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Espacio del Divulgador

# One-qubit purity in terms of the discrete Wigner transform

Pureza de un qubit en términos de la transformada discreta de Wigner

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## Abstract:

An explanation and an illustration of the meaning of a discrete phase-space is given. The class of a discrete Wigner transform (DWT) for the specific case of a one-qubit state is introduced. We derive the one-qubit state formalism around its formulation in terms of the DWT in detail. A novel structure of a one-qubit purity in terms of the DWT is introduced. We find a criterion for stating when a one-qubit state is either *mixed* or *pure*.

Keywords: Discrete phase space, Hilbert space, unbiased bases, discrete Wigner transform, purity.

## Resumen:

Se proporciona e ilustra una explicación del significado de espacio fase discreto dirigida a un lector no especialista. Se presenta también la clase de la transformada discreta de Wigner (TDW) para el caso específico de un estado de un *qubit*. Asimismo, derivamos detalladamente el formalismo involucrado en la formulación del estado de un *qubit* en términos de la TDW. En este contexto, se introduce una estructura novedosa de la pureza de un *qubit* en términos de la TDW y se halla un criterio para decidir cuando el estado de un *qubit* es puro o mixto.

Palabras clave: espacio fase discreto, espacio de Hilbert, bases imparciales, transformada discreta de Wigner, pureza.

## Introduction

The real-valued Wigner function $W(q, p)$ play the role of a quasi-probability distribution for continuous-variable quantum systems in continuous coordinates ($q$) versus momentum ($p$) (phase) space (Wigner, 1932; Hillary *et a*l., 1984). In spite of the fact that $W( q , p)$ allows us to calculate properties of a system through phase-space integrals weighted by it, however this cannot be interpreted as the positive-valued probability of simultaneously measuring observables $\hat{p}$ and $\hat{q}$ with eigenvalues $p_0$ and $q_0$. In fact, $W(q, p)$ could be negative in some phase-space regions (from there the term quasi-probability).

Buot (1974), Hannay and Berry (1980) were the first to propose the novel idea of the analogous Wigner function for a discrete (finite-dimensional) Hilbert space. Later on, such findings were rediscovered by Cohen and Scully (1986) and Feynman (1987) who defined a discrete Wigner function $W$ for a single qubit. The above works were extended by Wootters (1987) and Galetti and De Toledo Piza (1988) by introducing a Wigner function for prime-dimensional Hilbert spaces.

These extensions have been employed for teleportation protocols (Koniorczyk *et al.*, 2001; Paz, 2002), quantum algorithms (Bianucci *et al.*, 2002; Miquel *et al.*, 2002), and decoherence (Lopez, 2003).

There is scarce (almost null) information in the literature about the concept of purity of a single qubit from the point of view of the discrete Wigner transform. Within such a formalism it is difficult to find a criteria for stating whether a qubit state is pure. In the present paper we discuss both the discrete Wigner function for a single qubit and introduce the concept of purity for it.

## 1. One-qubit in terms of a discrete Wigner functio

The conventional definition of continuous phase space is a plane where the horizontal axis denotes the continuous position coordinate $q$ while the vertical axis represents the continuous linear momenta variable $p$. At this stage it arises the following question: what does a discrete phase space mean? By discrete phase space (DPS) we mean that the coordinate variable $q$ takes a finite set of real values $Q = \{q_1, q_2, \ldots, q_n\}$ while the momenta variable takes also a finite set of real values $P = \{p_1, p_2, \ldots, p_n\}$ in such a way that the DPS is the set $\{(q_i, p_j) | qi \in Q, p_j \in P\}$. We can state that the discrete analogue of phase space in a $d$-dimensional Hilbert space is a $d \times d$ real array. Thus, for the case of one qubit one has $d = 2$. Consequently a 2-dimensional Hilbert discrete phase space can be represented by the following four real numbers denoted by:

$$\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{matrix} \tag{1}$$

In the continuous phase space (continuous $XY$-plane) we can draw continuous lines while in the discrete case this is not the case. We define a "line" in the $d$-dimensional discrete phase space as a set of $d$ points in discrete phase space. Thus, a "line" in the case of a 2-dimensional discrete phase space is a set of 2 points in the discrete phase space of Eq. (1). The discrete phase space of Eq. (1) will contain the following six "lines"

