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Artículos originales

The asymptotic analysis of a Darcy-Stokes system coupled through a curved interface

Análisis asintótico de un sistema Darcy-Stokes acoplado a través de una interfaz curva

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Abstract: We present the asymptotic analysis of a Darcy-Stokes coupled system, modeling the fluid exchange between a narrow channel (Stokes flow) and a porous medium (Darcy flow), coupled through a C^2 curved interface. The channel is a cylindrical domain between the interface (Γ) and a parallel translation of itself $(\Gamma + \epsilon\,\widehat{e}_N, \epsilon > 0)$. The introduction of a change variable (to fix the domain geometry) and the introduction of two systems of coordinates: the Cartesian and a local one (consistent with the geometry of the surface), permit to find the limiting form of the system when the width of the channel tends to zero ($\epsilon \to 0$). The limit problem is a coupled system with Darcy flow in the porous medium and Brinkman flow on the curved interface (Γ).
MSC2010: 35K50, 35B25, 80A20, 35F15.

Keywords: porous media, curved interfaces, Darcy-Stokes system, Darcy-Brinkman system.

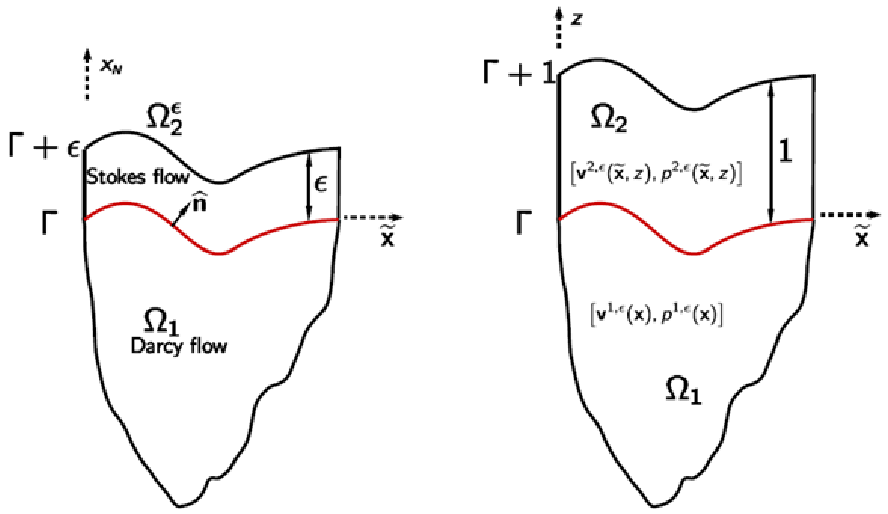
Resumen: En el trabajo se presenta el análisis asintótico de un sistema Darcy-Stokes acoplado a través de una interfaz curva. El sistema modela el intercambio de fluido entre un canal angosto (flujo Stokes) y un medio poroso (flujo Darcy). El canal es un dominio cilíndrico definido entre la interfaz (Γ) y una traslación paralela de dicha superficie $(\Gamma + \epsilon\,\widehat{e}_N, \epsilon > 0)$. Utilizando un cambio de variables para fijar un dominio de referencia e introduciendo dos sistemas de coordenadas, el Cartesiano canónico y el local (consistente con la geometría de la superficie), es posible encontrar la forma límite cuando el ancho del canal tiende a cero (ϵ - 0). El problema límite es un sistema acoplado con flujo Darcy en el medio poroso y flujo Brinkman en la interfaz (Γ).

Palabras clave: medio poroso, interfaces curvas, sistema Darcy-Stokes, sistema Darcy-Brinkman.

1. *Introduction*

In this paper we continue the work presented in [14], extending the result to a more general and realistic scenario. That is, we find the limiting form of a Darcy-Stokes (see equations (26)) coupled system, within a saturated domain Ω^ϵ in $\#^N$, consisting in three parts: a porous medium Ω_1 (Darcy flow), a narrow channel Ω_2^ϵ whose width is of order e (Stokes flow) and a coupling interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2^\epsilon$ (see Figure 1 (a)). In contrast with the system studied in [14], where the interface is flat, here the analysis is extended to *curved* interfaces. It will be seen that the limit is a fully-coupled system consisting of Darcy flow in the porous medium Ω_1 and a

Brinkman-type flow on the part Γ of its boundary, which now takes the form of a parametrized *N - 1* dimensional manifold.



(a) Original Domain

(b) Scaled Domain after the change of variable $\mathbf{x} \mapsto \varphi(\mathbf{x})$, with φ defined in Equation (10).

Figure 1

Figure (a) depicts the original domain with a thin channel on top, where we set the Stokes flow. Figure (b) depicts the domain after scaling by the change of variables x → ϕ(x), where ϕ is defined in Equation (10). This will be the domain of reference which is used for asymptotic analysis of the problem.

The central motivation in looking for the limiting problem of our Darcy-Stokes system is to attain a new model, free of the singularities present in (26). These are the narrowness of the channel $\mathcal{O}$ (ϵ) and the high velocity of the fluid in the channel $\mathcal{O}$ (ϵ), both (geometry and velocity) with respect to the porous medium. Both singularities have a substantial negative impact in the computational implementation of the system, such as numerical instability and poor quality of the solutions. Moreover, when considering the case of curved interfaces, the geometry of the surface aggravates these effects, making even more relevant the search for an approximate singularity-free system as it is done here.

The relevance of the Darcy-Stokes system itself, as well as its limiting form (a Darcy-Brinkman system) is confirmed by the numerous achievements reported in the literature: see [2], [4], [6] for the analytical approach, [3], [5], [9], [13] for the numerical analysis point of view, see [11], [21] for numerical experimental coupling and [12] for a broad perspective and references. Moreover, the modeling and scaling of the problem have already been extensively justified in [14]. Hence, this work is focused on addressing (rigorously) the interface geometry impact in the asymptotic analysis of the problem. It is important to consider the curvature of interfaces in the problem, rather than limiting the analysis to flat or periodic interfaces, because the fissures in a natural bedrock [where this phenomenon takes place] have wild geometry. In [7], [8] the analysis is made using homogenization techniques for periodically curved surfaces, which is the typical necessary assumption for this theory. In [17], [18] the analysis is made using boundary layer techniques, however no explicit results can be obtained, as usually with these methods. An early and simplified version of the present result can be found in [16], where incorporating the interface geometry in the asymptotic analysis of a multiscale Darcy-Darcy coupled system is done and a explicit description of the limiting problem is given.

