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Original articles

Continuous images of hereditarily indecomposable continua

Imágenes continuas de continuos hereditariamente indescomponibles

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Abstract: The theorem proven here is that every compact metric continuum is a continuous image of some hereditarily indecomposable metric continuum.
MSC2010: 54F15, 54F45, 54E45, 54C60.
Keywords: Continuous maps, continuum, hereditarily indecomposable.
Resumen: El teorema demostrado es que todo continuo métrico es imagen continua de algún continuo métrico hereditariamente indescomponible.
Palabras clave: Funciones continuas, continuo, hereditariamente indescomponible.

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1. Introduction

These definitions are needed in what follows and may or may not be familiar to everyone. A *continuum X* is a compact, connected metric space. A continuum X is *indecomposable* provided that whenever A and B are proper subcontinua of X, A # B is a proper subset of X; X is *hereditarily indecomposable* if, and only if, every subcontinuum of X is indecomposable. A *map* is a continuous function. A map f from a continuum X to a continuum Y is *weakly confluent* provided that given any continuum *M # Y* there exists a continuum W # X such that f (W) = M. When X is a continuum, C(X) is the hyperspace of subcontinua of X. If *a* and b are points in $\#^n$ with *a ≠ b, [a,b]* denotes the line segment from *a* to b. Let S^n denote the n dimensional sphere. An arc A # S^3 is *tame* if and only if there is a homeomorphism h: $S^3 \to S^3$ such that *h(A)* is an arc of a great circle in S^3 .

In [4] J. W. Rogers, Jr. asked whether every continuum is a continuous image of some indecomposable continuum. The author [1] gave an affirmative answer to this question.

Sometime later, in conversation, Rogers asked whether every continuum is a continuous image of some hereditarily indecomposable continuum. This article provides a proof that the answer to this question is also yes.

The author first announced this result in [1] but has not published it previously. It has come to my attention that in [4] this result has been extended to the non-metric case, building on the metric result.

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2. Necessary Lemmas

Lemma 2.1. *Let X and* Y *be continua. Then f: X → Y is weakly confluent if, and only if, the hyperspace map induced by f, C(f): C(X) → C(Y), is surjective.*

Proof. This is just a restatement of the definition of weakly confluent.

Lemma 2.2. *There exists a hereditarily indecomposable subcontinuum of* # [4] *which separates* # [4] *.*

Remark on proof. R. H. Bing [2] proved this not just for n = 4, but for every n > 1.

Lemma 2.3. *Each homotopically essential map from a continuum X to the three sphere,* S^3 *, is weakly confluent.*

Proof. This was essentially proven, although in a different context, by S. Mazurkiewicz in [5, Theoreme I, p. 328]. This argument gives the necessary details. Let X be a continuum, and suppose g: X → S^3 be a homotopically essential map. To prove that g is weakly confluent, it suffices to prove that every tame arc in S^3 is equal to g(M) for some continuum *M # X.* This follows from Lemma 2.1 because the set of tame arcs is dense in *C(*S^3*).*

First, set up some machinery and notation, as follows. Let J be a tame arc in S^3; let D_n be the closed disk in the complex plane with radius (1/n) centered at 0. Let E_n be the corresponding open disk, and let T_n be the circle $D_n \setminus E_n$. Let C_n be the solid cylinder D_n x [0,1]. Since J is a tame, there exists an embedding h of C into S^3 such that h({0} x [0,1]) = J. Consider C_n as a subset of S^3 by identifying C_1 with $h(C_1)$, and for each t # [0,1] let t denote the point h(0, t) # J.

Let F_n denote the manifold boundary of C_n , that is, F_n = (D_n x {0,1}) # (T_n x [0,1]). Note that given any n and any a, b # J there is an isotopy H: C_n x [0,1] → C_n satisfying the following:

(i) for each s # [0,1], H(J x {s}) = J;
(ii) for each x # F_n and each t # [0,1], H(x,t) = x;
(iii) for every x # C_n, H(x, 0) = x; and
(iv) H(b, 1) = a.