$$\begin{aligned} \alpha_{1,1} &= \{x_{11}, x_{21}\}, \\ \alpha_{1,2} &= \{x_{11}, x_{22}\}, \\ \alpha_{1,3} &= \{x_{11}, x_{12}\}, \\ \\ \alpha_{2,1} &= \{x_{12}, x_{21}\}, \\ \alpha_{2,2} &= \{x_{12}, x_{22}\}, \\ \alpha_{2,3} &= \{x_{21}, x_{22}\}. \end{aligned} \tag{2}$$

The discrete phase space can then be partitioned into a collection of parallel lines. By a parallel line it is understood disjoint sets of $d = 2$ phase space points. Such partitions are called striations (Wootters, 1987). According to Eq. (2), in the present case there will be the following $d + 1 = 2 + 1 = 3$ striations

$$\begin{aligned} S_1 &= \{\alpha_{1,1}, \alpha_{2,2}\}, \\ S_2 &= \{\alpha_{1,2}, \alpha_{2,1}\}, \\ S_3 &= \{\alpha_{1,3}, \alpha_{2,3}\}. \end{aligned} \tag{3}$$

Wootters' definition of discrete Wigner functions employs a special set of $d + 1 = 2 + 1 = 3$ bases for a $d$-dimensional Hilbert space. Such bases can be defined in terms of eigen-states of generalized Pauli operators. In this way, for the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the respective eigen-states should be:

$$\begin{aligned} |e_{x1}\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ |e_{x2}\rangle &= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned} \tag{4}$$

where we have employed the matrix notation $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the aboveequation by eigenstates of the Pauli operator $\sigma_x$ we mean the following where we have used that $\sigma_x |0\rangle = |1\rangle$ and $\sigma_x |1\rangle = |0\rangle$.

For the Pauli matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ the eigen-states are:

$$
\begin{aligned}
|e_{y1}\rangle &= \frac{1}{\sqrt{2}}\,(i|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \\
|e_{y2}\rangle &= \frac{1}{\sqrt{2}}\,(-i\,|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix},
\end{aligned} \tag{5}
$$

Clearly, $\sigma_y|e_{y1}\rangle = -|e_{y1}\rangle$ and $\sigma_y|e_{y2}\rangle = |e_{y2}\rangle$ .

While for $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the respective eigen-states will be:

$$
\begin{aligned}
|e_{z1}\rangle &= |0\rangle, \\
|e_{z2}\rangle &= |1\rangle.
\end{aligned} \tag{6}
$$

Clearly $\sigma_z|e_{z1}\rangle = |e_{z1}\rangle$ and $\sigma_z|e_{z2}\rangle = -|e_{z2}\rangle$

A set $B = \{|e_1\rangle, |e_2\rangle\}$ is a basis for a Hilbert space if any state $|\varphi\rangle$ of the space can be written as $|\varphi\rangle = a|e_1\rangle + b|e_2\rangle$ where $a$ and $b$ are complex numbers such that $|a|^2 + |b|^2 = 1$. The basis $B$ is orthonormal if $\langle e_i|e_j\rangle = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 0 \ \mathit{if}\ i \neq j \\ 1 \ \mathit{if}\ i = j \end{cases}$. Let us observe that $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

The states given by (5), (6), (7) define the following $d + 1 = 2 + 1 = 3$ different bases

$$
\begin{aligned}
B_x &= \{|e_{x1}\rangle, |e_{x2}\rangle\}, \\
B_y &= \{|e_{y1}\rangle, |e_{y2}\rangle\}, \\
B_z &= \{|e_{z1}\rangle, |e_{z2}\rangle\}.
\end{aligned} \tag{7}
$$

We must observe that there is a one-to-one correspondence between the striations of Eq. (3) and the bases of Eq. (7), that is

$$
\begin{aligned}
S_1 &\to B_x, \\
S_2 &\to B_y, \\
S_3 &\to B_z.
\end{aligned} \tag{8}
$$