The successful analysis of the present work is because of keeping an interplay between two coordinate systems: the Cartesian and a *local* one, consistent with the geometry of the interface r. While it is convenient to handle the independent variables in Cartesian coordinates, the asymptotic analysis of the flow fields in the free fluid region Ω_2^ϵ is more manageable when decomposed in normal and tangential directions to the interface (the local system). The a-priori estimates, the properties of weak limits, as well as the structure of the limiting problem will be more easily derived with this *double bookkeeping* of coordinate systems, rather than disposing of them for good. It is therefore a *strategic mistake* (not a mathematical one, of course) to seek a transformation flattening out the interface, as it is the usual approach in traces' theory for Sobolev spaces. The proposed method is significantly simpler than other techniques and it is precisely this *simplicity* which permits to obtain the limiting form explicit description for a problem of such complexity, as a multiscale coupled Darcy-Stokes.

Notation

We shall use standard function spaces [see [1], [20]). For any smooth bounded region G in $\#^N$ with boundary ∂G, the space of square integrable functions is denoted by $L^2(G)$ and the Sobolev space $H^1(G)$ consists of those functions in $L^2(G)$ for which each of its first-order weak partial derivatives belongs to $L^2(G)$. The *trace* is the continuous linear function $\gamma : H^1(G) - L^2(\partial G)$ which agrees with the restriction to the boundary on smooth functions, i.e., $\gamma(w) = w|_{\partial G}$ if $w \in C(\text{cl}(G))$. Its kernel is $H_0^1(G) \overset{\text{def}}{=} \{w \in H^1(G) : \gamma(w) = 0\}$. The trace space is $H^{1/2}(\partial G) \overset{\text{def}}{=} \gamma(H^1(G))$, the range of γ endowed with the usual norm from the quotient space $H^1(G)/H_0^1(G)$, and we denote by $H^{-1/2}(\partial G)$ its topological dual. Column vectors and corresponding vector-valued functions will be denoted by boldface symbols, e.g., we denote the product space $[L^2(G)]^N$ by $L^2(G)$ and the respective N-tuple of Sobolev spaces by $\mathbf{H}^1(G) \overset{\text{def}}{=} [H^1(G)]^N$. Each $w \# L^2(G)$ has *gradient* $\nabla w = (\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_N}) \in \mathbf{L}^2(G)$, furthermore we understand it as a row vector. We shall also use the space $H_{div}(G)$ of vector functions w # $L^2(G)$ whose weak divergence # • w belongs to $L^2(G)$. The symbol $\widehat{\boldsymbol{n}}$ stands for the unit outward normal vector on ∂G. If w is a vector function on ∂G, we indicate its normal component by $\mathbf{w}_{\widehat{\boldsymbol{n}}} = \gamma(\mathbf{w}) \cdot \widehat{\boldsymbol{n}}$, and its normal projection by $\mathbf{w}(\widehat{\boldsymbol{n}}) = \mathbf{w}_{\widehat{\boldsymbol{n}}}\, \widehat{\boldsymbol{n}}$. The tangential component is denoted by $\mathbf{w}(\text{tg}) = \mathbf{w} - \mathbf{w}(\widehat{\boldsymbol{n}})$. The notations w_N, w_T indicate respectively, the last component and the first N - 1 components of the vector function w in the canonical basis. For the functions w # $H_{div}(G)$, there is a *normal trace* defined on the boundary values, which will be denoted by $\mathbf{w} \cdot \widehat{\boldsymbol{n}} \in H^{-1/2}(\partial G)$. For those w # G $H^1(G)$ this agrees with $\gamma(\mathbf{w}) \cdot \widehat{\boldsymbol{n}}$. Greek letters are used to denote general second-order tensors. The contraction of two tensors is given by $\sigma : \kappa = \sum_{i,j} \sigma_{ij}\kappa_{ij}$. For a tensor-valued function *k* on *∂G,* we denote the normal component (vector) by $\kappa(\widehat{\boldsymbol{n}}) \overset{\text{def}}{=} \sum_j \kappa_{ij}\,\widehat{n}_j \in \mathbb{R}^N$, and its normal and tangential parts by $\kappa(\widehat{\boldsymbol{n}}) \cdot \widehat{\boldsymbol{n}} = \kappa(\widehat{\boldsymbol{n}})_{\widehat{\mathbf{n}}} \overset{\text{def}}{=} \sum_{i,j} \kappa_{ij}\widehat{n}_i\widehat{n}_j$, $\kappa(\widehat{\boldsymbol{n}})_{\text{tg}} \overset{\text{def}}{=} \kappa(\widehat{\boldsymbol{n}}) - \kappa_{\widehat{\mathbf{n}}}\,\widehat{\boldsymbol{n}}$, respectively. For a vector function w # $H^1(G)$, the tensor $(\nabla \mathbf{w})_{ij} = \frac{\partial w_i}{\partial x_j}$ is the *gradient* of w and the tensor (£(w)) = ^(ff^ + is the *symmetric gradient.*

The set $\mathcal{B}_0 \overset{\text{def}}{=} \{\widehat{e}_1, \ldots, \widehat{e}_N\}$ indicates the standard canonical basis in $\#^N$. For a column vector x = (x_1,..., x_{N-1}, x_N) # $\#^N$ we denote by $\widetilde{\mathbf{x}} = (x_1, \ldots, x_{N-1})$ the vector in $\#^{N-1}$ consisting of the first N - 1 components of x. In addition, we identify $\#^{N-1}$ x {0} with $\#^{N-1}$ by $\mathbf{x} = (\widetilde{\mathbf{x}}, x_N)$. The operators $\boldsymbol{\nabla}_T$, $\boldsymbol{\nabla}_T \cdot$ denote respectively the $\#^{N-1}$-gradient and the $\#^{N-1}$-divergence in the first N - 1-canonical directions, i.e., $\boldsymbol{\nabla}_T \overset{\text{def}}{=} \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{N-1}}\right)$; moreover, we regard $\#_T$ as a row vector. Finally, $\boldsymbol{\nabla}^t, \boldsymbol{\nabla}_T^t$ denote the corresponding operators written as column vectors.

Remark 1.1. It shall be noticed that different notations have been chosen to indicate the first N - 1 components: we use $\widetilde{\mathbf{x}}$ for a vector variable as x, while we use w_T for a vector function w (or the operator $\#_T$, #). This difference in notation will ease keeping track of the involved variables and will not introduce confusion.

