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A brief description of operators associated to the quantum harmonic oscillator on Schatten-von Neumann classes

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Abstract. In this note we study pseudo-multipliers associated to the harmonic oscillator (also called Hermite multipliers) belonging to Schatten classes on $L^2(\mathbb{R}^n)$. We also investigate the spectral trace of these operators.
Keywords: Harmonic oscillator, Fourier multiplier, Hermite multiplier, nuclear operator, traces.
MSC2010: 81Q10, 47B10, 81Q05.

Una descripción breve de operadores asociados al oscilador armónico cuántico sobre las clases de Schatten-von Neumann

Resumen. En esta nota se estudia una clase de operadores definidos a través del espectro del oscilador armónico y conocidos en la literatura como pseudo multiplicadores (pseudo multiplicadores de Hermite). Se analizan criterios óptimos para clasificar estos operadores en las clases de Schatten-von Neumann sobre $L^2(\mathbb{R}^n)$. El trabajo culmina con una investigación sobre la traza espectral y/o nuclear de tales operadores.
Palabras clave: Oscilador armónico, multiplicador de Fourier, multiplicadores de Hermite, operador nuclear, trazas.

1. Introduction

1.1. Outline of the paper

Pseudo-multipliers and multipliers associated to the harmonic oscillator arise from the study of Hermite expansions for complex functions on $\mathbb{R}^n$ (see Thangavelu [23], [24],

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[25], [26] [27], [28], Epperson [11] and Bagchi and Thangavelu [1]). At the same time, we note that pseudo-multipliers are pseudo-differential operators on $\mathbb{R}^n$ in view of the quantization process developed by Ruzhansky and Tokmagambetov in [17] and [18] when the reference operator is the harmonic oscillator. In this note, we are interested in the membership of pseudo-multipliers associated to the harmonic oscillator (also called Hermite pseudo-multipliers) in the Schatten classes, $S_r(L^2)$ on $L^2(\mathbb{R}^n)$. With this paper we finish the classification of pseudo-multipliers in classes of r-nuclear operators on L^p-spaces (see Barraza and Cardona [2], [3]), which on $L^2(\mathbb{R}^n)$ coincide with the Schatten-von Neumann classes of order r. Our main result is Theorem 1.1 where we establish some criteria in order that pseudo-multipliers belong to the classes $S_r(L^2)$, $0 < r \leq 2$. In order to present our main result we recall some notions. Let us consider the sequence of Hermite functions on $\mathbb{R}^n$,

$$\phi_\nu = \Pi_{j=1}^n \phi_{\nu_j}, \ \ \phi_{\nu_j}(x_j) = (2^{\nu_j}\nu_j!\sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j) e^{-\frac{1}{2}x_j^2} \tag{1}$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$, and $H_{\nu_j}(x_j)$ denotes the Hermite polynomial of order ν_j. It is well known that the Hermite functions provide a complete and orthonormal system in $L^2(\mathbb{R}^n)$. If we consider the operator $L = -\Delta + |x|^2$ acting on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, where Δ is the standard Laplace operator on $\mathbb{R}^n$, then we have the relation $L\phi_\nu = \lambda_\nu \phi_\nu, \ \nu \in \mathbb{N}_0^n$. The operator L is symmetric and positive in $L^2(\mathbb{R}^n)$ and admits a self-adjoint extension H whose domain is given by

$$\mathrm{Dom}(H) = \left\{ \sum_{\nu \in \mathbb{N}_0^n} \langle f, \phi_\nu \rangle_{L^2} \phi_\nu : \sum_{\nu \in \mathbb{N}_0^n} |\lambda_\nu \langle f, \phi_\nu \rangle_{L^2}|^2 < \infty \right\}. \tag{2}$$

So, for $f \in \mathrm{Dom}(H)$, we have

$$(Hf)(x) = \sum_{\nu \in \mathbb{N}_0} \lambda_\nu \widehat{f}(\phi_\nu)\phi_\nu(x), \quad \widehat{f}(\phi_\nu) = \langle f, \phi_\nu \rangle_{L^2}. \tag{3}$$