On the other hand, the bases of Eq. (7) are *unbiased*, that is

$$
|\langle e_{i,\,j} \mid e_{k,\,l}\rangle|^2 = \frac{1}{d} = \frac{1}{2} \qquad i \neq k \tag{9}
$$

## 2. Discrete Wigner functions

We have the necessary ingredients to define a class of discrete Wigner functions. On one hand, there is a $d + 1 = 2 + 1 = 3$ mutually unbiased bases $\{B_x, B_y, B_z\}$ and on the other a set of $d + 1 = 2 + 1 = 3$ striations $\{S_1, S_2, S_3\}$ of the $d \times d = 2 \times 2 = 4$ phase space into $d = 2$ parallel lines. We need to choose a one-to-one maps as follows:

I. Each basis set $B_i$ is associated with one striation $S_i$

II. Each basis vector $|e_{i,j}\rangle$ is associated with a line $\alpha_{i,\,j}$ (the jth line of the ith striation).

With the above association, the Wigner function $W$ is uniquely defined if we demand that

$$Tr(|e_{i,j}\rangle\langle e_{i,j}|\rho) = \sum_{e\in\propto_{i,j}} W_e, \tag{10}$$

where $|e_{i,j}\rangle$ are the basis states of Eqs. (5)-(7) and $p$ is the one-qubit density state (Nielsen, & Chuang, 2000). Let us observe that Eq. (10) gives rise to the following Wigner functions associated to the four real numbers of Eq. (1)

$$\begin{aligned}
Tr(|e_{x,1}\rangle\langle e_{x,1}|\rho) &= \sum_{e\in\propto_{1,1}} W_e = W(x_{11}) + W(x_{21}) = p_{1,1},\\
Tr(|e_{x,2}\rangle\langle e_{x,2}|\rho) &= \sum_{e\in\propto_{1,2}} W_e = W(x_{11}) + W(x_{22}) = p_{1,2},\\
Tr(|e_{y,1}\rangle\langle e_{y,1}|\rho) &= \sum_{e\in\propto_{1,3}} W_e = W(x_{11}) + W(x_{12}) = p_{2,1},\\
Tr(|e_{y,2}\rangle\langle e_{y,2}|\rho) &= \sum_{e\in\propto_{2,1}} W_e = W(x_{12}) + W(x_{21}) = p_{2,2},\\
Tr(|e_{z,1}\rangle\langle e_{z,1}|\rho) &= \sum_{e\in\propto_{2,2}} W_e = W(x_{12}) + W(x_{22}) = p_{3,1},\\
Tr(|e_{z,2}\rangle\langle e_{z,2}|\rho) &= \sum_{e\in\propto_{2,3}} W_e = W(x_{21}) + W(x_{22}) = p_{3,2}.
\end{aligned} \tag{11}$$

where the probabilities satisfy

$$\sum_{i,j} p_{i,j} = 1 \tag{12}$$

Eq. (11) define uniquely the Wigner function $W$ in terms of the probabilities

$$\begin{aligned}
W(x_{11}) &= \frac{1}{2}\,(p_{1,1} + p_{2,1} + p_{3,1} - 1),\\
W(x_{12}) &= \frac{1}{2}\,(p_{1,1} + p_{2,2} + p_{3,2} - 1),\\
W(x_{21}) &= \frac{1}{2}\,(p_{1,2} + p_{2,1} + p_{3,2} - 1),\\
W(x_{22}) &= \frac{1}{2}\,(p_{1,2} + p_{2,2} + p_{3,1} - 1).
\end{aligned} \tag{13}$$