Preliminary Results

We close this section recalling some classic results.

Lemma 1.2. *Let* G # $\#^N$ *be an open set with Lipschitz boundary, and* $\widehat{\boldsymbol{n}}$ be *the unit outward normal vector on* ∂G*. Let the normal trace operator* $\mathbf{u} \in \mathbf{H}_{\text{div}}(G) \mapsto \mathbf{u} \cdot \widehat{n} \in H^{-1/2}(\partial G)$ *be defined by*

$$\langle \mathbf{u} \cdot \widehat{n}, \phi \rangle_{H^{-1/2}(\partial G),\, H^{1/2}(\partial G)} \overset{\text{def}}{=} \int_G \left(\mathbf{u} \cdot \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \cdot \mathbf{u}\,\phi \right) dx, \quad \phi \in H^1(G). \tag{1}$$

For any g # $H^{-1/2}$(∂G) *there exists* u G H_{div}(G) *such that* $\mathbf{u} \cdot \widehat{\boldsymbol{n}} = g$ *on* ∂*G and* $\|\mathbf{u}\|_{\mathbf{H}_{\text{div}}(G)} \leq K \|g\|_{H^{-1/2}(\partial G)}$, *with K depending only on the domain G. In particular, if g belongs to* L^2*(∂G), the function* **u** *satisfies the estimate* $\|\mathbf{u}\|_{\mathbf{H}_{\text{div}}(G)} \leq K \|g\|_{0,\partial G}$.

Proof. See Lemma 20.2 in [19].

Next we recall a central result to be used in this work

Theorem 1.3. *Let* X, Y, X', Y' *be Hilbert spaces and their corresponding topological duals. Let A* : X → X', B : X → Y', C : Y → Y' *be linear and continuous operators satisfying the following conditions*

I. *A is non-negative and X-coercive on* ker(*B*);

II. B *satisfies the inf-sup condition*

$$\inf_{\mathbf{y} \in \mathbf{Y}} \sup_{\mathbf{x} \in \mathbf{X}} \frac{|\mathcal{B}\mathbf{x}(\mathbf{y})|}{\|\mathbf{x}\|_{\mathbf{X}} \, \|\mathbf{y}\|_{\mathbf{Y}}} > 0\,; \tag{2}$$

III. *C is non-negative and symmetric.*

Then, for every F_1 *#* X' *and* F_2 *#* Y', *the problem* (3) *below has a unique solution* (x, y) # X x Y:

$$(\mathbf{x},\mathbf{y}) \in \mathbf{X}\times\mathbf{Y}: \quad \begin{aligned} \mathcal{A}\mathbf{x} + \mathcal{B}'\mathbf{y} &= F_1 \quad in\ \mathbf{X}', \\ -\mathcal{B}\mathbf{x} + \mathcal{C}\mathbf{y} &= F_2 \quad in\ \mathbf{Y}'. \end{aligned} \tag{3}$$

Moreover, the solution satisfies the estimate

$$\|\mathbf{x}\|_{\mathbf{X}} + \|\mathbf{y}\|_{\mathbf{Y}} \leq c\left(\|F_1\|_{\mathbf{X}'} + \|F_2\|_{\mathbf{Y}'}\right), \tag{4}$$

for a positive constant c depending only on the preceding assumptions on A, B, and C.

Proof. See Section 4 in [10].

2. Geometric setting and formulation of the problem

In this section we introduce the Darcy-Stokes coupled system when the interface is curved, analogous to the one presented in [14]. We begin with the geometric setting

2.1. Geometric setting and change of coordinates

We describe here the geometry of the domains to be used in the present work; see Figure 1 (a) for the case N =2. The ϵ-domain $\Omega^\epsilon \overset{\text{def}}{=} \Omega_1 \cup \Gamma \cup \Omega_2^\epsilon$ is composed of two disjoint bounded open sets Ω_1 and Ω_2^ϵ in $\#^N$ sharing a common *interface* $\Gamma \overset{\text{def}}{=} \partial\Omega_1 \cap \partial\Omega_2^\epsilon \subseteq \mathbb{R}^N$. The domain Ω1 is the porous medium and Ω_2^ϵ is the free fluid region. For simplicity we have assumed that the domain Ω_2^ϵ is a cylinder defined by the interface Γ and a small height ϵ > 0. It follows that the interface must verify specific requirements for a successful analysis

Hypothesis 1. There exist G_0, G bounded open connected domains in $\#^{N-1}$ such that cl(G) # G_0, and a function ζ : G_0 → #, in $C^2(G_0)$, such that the interface Γ can be described by

$$\Gamma \overset{\text{def}}{=} \left\{\left(\widetilde{\mathbf{x}},\, \zeta\left(\widetilde{\mathbf{x}}\right)\right) : \widetilde{\mathbf{x}} \in G\right\}. \tag{5}$$

That is, Γ is aparametrized N - 1 manifold in $\#^N$. In addition, the domain Ω_2^ϵ is described by

$$\Omega_2^\epsilon \overset{\text{def}}{=} \left\{\left(\widetilde{\mathbf{x}},\, y\right) : \zeta\left(\widetilde{\mathbf{x}}\right) < y < \zeta\left(\widetilde{\mathbf{x}}\right) + \epsilon,\, \widetilde{\mathbf{x}} \in G\right\}. \tag{6}$$

Remark 2.1. I. Observe that the domain G is the orthogonal projection of the open surface Γ # $\#^N$ into $\#^{N-1}$.

II. Notice that due to the properties of ζ it must hold that if $\widehat{n} = \widehat{n}(\widetilde{\mathbf{x}})$ is the upwards unitary vector, orthogonal to the surface Γ, then

$$\delta \overset{\text{def}}{=} \inf\left\{\widehat{n}(\widetilde{\mathbf{x}}) \cdot \widehat{e}_N : \widetilde{\mathbf{x}} \in G\right\} > 0. \tag{7}$$

Here $\widehat{e}_N$ is the last element of the standard canonical basis in $\#^N$

For simplicity of notation in the following we write

III.

$$\Gamma + \epsilon \overset{\text{def}}{=} \{(\widetilde{\mathrm{x}}, \zeta(\widetilde{\mathrm{x}}) + \epsilon) : \widetilde{\mathrm{x}} \in G\}, \tag{8}$$

IV.

$$\Omega_2 \overset{\text{def}}{=} \Omega_2^1, \qquad \Omega \overset{\text{def}}{=} \Omega^1. \tag{9}$$

In order to conduct the asymptotic analysis of the coupled system, a domain of reference n needs to be settled (see Figure 1 (b)). Therefore, we adopt a bijection between domains and account for the changes in the differential operators.