The operator H is precisely the quantum harmonic oscillator on $\mathbb{R}^n$ (see [15]). The sequence $\{\widehat{f}(\phi_v)\}$ determines the Fourier-Hermite transform of f, with corresponding inversion formula

$$f(x) = \sum_{\nu \in \mathbb{N}_0^n} \widehat{f}(\phi_v)\phi_\nu(x). \tag{4}$$

On the other hand, pseudo-multipliers are defined by the quantization process that associates to a function m on $\mathbb{R}^n \times \mathbb{N}_0^n$ a linear operator T_m of the form

$$T_m f(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu)\widehat{f}(\phi_\nu)\phi_\nu(x), \quad f \in \mathrm{Dom}(T_m). \tag{5}$$

The function m on $\mathbb{R}^n \times \mathbb{N}_0^n$ is called the symbol of the pseudo-multiplier T_m. If in (5), $m(x, \nu) = m(\nu)$ for all x, the operator T_m is called a multiplier. Multipliers and pseudo-multipliers have been studied, for example, in the works [1], [20], [21], [22], [23], [24] (and references therein) principally by its mapping properties on L^p spaces. In order that the operator $T_m : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ belongs to the Schatten class $S_r(L^2)$, in this paper we provide some (sharp) conditions on the symbol m.

1.2. Pseudo-multipliers in Schatten classes

By following A. Grothendieck [12], we can recall that a linear operator $T: E \to F$ (E and F Banach spaces) is r-nuclear, if there exist sequences $(e'_n)_{n\in\mathbb{N}_0}$ in E' (the dual space of E) and $(y_n)_{n\in\mathbb{N}_0}$ in F such that

$$Tf = \sum_{n\in\mathbb{N}_0} e'_n(f) y_n, \quad \text{and} \quad \sum_{n\in\mathbb{N}_0} \|e'_n\|^r_{E'} \|y_n\|^r_F < \infty. \tag{6}$$

The class of $r-$nuclear operators is usually endowed with the quasi-norm

$$n_r(T) := \inf \left\{ \left\{ \sum_n \|e'_n\|^r_{E'} \|y_n\|^r_F \right\}^{\frac{1}{r}} : T = \sum_n e'_n \otimes y_n \right\}. \tag{7}$$

In addition, when $E = F$ is a Hilbert space and $r = 1$ (resp. $r = 2$), the definition above agrees with the concept of trace class operators (resp. Hilbert-Schmidt). For the case of Hilbert spaces H, the set of r-nuclear operators agrees with the Schatten-von Neumann class of order r (see Pietsch [13], [14]). We recall that a linear operator T on a Hilbert space H belong to the Schatten class of order r, $S_r(H)$, if

$$s_r(T) := \sum_{n\in\mathbb{N}_0} \lambda_n(T)^r < \infty, \tag{8}$$

where $\{\lambda_n(T)\}$ denotes the sequence of singular values of T, which are the eigenvalues of the operator $\sqrt{T^*T}$. It was proved in [2] that a multiplier T_m, with symbol satisfying conditions of the form

- $$\varkappa(m, p_1, p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} \alpha_{r,p_1,p_2}(s,\nu)|m(\nu)|^r < \infty, \tag{9}$$

where $\{I_s\}_{s=0}^n$ is a suitable partition of $\mathbb{N}_0^n$, and $\alpha_{r,p_1,p_2}(s,\nu)$ is a suitable kernel, can be extended to a r-nuclear operator from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$. Although is easy to see that similar necessary conditions apply for pseudo-multipliers, the r-nuclearity for these operators in L^p-spaces was characterized in [3] by the following condition:

- a pseudo-multiplier T_m can be extended to a r-nuclear operator from L^{p_1} into L^{p_2} if and only if there exist functions h_k and g_k satisfying

$$m(x,\nu) = \phi_\nu(x)^{-1} \sum_{k=1}^{\infty} h_k(x)\widehat{g}(\phi_\nu), \;\; \phi_\nu(x) \neq 0, \text{ with } \sum_{k=0}^{\infty} \|g_k\|^r_{L^{p'_1}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{10}$$