## 3. Wigner formulation of one-qubit purity

In the literature is scarce the information on the one-qubit purity. We then propose the following formulation of the purity in terms of the Wigner function

$$\pi_{i,j} = Tr(|e_{i,j}\rangle\langle e_{i,j}|\rho^2) = \sum_{e\in\propto_{i,j}} c_e W_e, \tag{14}$$

where $0 \leq c_e \leq 1$ are real numbers. Eq. (14) can be expanded in terms of the following six equations as follows

$$
\begin{aligned}
Tr(|e_{x,1}\rangle\langle e_{x,1}|\rho^2|) &= \sum_{e\in\propto_{1,1}} c_e W_e = c_{11}W(x_{11}) + c_{21}W(x_{21}) = \pi_{1,1},\\
Tr(|e_{x,2}\rangle\langle e_{x,2}|\rho^2) &= \sum_{e\in\propto_{1,2}} c_e W_e = c_{11}W(x_{11}) + c_{22}W(x_{22}) = \pi_{1,2},\\
Tr(|e_{y,1}\rangle\langle e_{y,1}|\rho^2) &= \sum_{e\in\propto_{1,3}} c_e W_e = c_{11}W(x_{11}) + c_{12}W(x_{12}) = \pi_{2,1},\\
Tr(|e_{y,2}\rangle\langle e_{y,2}|\rho^2) &= \sum_{e\in\propto_{2,1}} c_e W_e = c_{12}W(x_{12}) + c_{21}W(x_{21}) = \pi_{2,2},\\
Tr(|e_{z,1}\rangle\langle e_{z,1}|\rho^2) &= \sum_{e\in\propto_{2,2}} c_e W_e = c_{12}W(x_{12}) + c_{22}W(x_{22}) = \pi_{3,1},\\
Tr(|e_{z,2}\rangle\langle e_{z,2}|\rho^2) &= \sum_{e\in\propto_{2,3}} c_e W_e = c_{21}W(x_{21}) + c_{22}W(x_{22}) = \pi_{3,2}.
\end{aligned} \tag{15}
$$

We say that one qubit is in a *mixed state* if

$$\sum_{i,j} \pi_{i,j} < 1. \tag{16}$$

On the other hand, one qubit is in a *pure state* if

$$\sum_{i,j} \pi_{i,j} = 1. \tag{17}$$

## Conclusions

We have formulated the one-qubit state in terms of the Wigner transform operating on a discrete phase-space. In classical approaches the phase-space is usually understood as the composition of both the continuous spatial coordinates {X} and the continuous momentum space $\{P\}$, that is, a $\{X, P\}$ continuous generalized coordinates space. In the present approach we focus on two spatial coordinates $\{q_1, q_2\}$ versus two momenta coordinates $\{p_1, p_2\}$ is such a way that the two above sets generates the following four elements set $\{q_1 p_1, q_1 p_2, q2\,p1, q2\,p2\} \equiv \{x11, x12, x21, x22\}$ defining the discrete phase-space of Eq. (1). An example of a one possible phase-space is the following {-17.21, 8.3, -2.1, -0.17}.

The striations of Eq. (3) can be generalized for a prime $d$-dimensional phase-space. In particular, we have considered a one-qubit state where $d = 2$. Let us note that for the one-qutrit state one must have $d = 3$ Eq. (10) defines uniquely a one-qubit state in terms of the Wigner discrete transform $W$. Such a definition is equivalent to the conventional definition of a one-qubit state if one observe from Eq. (10) that $\rho = (a_0|0\rangle + a_1|1\rangle)(\mathrm{a}_0^*\langle 0| + \mathrm{a}_1^*\langle 1|) = \begin{pmatrix} a_0 a_0^* & a_0\, a_1^* \\ a_1 a_0^* & a_1\, a_1^* \end{pmatrix}$ where $Tr\,\rho = |a_0|^2 + |a_1|^2 = 1$. On the other hand, the main achievement of the present work is the formulation of the *purity* of a one-qubit state in terms of the Wigner discrete transform $W$ through Eq. (14). It is worth to mention that such a formulation is absent in the literature. With Eqs. (14) and (17) we can state a criteria for concluding when a one-qubit is in a *pure state*.

## Prospective analysis

The formulation of a one-qubit state in terms of $W$ is elegant and allows to geometrize the probabilities $p(_i, _j)$ of Eq. (11). Indeed, the constrain of Eq. (12) on the probabilities $p_{i\,,\,j}$ implies that they can be represented in a Bloch sphere.

On the other hand, the understanding of the formulation of the purity of a one-qubit state in terms of the discrete Wigner transform as stated from Eqs. (15)-(17) could help in the future to understand intriguing properties of qubits such as the relation between $W$ and the speed of quantum information processing.

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