Definition 2.2. Let $\varphi : \Omega_2^\epsilon \to \Omega$ be the change of variables defined by

$$\varphi(y_1, \ldots, y_{N-1}, y_N) \overset{\text{def}}{=} \begin{Bmatrix} y_1 \\ \vdots \\ y_{N-1} \\ \epsilon^{-1}(y_N - \zeta(y_1, \ldots, y_{N-1})) + \zeta(y_1, \ldots, y_{N-1}) \end{Bmatrix}, \tag{10}$$

with $\mathbf{y} = (y_1, \ldots, y_{N-1}, y_N) \in \Omega_2^\epsilon$. Also, denote $\mathbf{x} = (x_1, \ldots, x_{N-1}, z) \overset{\text{def}}{=} \varphi(\mathbf{y})$, i.e.,

$$\mathbf{x} = (x_1, \ldots, x_{N-1}, z) \in \Omega_2, \qquad \mathbf{x} \cdot \widehat{e}_\ell = \varphi(\mathbf{y}) \cdot \widehat{e}_\ell \text{ for all } \ell = 1, \ldots N, \tag{11}$$

where $\widehat{e}_1, \ldots, \widehat{e}_N$ are the standard canonical basis in $\#^{N}$.

Remark 2.3. Observe that $\varphi : \Omega_2^\epsilon \to \Omega_2$ is a bijective map (see Figure 1 (b)).

Gradient operator

Denote by ^{y}, $^{x}\#$ the gradient operators with respect to the variables y and x respectively. Due to the convention of equation 9 above, a direct computation shows that these operators satisfy the relationship

$$^{\mathrm{y}}\boldsymbol{\nabla}^t = \begin{Bmatrix} ^{\mathrm{y}}\boldsymbol{\nabla}_T^t \\ \frac{\partial}{\partial y_N} \end{Bmatrix} = \begin{bmatrix} I & (1-\epsilon^{-1})\, ^{\mathrm{x}}\boldsymbol{\nabla}_T{}^t\zeta \\ \mathbf{0} & \epsilon^{-1} \end{bmatrix} \begin{Bmatrix} ^{\mathrm{x}}\boldsymbol{\nabla}_T{}^t \\ \partial_z \end{Bmatrix}. \tag{12}$$

In the block matrix notation above, it is understood that I is the identity matrix in $\mathbb{R}^{(N-1)\times(N-1)}$, $\boldsymbol{\nabla}_T\zeta, \mathbf{0}$ are vectors in $\#^{N-1}$ and $\partial_z = \frac{\partial}{\partial z}$. In order to apply these changes to the gradient of a vector function **w**, we recall the matrix notation

$$^{\mathrm{y}}\boldsymbol{\nabla}\,\mathbf{w} = \begin{bmatrix} ^{\mathrm{y}}\boldsymbol{\nabla} w_1 \\ \vdots \\ ^{\mathrm{y}}\boldsymbol{\nabla} w_N \end{bmatrix} = \begin{bmatrix} ^{\mathrm{x}}\boldsymbol{\nabla}_T w_1 + (1-\epsilon^{-1})\partial_z w_1\, ^{\mathrm{x}}\boldsymbol{\nabla}_T\zeta & \epsilon^{-1}\,\partial_z w_1 \\ \vdots & \vdots \\ ^{\mathrm{x}}\boldsymbol{\nabla}_T w_N + (1-\epsilon^{-1})\partial_z w_N\, ^{\mathrm{x}}\boldsymbol{\nabla}_T\zeta & \epsilon^{-1}\,\partial_z w_N \end{bmatrix}. \tag{13}$$

Reordering we get

$$^{\mathrm{y}}\boldsymbol{\nabla}\mathbf{w}\,(\widetilde{\mathrm{x}}, x_N) = \begin{bmatrix} ^{\mathrm{x}}\boldsymbol{D}^\epsilon\mathbf{w} & \frac{1}{\epsilon}\,\partial_z\,\mathbf{w} \end{bmatrix}. \tag{14}$$

Here, the operator $^{x}£)^{e}$ is defined by

$$^{\mathrm{x}}\boldsymbol{D}^\epsilon\mathbf{w} \overset{\text{def}}{=} {}^{\mathrm{x}}\boldsymbol{\nabla}_T\mathbf{w} + \Big(1 - \frac{1}{\epsilon}\Big)\partial_z\mathbf{w}\, ^{\mathrm{x}}\boldsymbol{\nabla}_T{}^t\zeta, \tag{15}$$

i.e., $^{\mathrm{x}}\boldsymbol{D}^\epsilon\,\mathbf{w} \in \mathbb{R}^{N\times(N-1)}$; it is introduced to have a more efficient notation. In the next section we address the interface conditions.

Divergence operator

Observing the diagonal of the matrix in (13) we have

$$^{\mathrm{y}}\boldsymbol{\nabla}\cdot\mathbf{w}\,(\widetilde{\mathrm{x}}, x_N) = \Big(^{\mathrm{x}}\boldsymbol{\nabla}_T\cdot\mathbf{w}_T + \Big(1 - \frac{1}{\epsilon}\Big)\partial_z\,\mathbf{w}_T\cdot{}^{\mathrm{x}}\boldsymbol{\nabla}_T\cdot\zeta\Big)(\widetilde{\mathrm{x}}, z) + \frac{1}{\epsilon}\partial_z\mathbf{w}_N\,(\widetilde{\mathrm{x}}, z). \tag{16}$$

Remark 2.4. The prescript indexes y, x written on the operators above were used only to derive the relation between them; however, they will be dropped once the context is clear.

Local vs global vector basis

It shall be seen later on, that the velocities in the channel need to be expressed in terms of an orthonormal basis B, such that the normal vector $\widehat{n}$ belongs to B, and the remaining vectors are locally tangent to the interface Γ. Since ζ : G → # is a C^2 function, it follows that $\widetilde{\mathrm{x}} \mapsto \widehat{n}(\widetilde{\mathrm{x}})$ is at least C^1.

