase contained in $\mathscr{V}$

Greek symbols

$\langle \Delta p_{km} \rangle$	defined in Eq. (52)
ε_κ	volume fraction of phase *k*
γ	cell radius respect to eccentric coordinates
ϕ	local potential
$\tilde{\phi}$	spatial deviations of potential
$\langle \phi \rangle^l$	average potential
ρ_k	density of phase *k*

Subscripts

c	concentric
e	eccentric
g	disperse phase
k	*g* or *l* ($k \neq m$)
l	continuous phase
m	*g* or *l* ($m \neq k$)

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