If we consider $p_1 = p_2 = 2$, and a multiplier T_m, the conditions above can be replaced by the following more simple one,

$$\varkappa(m, 2, r) := \sum_{\nu\in\mathbb{N}_0} |m(\nu)|^r < \infty, \tag{11}$$

because the set of singular values of a multiplier T_m consists of the elements in the sequence $\{|m(\nu)|\}_{\nu\in\mathbb{N}_0^n}$. The condition (10) characterizes the membership of pseudo-multipliers in Schatten classes in terms of the existence of certain measurable functions. However, in this paper we provide explicit conditions on m in order to guarantee that $T_m \in S_r(L^2)$, because explicit conditions allow us to known information about the distribution of the spectrum of these operators. Our main result is the following theorem.

Theorem 1.1. *Let T_m be a pseudo-multiplier with symbol m defined on $\mathbb{R}^n \times \mathbb{N}_0^n$. Then we have:*

- *T_m is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, i.e., $T_m \in S_2(L^2)$, if and only if*

$$\sum_{\nu\in\mathbb{N}_0^n} \int_{\mathbb{R}^n} |m(x,\nu)|^2 \phi_\nu(x)^2 dx < \infty. \tag{12}$$

- *If T_m is a positive operator, then T_m is trace class, i.e., $T_m \in S_1(L^2)$, if and only if*

$$\sum_{\nu\in\mathbb{N}_0^n} \int_{\mathbb{R}^n} m(x,\nu) \phi_\nu(x)^2 dx < \infty. \tag{13}$$

- *$T_m \in S_r(L^2)$, $0 < r \leq 1$, if*

$$\sum_{\nu\in\mathbb{N}_0^n} \left(\int_{\mathbb{R}^n} |m(x,\nu)|^2 \phi_\nu(x)^2 dx \right)^{\frac{r}{2}} < \infty. \tag{14}$$

- *If $1 < r < 2$ and there exists $\sigma > n(\frac{1}{r} - \frac{1}{2})$ such that*

$$\sum_{\nu\in\mathbb{N}_0^n} |\nu|^{2\sigma} \int_{\mathbb{R}^n} |m(x,\nu)|^2 \phi_\nu(x)^2 dx < \infty, \tag{15}$$

  then $T_m \in S_r(L^2)$.

In general, on a Banach space compact linear operators are bounded operators. Taking into account that Schatten-von Neumann classes on Hilbert spaces are families of compact operators, our main theorem gives conditions for the $L^2(\mathbb{R}^n)$-continuity of pseudo-multipliers. The problem of finding "satisfactory" conditions for the $L^2(\mathbb{R}^n)$-boundedness of pseudo-multipliers remains open, and it was proposed by Bagchi and Thangavelu in [1]; with our main result and the conditions proposed in Cardona and Barraza [3], we solve partially such problem. However, Bagchi-Thangavelu's problem will be "satisfactorily" solved in the work Cardona and Ruzhansky [4].

1.3. Related works

Now, we include some references on the subject. Sufficient conditions for the r-nuclearity of spectral multipliers associated to the harmonic oscillator, but in modulation spaces and Wiener amalgam spaces, have been considered by J. Delgado, M. Ruzhansky and B.

Wang in [8], [9]. The Properties of these multipliers in L^p-spaces have been investigated in the references S. Bagchi, S. Thangavelu [1], J. Epperson [11], K. Stempak and J.L. Torrea [20], [21], [22], S. Thangavelu [23], [24] and references therein. Hermite expansions for distributions can be found in B. Simon [19]. The r-nuclearity and Grothendieck-Lidskii formulae for multipliers and other types of integral operators can be found in [7], [9]. On Hilbert spaces the class of r-nuclear operators agrees with the Schatten-von Neumann class $S_r(H)$; in this context operators with integral kernel on Lebesgue spaces and, in particular, operators with kernel acting of a special way with anharmonic oscillators of the form $E_a = -\Delta_x + |x|^a$, $a > 0$, has been considered on Schatten classes on $L^2(\mathbb{R}^n)$ in J. Delgado and M. Ruzhansky [10]. A complete treatment for L^p-boundedness and L^p-compactness properties in terms of the Littlewood-Paley theory and the Hörmander condition will be considered in Cardona and Ruzhansky [4]. The proof of our results will be presented in the next section.