Definition 2.5. Let $\mathcal{B}_0 \overset{\text{def}}{=} \{\widehat{e}_1, \ldots, \widehat{e}_{N-1}, \widehat{e}_N\}$ be the standard canonical basis in $\#^N$. For any $\widetilde{\mathrm{x}} \in G$ let $\mathcal{B} = \mathcal{B}(\widetilde{\mathrm{x}}) \overset{\text{def}}{=} \{\widehat{\nu}_1, \ldots, \widehat{\nu}_{N-1}, \widehat{n}\}$ be an orthonormal basis in $\#^N$. Define the linear map $U(\widetilde{\mathrm{x}}) : \mathbb{R}^N \to \mathbb{R}^N$ by

$$U(\widetilde{\mathrm{x}})\widehat{\nu}_i \overset{\text{def}}{=} \widehat{e}_i, \text{ for } i = 1, \ldots, N-1, \qquad U(\widetilde{\mathrm{x}})\widehat{n} \overset{\text{def}}{=} \widehat{e}_N. \tag{17}$$

We say the map $\widetilde{\mathrm{x}} \mapsto U(\widetilde{\mathrm{x}})$ is a **stream line localizer** if it is of class C^1. In the sequel we write it with the following block matrix notation:

$$U(\widetilde{\mathrm{x}}) \overset{\text{def}}{=} \begin{bmatrix} U^{T,\mathrm{tg}}(\widetilde{\mathrm{x}}) & U^{T,\widehat{n}}(\widetilde{\mathrm{x}}) \\ U^{N,\mathrm{tg}}(\widetilde{\mathrm{x}}) & U^{N,\widehat{n}}(\widetilde{\mathrm{x}}) \end{bmatrix}. \tag{18}$$

Here, the indexes T and N stand for the first N - 1 components and the last component of the vector field. The indexes tg and $\widehat{n}$ indicate the tangent and normal directions to the interface Γ.

Remark 2.6. I. Since ζ is bounded an C^2(C), clearly for each $\widetilde{\mathrm{x}} \in G$, a basis $\mathcal{B} = \{\widehat{\nu}_1, \ldots, \widehat{\nu}_{N-1}, \widehat{n}\}$ can be chosen so that $\widetilde{\mathrm{x}} \mapsto U(\widetilde{\mathrm{x}})$ is C^1. In the following it will be assumed that U is a stream line localizer.

II. Notice that by definition $U(\widetilde{\mathrm{x}})$ is an orthogonal matrix for all $\widetilde{\mathrm{x}} \in G$.

Next, we express the velocity fields w^2 in terms of the normal and tangential components, using the following relations:

$$\mathbf{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} = \mathbf{w}^2 \cdot \widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}}), \tag{19a}$$

$$\mathbf{w}^2_{\mathrm{tg}(\widetilde{x})} = \begin{Bmatrix} \mathbf{w}^2 \cdot \widehat{\boldsymbol{\nu}}_1(\widetilde{\mathbf{x}}) \\ \vdots \\ \mathbf{w}^2 \cdot \widehat{\boldsymbol{\nu}}_{N-1}(\widetilde{\mathbf{x}}) \end{Bmatrix}. \tag{19b}$$

Clearly, if $\mathrm{w}^2 = \mathrm{w}^2(\widetilde{\mathrm{x}}, x_N)$ is expressed in terms of the canonical basis, the relationship between velocities is given by

$$\begin{aligned} \mathrm{w}^2(\widetilde{\mathrm{x}}, x_N) &= U(\widetilde{\mathrm{x}}) \begin{Bmatrix} \mathrm{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \mathrm{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} (\widetilde{\mathrm{x}}, x_N) \\ &= \begin{bmatrix} U^{T,\mathrm{tg}}(\widetilde{\mathrm{x}}) & U^{T,\widehat{n}}(\widetilde{\mathrm{x}}) \\ U^{N,\mathrm{tg}}(\widetilde{\mathrm{x}}) & U^{N,\widehat{n}}(\widetilde{\mathrm{x}}) \end{bmatrix} \begin{Bmatrix} \mathrm{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \mathrm{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} (\widetilde{\mathrm{x}}, x_N). \end{aligned} \tag{20}$$

Remark 2.7. We stress the following observations

I. The procedure above does not modify the dependence of the variables; only the way velocity fields are expressed as linear combinations of a convenient (stream line) orthonormal basis.

II. The fact that U is a smooth function allows to claim that $\mathrm{w}^2_{\mathrm{tg}}$ belongs to $[H^1(\Omega_2)]^{N-1}$ and $\mathrm{w}^2_{\widehat{\mathbf{n}}} \in H^1(\Omega_2)$.

III. In order to keep notation as light as possible, the dependence of the matrix U with respect to $\widetilde{\mathbf{x}}$, as well as the normal and tangential directions $\widehat{n}$, tg will be omitted whenever is not necessary to write explicitly these parameters.

IV. Recall that for any vector field $\mathbf{v}$, $\mathbf{v}(\widehat{n}) = (\mathbf{v}\cdot\widehat{n})\widehat{n}$ denotes its normal projection on the direction $\widehat{n}$, while v(tg) = v - v($\widehat{n}$), i.e., the component orthogonal to $\widehat{n}$ (and tangent to Γ). Considering the previous, given any two flow fields $\mathbf{u}^2$, $\mathbf{w}^2$, the following isometric identities hold:

$$\begin{aligned} \mathbf{u}^2_{\mathrm{tg}}\cdot\mathbf{w}^2_{\mathrm{tg}} &= \mathbf{u}^2(\mathrm{tg})\cdot\mathbf{w}^2(\mathrm{tg}),\\ \mathbf{u}^2_{\widehat{\mathbf{n}}}\cdot\mathbf{w}^2_{\widehat{\mathbf{n}}} &= \mathbf{u}^2(\widehat{n})\cdot\mathbf{w}^2(\widehat{n}),\\ \mathbf{u}^2\cdot\mathbf{w}^2 &= \mathbf{u}^2(\mathrm{tg})\cdot\mathbf{w}^2(\mathrm{tg}) + \mathbf{u}^2(\widehat{n})\cdot\mathbf{w}^2(\widehat{n}) = \mathbf{u}^2_{\mathrm{tg}}\cdot\mathbf{w}^2_{\mathrm{tg}} + \mathbf{u}^2_{\widehat{\mathbf{n}}}\mathbf{w}^2_{\widehat{\mathbf{n}}}. \end{aligned} \tag{21}$$

Proposition 2.8. *Let* w^2 # $\mathrm{H}^1(\Omega_2)$; *and let* $\mathbf{w}^2_{\widehat{\mathbf{n}}}, \mathbf{w}^2_{\mathrm{tg}}$ *be as defined in* (19). *Then,*

I.

$$\partial_z \mathbf{w}^2(\widetilde{\mathbf{x}}, x_N) = U(\widetilde{\mathbf{x}}) \begin{Bmatrix} \partial_z\, \mathbf{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \partial_z\, \mathbf{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_N). \tag{22}$$

II.

$$\begin{aligned} \|\partial_z \mathbf{w}^2\|^2_{0,\Omega_2} &= \|\partial_z \mathbf{w}^2_T\|^2_{0,\Omega_2} + \|\partial_z \mathbf{w}^2_N\|^2_{0,\Omega_2} \\ &= \|\partial_z \mathbf{w}^2_{\mathrm{tg}}\|^2_{0,\Omega_2} + \|\partial_z \mathbf{w}^2_{\widehat{\mathbf{n}}}\|^2_{0,\Omega_2}. \end{aligned} \tag{23}$$

Proof. I. It suffices to observe that the orthogonal matrix *U* defined in (20) is independent from *z*.