By setting H(x, t) = x for every x # S^3 \ Cn, and every t # [0, 1], H can be considered to be a function (hence an isotopy) from S^3 x [0, 1] to S^3.

Now, suppose X is a continuum and let g: X → S^3 be a homotopically essential map. To prove that g is weakly confluent, it suffices to prove that there exists a continum M # X such that g(M) = J.

Proceed by contradiction; assume there is no such M. Then no component of g^{-1}(J) intersects both g^{-1}(0) and g^{-1}(1). By compactness, there is a separation, R_0 # R_1 of g^{-1}(J) satisfying g^{-1}(0) # R_0 and g^{-1}(1) # R_1. Since R_0 and R_1 are disjoint closed sets in X, there exist open subsets S_0 and S_1 of X such that R_0 # S_0 and R_1 # S_1 and Cl(S_0) # Cl(S_1) = #. There exists n such that g^{-1}(Cn) # S_0 # S_1. Let p = inf g(R_1) and let q = sup g(R_0), and let a, b # J be such that 0 < a < p and q < b < 1. If p > q, then g is not surjective and hence not essential, so 0 <a<p ≤ q<b< 1. #sing the number n and the points a and b just chosen, let H: S^3 x [0, 1] → S^3 be the isotopy described above. Define a homotopy G: X x [0,1] → S^3 by G(x, t) = g(x) if x # X \ S_0 and G(x,t) = H(g(x),t) if x # Cl(S_0). Define f: X → S^3 by f (x) = G(x, 1).

Then, note that if y # J and a < y < p, then there does not exist z # X such that f (z) = y, so f is nonsurjective. Hence, f is inessential. Since g is homotopic to f, g is inessential also, a contradiction, which completes the proof.

Lemma 2.4. *A continuum* X # R^4 *admits a homotopically essential map onto* S^3 *if, and only if,* R^4 X *is not connected* S^3.

Remark on Proof. This is a special case of the Borsuk separation theorem. I do not have a reference to the original proof, but a proof can be found in almost any advanced topology or algebraic topology book.

Lemma 2.5. *Given any continuum* Y*, there is a continuum* X # S^3 *that admits a continuous surjection* f : X → Y.

Proof. Let Y be a continuum and let C and D be Cantor sets in R^3 such that C and D lie on lines skew to each other. Then, whenever a, p # C and b, q # D, and a, p, b, and q are all different, the line segments [a, b] and [p, q] are disjoint. Let g: C # D → Y be a map such that g|C: C → Y and g| D: D → Y are both onto. Such a g exists since a Cantor set can be mapped onto every compact metric space. Define $X = \bigcup\{[a, b] : a \in C; b \in D$ and g(a) = g(b)}. Then X is a continuum in R^3. For each x # X, let [a(x), b(x)] be a segment in X satisfying a(x) # C; b(x) # D, and x # [a(x), b(x)]. (This segment is unique unless x = a(x) or x = b(x).) Define f: X → Y by f (x) = g(a(x)) = g(b(x)). It is straightforward to verify that f : X → Y is continuous and onto. Since for any point p # S^3, S^3 \ {p} is a copy of R^3, X can be treated as a subcontinuum of S^3.

3. Main Result

Theorem 3.1. *Let* Y *be an arbitrary continuum. There exists a hereditarily indecomposable continuum* K *that admits a surjective map* f: *K* → Y.

Proof. Let Y be a continuum. By Lemma 2.5, there is a continuum T # S^3 and an onto map g : T → Y. By Lemma 2.2, there exists a hereditarily indecomposable continuum L # R^4 that separates R^4. Thus by Lemma 2.4, there is a homotopically essential map h: L → S^3. By Lemma 2.3, h is weakly confluent, so there exists a continuum K C L such that h(K) = T. Let f = g o (h|K). Then f: K → Y is the desired map; K is hereditarily indecomposable since it is a subcontinuum of L.

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Notes

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