2. Pseudo-multipliers in Schatten-von Neumann classes

In this section we prove our main result for pseudo-multipliers T_m. Our criteria will be formulated in terms of the symbols m. First, let us observe that every pseudo-multiplier T_m is an operator with kernel $K_m(x,y)$. In fact, straightforward computation shows that

$$T_m f(x) = \int_{\mathbb{R}^n} K_m(x,y)f(y)dy,\ K_m(x,y) := \sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)\phi_\nu(y) \tag{16}$$

for every $f \in \mathscr{D}(\mathbb{R}^n)$. We will use the following result (see J. Delgado [5], [6]).

Theorem 2.1. *Let us consider $1 \leq p_1, p_2 < \infty$, $0 < r \leq 1$ and let q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Let (X_1,μ_1) and (X_2,μ_2) be σ-finite measure spaces. An operator $T : L^{p_1}(X_1,\mu_1) \to L^{p_2}(X_2,\mu_2)$ is r-nuclear if and only if there exist sequences $(g_n)_n$ in $L^{p_2}(\mu_2)$, and (h_n) in $L^{q_1}(\mu_1)$, such that*

$$\sum_n \|g_n\|^r_{L^{p_2}}\|h_n\|^r_{L^{q_1}} < \infty, \text{ and } Tf(x) = \int (\sum_n g_n(x)h_n(y))f(y)d\mu_1(y), \text{ a.e.w. } x, \tag{17}$$

for every $f \in L^{p_1}(\mu_1)$. In this case, if $p_1 = p_2$ (see Section 3 of *[5]) the nuclear trace of T is given by*

$$\mathrm{Tr}(T) := \int \sum_n g_n(x)h_n(x)d\mu_1(x). \tag{18}$$

Now, we prove our main theorem.

Proof of Theorem 1.1. Let us consider a pseudo-multiplier T_m. By definition, T_m is a Hilbert-Schmidt operator if and only if there exists an orthonormal basis $\{e_\nu\}_\nu$ of $L^2(\mathbb{R}^n)$ such that

$$\sum_\nu \|T_m e_\nu\|^2_{L^2} < \infty. \tag{19}$$

In particular, if we choose the system of Hermite functions $\{\phi_\nu\}$, which provides an orthonormal basis of $L^2(\mathbb{R}^n)$, from the relation $T_m(\phi_\nu) = m(x,\nu)\phi_\nu$ we conclude that

T_m is of Hilbert-Schmidt type, if and only if

$$\sum_{\nu} \|m(\cdot,\nu)\phi_\nu\|_{L^2}^2 = \sum_{\nu\in\mathbb{N}_0^n} \int_{\mathbb{R}^n} |m(x,\nu)|^2\phi_\nu(x)^2 dx < \infty. \tag{20}$$

So, we have proved the first statement. Now, if we assume that T_m is positive, then T_m is of class trace if and only if there exists an orthonormal basis $\{e_\nu\}_\nu$ of $L^2(\mathbb{R}^n)$ such that

$$\sum_{\nu} \langle T_m e_\nu, e_\nu\rangle_{L^2} < \infty. \tag{21}$$

As in the first assertion, if we choose the basis formed by the Hermite functions, T_m is of class trace if and only if

$$\sum_{\nu} \langle T_m e_\nu, e_\nu\rangle_{L^2} = \sum_{\nu\in\mathbb{N}_0^n} \int_{\mathbb{R}^n} m(x,\nu)\phi_\nu(x)^2 dx < \infty, \tag{22}$$

which proves the second assertion. Now, we will verify that (14) implies that $T_m \in S_r(L^2)$ for $0 < r \leq 1$. For this, we will use Delgado's Theorem (Theorem 2.1) to the representation (16) of K_m,

$$K_m(x,y) := \sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)\phi_\nu(y). \tag{23}$$