II. Due to (22), we have

$$\begin{aligned} \left|\partial_z \mathbf{w}^2(\widetilde{\mathbf{x}}, x_N)\right|^2 &= \partial_z \mathbf{w}^2(\widetilde{\mathbf{x}}, x_N)\cdot\partial_z \mathbf{w}^2(\widetilde{\mathbf{x}}, x_N)\\ &= U(\widetilde{\mathbf{x}}) \begin{Bmatrix} \partial_z\, \mathbf{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \partial_z\, \mathbf{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_N)\cdot U(\widetilde{\mathbf{x}}) \begin{Bmatrix} \partial_z\, \mathbf{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \partial_z\, \mathbf{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_N)\\ &= \left| \begin{Bmatrix} \partial_z\, \mathbf{w}^2_{\mathrm{tg}(\widetilde{x})} \\ \partial_z\, \mathbf{w}^2_{\widehat{\mathbf{n}}(\widetilde{x})} \end{Bmatrix} \right|^2. \end{aligned}$$

The last equality holds true because the matrix U ($\widetilde{\mathbf{x}}$) is orthogonal at each point $\widetilde{\mathbf{x}} \in G$, therefore it is an isometry in the Hilbert space $\#^{\mathrm{N}}$ endowed with the standard inner product. Recalling that $|\partial_z \mathbf{w}^2_T(\widetilde{\mathbf{x}}, x_N)|^2 + |\partial_z \mathbf{w}^2_N(\widetilde{\mathbf{x}}, x_N)|^2 = |\partial_z \mathbf{w}^2(\widetilde{\mathbf{x}}, x_N)|^2$ for all $\mathbf{x} = (\widetilde{\mathbf{x}}, x_N)$, the result follows.

2.2. Interface conditions and the strong form

The interface conditions need to account for stress and mass balance. We start decomposing the stress in its tangential and normal components; the former is handled by the *Beavers-Joseph-Saffman* (24a) condition and the latter by the classical *Robin* boundary condition (24b); this gives

$$\sigma^2(\widehat{n})_{\mathrm{tg}} = \epsilon^2\, \beta\, \sqrt{Q}\, \mathbf{v}^2(\mathrm{tg})\,, \tag{24a}$$

$$\sigma^2(\widehat{n})_{\widehat{\mathbf{n}}} - p^2 + p^1 = \alpha\, \mathbf{v}^1\cdot\widehat{n} \text{ on } \Gamma. \tag{24b}$$

In the expression (24a) above, e^2 is a scaling factor introduced to balance out the geometric singularity coming from the thinness of the channel. In addition, the coefficient $\alpha \geq 0$ in (24b) is the *fluid entry resistance.*

Next, recall that the stress satisfies $\sigma^2 = 2\epsilon\mu\boldsymbol{\mathcal{E}}(\mathrm{v}^2)$ (where the scale e is introduced according to the thinness of the channel and μ > 0 is the shear viscosity of the fluid; see also Hypothesis 2) and that #· v^2 =0 (since the system is conservative); then we have

$$\nabla \cdot \sigma^2 = \nabla \cdot \left[2\,\epsilon\,\mu\,\boldsymbol{\mathcal{E}}(\mathbf{v}^2)\right] = \epsilon\,\mu\nabla \cdot \nabla \mathbf{v}^2.$$

Replacing in the equations (24) we derive the following set of interface conditions:

$$\epsilon\mu\left(\frac{\partial\,\mathbf{v}^2}{\partial\,\widehat{n}} - \left(\frac{\partial\,\mathbf{v}^2}{\partial\,\widehat{n}}\cdot\widehat{n}\right)\widehat{n}\right) = \epsilon^2\beta\,\sqrt{Q}\,\mathbf{v}^2(\mathrm{tg})\,, \tag{25a}$$

$$\epsilon\mu\left(\frac{\partial\,\mathbf{v}^2}{\partial\,\widehat{n}}\cdot\widehat{n}\right) - p^2 + p^1 = \alpha\,\mathbf{v}^1\cdot\widehat{n}, \tag{25b}$$

$$\mathbf{v}^1\cdot\widehat{n} = \mathbf{v}^2\cdot\widehat{n}\ \text{on}\ \Gamma. \tag{25c}$$

The condition (25c) states the fluid flow (or mass) balance.

With the previous considerations, the Darcy-Stokes coupled system in terms of the velocity v and the pressure *p* is given by

$$\nabla\cdot\mathbf{v}^1 = h_1\,, \tag{26a}$$

$$Q\,\mathbf{v}^1 + \nabla p^1 = \mathbf{0}\,,\quad \text{in}\ \Omega_1. \tag{26b}$$

$$\nabla\cdot\mathbf{v}^2 = 0, \tag{26c}$$

$$-\nabla\cdot 2\epsilon\mu\boldsymbol{\mathcal{E}}(\mathbf{v}^2) + \nabla p^2 = \mathrm{f}_2\,,\quad \text{in}\ \Omega_2. \tag{26d}$$

Here, equations (26a), (26b) correspond to the Darcy flow filtration through the porous medium, while equations (26c) and (26d) stand for the Stokes free flow. Finally, we adopt the following boundary conditions:

$$p^1 = 0\quad \text{on}\ \partial\Omega_1 - \Gamma. \tag{27a}$$

$$\mathbf{v}^2 = 0\quad \text{on}\ \partial\Omega_2^{\epsilon} - (\Gamma+\epsilon). \tag{27b}$$

$$\frac{\partial\,\mathbf{v}^2}{\partial\,\widehat{n}} - \left(\frac{\partial\,\mathbf{v}^2}{\partial\,\widehat{n}}\cdot\widehat{n}\right)\widehat{n} = 0\quad \text{on}\ \Gamma+\epsilon, \tag{27c}$$

$$\mathbf{v}^2\cdot\widehat{n} = \mathbf{v}^2_{\widehat{n}} = 0\quad \text{on}\ \Gamma+\epsilon. \tag{27d}$$

The system of equations (26), (27) and (25) constitute the strong form of the Darcy-Stokes coupled system.