So, $T_m \in S_r(L^2)$ if

$$\sum_{\nu} \|m(\cdot,\nu)\|_{L^2}^r \|\phi_\nu\|_{L^2}^r = \sum_{\nu\in\mathbb{N}_0^n} \left(\int_{\mathbb{R}^n} |m(x,\nu)|^2\phi_\nu(x)^2 dx\right)^{\frac{r}{2}} < \infty, \tag{24}$$

where we have used that the L^2-norm of every Hermite function ϕ_ν is normalised. In order to finish the proof, we only need to prove that (15) assures that $T_m \in S_r(L^2)$ for $1 < r < 2$. This can be proved by using the following multiplication property on Schatten classes:

$$S_p(H)S_q(H) \subset S_r(H), \;\; \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{25}$$

So, we will factorize T_m as

$$T_m = T_m H^\sigma H^{-\sigma}, \;\; \sigma > 0, \tag{26}$$

where H is the harmonic oscillator. Let us note that the symbol of $A = T_m H^\sigma$ is given by $a(x,\nu) = m(x,\nu)(2|\nu|+n)^\sigma$. So, from the first assertion, $A \in S_2(L^2)$ if and only if

$$\sum_{\nu\in\mathbb{N}_0^n} |\nu|^{2\sigma} \int_{\mathbb{R}^n} |m(x,\nu)|^2\phi_\nu(x)^2 dx \asymp \sum_{\nu\in\mathbb{N}_0^n} (2|\nu|+n)^{2\sigma} \int_{\mathbb{R}^n} |m(x,\nu)|^2\phi_\nu(x)^2 dx < \infty.$$

In order to prove that $T_m \in S_r(L^2)$, in view of the multiplication property

$$S_2(L^2)S_{\frac{2r}{2-r}}(L^2) \subset S_r(L^2), \;\; \frac{1}{r} = \frac{1}{2r/(2-r)} + \frac{1}{2}, \tag{27}$$

we only need to prove that $H^{-\sigma} \in S_p(L^2)$ with $p = \frac{2r}{2-r}$. The symbol of $H^{-\sigma}$ is given by $a'(\nu) = (2|\nu| + n)^{-\sigma}$. By using the hypothesis $\sigma > n(\frac{1}{r} - \frac{1}{2})$ we have that

$$\sum_{\nu} |a'(\nu)|^p = \sum_{\nu} (2|\nu| + n)^{-\sigma p} < \infty,$$

because $\sigma p = \sigma(\frac{1}{r} - \frac{1}{2})^{-1} > n$. So, we finish the proof. ☑

2.1. Trace class pseudo-multipliers of the harmonic oscillator

In order to determinate a relation with the eigenvalues of T_m we recall the following result (see [16]).

Theorem 2.2. *Let $T : L^p(\mu) \to L^p(\mu)$ be a r-nuclear operator as in* (6)*. If $\frac{1}{r} = 1 + |\frac{1}{p} - \frac{1}{2}|$, then,*

$$\mathrm{Tr}(T) := \sum_{n \in \mathbb{N}_0^n} e_n'(f_n) = \sum_n \lambda_n(T), \tag{28}$$

where $\lambda_n(T)$, $n \in \mathbb{N}$, is the sequence of eigenvalues of T with multiplicities taken into account.

As an immediate consequence of the preceding theorem (or the classical Grothendieck-Lidskii Theorem), if $T_m : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is trace class (1-nuclear) then,

$$\mathrm{Tr}(T_m) = \int_{\mathbb{R}^n} \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu)\phi_\nu(x)^2 dx = \sum_n \lambda_n(T), \tag{29}$$

where $\lambda_n(T)$, $n \in \mathbb{N}$, is the sequence of eigenvalues of T_m with multiplicities taken into account.

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