Remark 2.9. i. For a detailed exposition on the system's scaling, namely, the fluid stress tensor $\sigma^2 = 2\epsilon\mu\boldsymbol{\mathcal{E}}(\mathrm{v}^2)$ and the Beavers-Joseph-Saffman condition (24a), together with the *formal* asymptotic analysis, we refer to [15].

II. A deep discussion on the role of each physical variable and parameter in equations (26), as well as the *meaning* of the boundary conditions (27), can be found in Sections 1.2, 1.3 and 1.4 in [14].

2.3. Weak variational formulation and a reference domain

In this section we present the weak variational formulation of the problem defined by the system of equations (26), (27) and (25), on the domain Ω^{ϵ}. Next, we rescale Ω_2^{ϵ} to get a uniform domain of reference. We begin defining the function spaces where the problem is modeled.

Definition 2.10. Let $\Omega, \Omega_1, \Omega_2^{\epsilon}, \Gamma$ be as introduced in Section 2.1; in particular, Ω_2 and Γ satisfy Hypothesis 1. Define the spaces

$$\mathbf{X}_2^{\epsilon} \overset{\text{def}}{=} \{\mathbf{v} \in \mathbf{H}^1(\Omega_2^{\epsilon}) : \mathbf{v} = 0 \text{ on } \partial\Omega_2^{\epsilon} - (\Gamma + \epsilon),\ \mathbf{v} \cdot \widehat{n} = 0 \text{ on } \Gamma + \epsilon\}, \tag{28a}$$

$$\begin{aligned} \mathbf{X}^{\epsilon} &\overset{\text{def}}{=} \{[\mathbf{v}^1, \mathbf{v}^2] \in \mathbf{H}_{\text{div}}(\Omega_1^{\epsilon}) \times \mathbf{X}_2^{\epsilon} : \mathbf{v}^1 \cdot \widehat{n} = \mathbf{v}^2 \cdot \widehat{n} \text{ on } \Gamma\} \\ &= \{\mathbf{v} \in \mathbf{H}_{\text{div}}(\Omega^{\epsilon}) : \mathbf{v}^2 \in \mathbf{X}_2^{\epsilon}\}, \end{aligned} \tag{28b}$$

$$\mathbf{Y}^{\epsilon} \overset{\text{def}}{=} L^2(\Omega^{\epsilon}), \tag{28c}$$

endowed with their respective natural norms. Moreover, for $\epsilon = 1$ we simply write X, X_2 and Y.

In order to attain well-posedness of the problem, the following hypothesis is adopted.

Hypothesis 2. It will be assumed that μ, > 0 and that the coefficients β, α are non-negative and bounded almost everywhere. Moreover, the tensor $\mathcal{Q}$ is elliptic, i.e., there exists a positive constant C_Q such that $(\mathcal{Q}\mathbf{x}) \cdot \mathbf{x} \geq C_{\mathcal{Q}} \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^N$.

Theorem 2.11. *Consider the boundary-value problem defined by the equations* (26), *the interface coupling conditions* (25) *and the boundary conditions* (27); *then,*

I. *A weak variational formulation of the problem is given by*

$$[\mathbf{v}^{\epsilon}, p^{\epsilon}] \in \mathbf{X}^{\epsilon} \times \mathbf{Y}^{\epsilon} :$$

$$\begin{aligned} &\int_{\Omega_1} (\mathcal{Q}\,\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1 - p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}^1)\, dy + \int_{\Omega_2^{\epsilon}} (\epsilon\, \mu \boldsymbol{\nabla}\, \mathbf{v}^{2,\epsilon} - p^{2,\epsilon}\delta) : \boldsymbol{\nabla} \mathbf{w}^2\, d\widetilde{\mathbf{y}} d\mathbf{y}_N \\ &\quad + \alpha \int_{\Gamma} (\mathbf{v}^{2,\epsilon} \cdot \widehat{n})\, (\mathbf{w}^2 \cdot \widehat{n})\, dS + \int_{\Gamma} \epsilon^2 \beta \sqrt{\mathcal{Q}}\, \mathbf{v}_{\text{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\text{tg}}^2\, dS \\ &\qquad = \int_{\Omega_2^{\epsilon}} \mathbf{f}^{2,\epsilon} \cdot \mathbf{w}^2\, d\widetilde{\mathbf{y}}\, d\mathbf{y}_N, \end{aligned} \tag{29a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} \varphi^1\, dy + \int_{\Omega_2^{\epsilon}} \boldsymbol{\nabla} \cdot \mathbf{v}^{2,\epsilon} \varphi^2\, d\widetilde{\mathbf{y}}\, d\mathbf{y}_N = \int_{\Omega_1} h^{1,\epsilon} \varphi^1\, dy, \tag{29b}$$

$$\textit{for all } [\mathbf{w}, \varphi] \in \mathbf{X}^{\epsilon} \times \mathbf{Y}^{\epsilon}.$$

II. *The problem* (29) *is well-posed.*

III. *The problem* (29) *is equivalent to*

$$[\mathbf{v}^{\epsilon}, \mathbf{p}^{\epsilon}] \in \mathbf{X} \times \mathbf{Y} :$$

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1\, d\mathbf{x} - \int_{\Omega_1} p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}^1\, d\mathbf{x} - \epsilon \int_{\Omega_2} p^{2,\epsilon}\, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2\, d\widetilde{\mathbf{x}}\, dz \\ &\quad - \epsilon\left(1 - \frac{1}{\epsilon}\right) \int_{\Omega_2} p^{2,\epsilon}\, \partial_z\, \mathbf{w}_T^2 \cdot \boldsymbol{\nabla}_T \zeta\, d\widetilde{\mathbf{x}}\, dz - \int_{\Omega_2} p^{2,\epsilon}\, \partial_z \mathbf{w}_N^2\, d\widetilde{\mathbf{x}}\, dz \\ &\quad + \epsilon^2 \int_{\Omega_2} \mu\, \boldsymbol{D}^{\epsilon} \mathbf{v}^{2,\epsilon} : \boldsymbol{D}^{\epsilon} \mathbf{w}^2\, d\widetilde{\mathbf{x}}\, dz + \int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}^{2,\epsilon} \cdot \partial_z\, \mathbf{w}^2\, d\widetilde{\mathbf{x}}\, dz \\ &\quad + \alpha \int_{\Gamma} (\mathbf{v}^{1,\epsilon} \cdot \widehat{n})\, (\mathbf{w}^1 \cdot \widehat{n})\, dS + \epsilon^2 \int_{\Gamma} \beta \sqrt{\mathcal{Q}}\, \mathbf{v}_{\text{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\text{tg}}^2\, dS \\ &\qquad = \epsilon \int_{\Omega_2} \mathbf{f}^{2,\epsilon} \cdot \mathbf{w}^2\, d\widetilde{\mathbf{x}}\, dz, \end{aligned} \tag{30a}$$

$$\begin{aligned} &\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} \varphi^1\, d\mathbf{x} + \epsilon \int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \mathbf{v}_T^{2,\epsilon} \varphi^2\, d\widetilde{\mathbf{x}}\, dz \\ &\quad + \epsilon\left(1 - \frac{1}{\epsilon}\right) \int_{\Omega_2} \partial_z \mathbf{v}_T^{2,\epsilon} \cdot \boldsymbol{\nabla}_T \zeta\, \varphi^2\, d\widetilde{\mathbf{x}}\, dz + \int_{\Omega_2} \partial_z \mathbf{v}_N^{2,\epsilon} \varphi^2\, d\widetilde{\mathbf{x}}\, dz = \int_{\Omega_1} h^{1,\epsilon} \varphi^1\, d\mathbf{x}, \end{aligned} \tag{30b}$$

$$\textit{for all } [\mathbf{w}, \Phi] \in \mathbf{X} \times \mathbf{Y}.$$

Proof. I. See Proposition 3 in [14]. We simply highlight that the term $\int_{\Omega_2} \epsilon^2 \beta \sqrt{Q}\, \mathbf{v}_T^{2,\epsilon} \cdot \mathbf{w}_T^2\, dS$ has been replaced by $\int_{\Omega_2} \epsilon^2 \beta \sqrt{Q}\, \mathbf{v}_{\text{tg}}^2 \cdot \mathbf{w}_{\text{tg}}^2\, dS,$ due to the isometric identities [21].

II. See Theorem 6 in [14]. The technique identifies the operators A, *B*, *C* in the variational statements (29a) and (29b), then it verifies that these operators satisfy the hypotheses of Theorem 1.3; this result delivers well-posedness.

III. A direct substitution of the expressions (14) and (16) in the statements (29), combined with the definition (15) yields the system (30). (Also notice that the determinant of the matrix in the right hand side of the equation (14) is equal to ϵ^{-1}.) Finally, observe that the boundary conditions of space $\mathbf{X}_2^\epsilon$, defined in (28a) are transformed into the boundary conditions of X_2 because none of them involve derivatives.

Remark 2.1f. In order to prevent heavy notation, from now on we denote the volume integrals by $\int_{\Omega_1} F = \int_{\Omega_1} F\, dx$ and $\int_{\Omega_2} F = \int_{\Omega_2} F\, d\widetilde{x}\, dz.$ We will use the explicit notation $\int_{\Omega_2} F\, d\widetilde{x}\, dz$ only when specific calculations are needed. Both notations will be clear from the context.

3. *Asymptotic analysis*

In this section, we present the asymptotic analysis of the problem, i.e., we obtain a-priori estimates for the solutions $((v^\epsilon, p^\epsilon) : \epsilon > 0)$, derive weak limits and conclude features about them (velocity and pressure). We start recalling a classical space.

Definition 3.1. Let Ω_2 be as in Definition 1 and define the Hilbert spaces

$$H(\partial_z, \Omega_2) \overset{\text{def}}{=} \{w \in L^2(\Omega_2) : \partial_z w \in L^2(\Omega_2)\}, \tag{31a}$$

$$\mathbf{H}(\partial_z, \Omega_2) \overset{\text{def}}{=} \{\mathbf{w} \in L^2(\Omega_2) : \partial_z \mathbf{w} \in L^2(\Omega_2)\}, \tag{31b}$$

endowed with the corresponding inner products

$$\langle u, v \rangle_{H(\partial_z, \Omega_2)} \overset{\text{def}}{=} \int_{\Omega_2} (u\, v + \partial_z u\, \partial_z v), \tag{31c}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{H}(\partial_z, \Omega_2)} \overset{\text{def}}{=} \int_{\Omega_2} (\mathbf{u}\, \mathbf{v} + \partial_z \mathbf{u}\, \partial_z \mathbf{v}). \tag{31d}$$

Lemma 3.2. 1. *Let* $H(\partial_z, \Omega_2)$ *be the space introduced in Definition 3.1; then, the trace map* $w \mapsto w|_\Gamma$ *from* $H(\partial_z, \Omega_2)$ *to* $L^2(\Gamma)$ *is well-defined. Moreover, the following Poincaré-type inequalities hold in this space:*

$$\|w\|_{0,\Gamma} \leq \sqrt{2}\left(\|w\|_{0,\Omega_2} + \|\partial_z w\|_{0,\Omega_2}\right), \tag{32a}$$

$$\|w\|_{0,\Omega_2} \leq \sqrt{2}\left(\|\partial_z w\|_{0,\Omega_2} + \|w\|_{0,\Gamma}\right), \tag{32b}$$

for all w # $H(\partial_z, \Omega_2)$.

II. *Let* $\mathbf{H}(\partial_z, \Omega_2)$ *be the vector space introduced in Definition 3.1; then, for any w #* $\mathbf{H}(\partial_z, \Omega_2)$ *the estimates analogous to* (32) *hold.*

III. *Let* w^2 # $H^1(\Omega_2)$ # $H(\partial_z, \Omega_2)$ *and let* $w^2_{\hat{n}}, w^2_{tg}$ *be as defined in* (19); *then,*

$$\| w^2_{\hat{n}} \|_{0,\Omega_2} \leq \| \partial_z w^2_{\hat{n}} \|_{0,\Omega_2} + 2 \| w^2_{\hat{n}} \|_{0,\Gamma}, \tag{33a}$$

$$\| w^2_{tg} \|_{0,\Omega_2} \leq \| \partial_z w^2_{tg} \|_{0,\Omega_2} + 2 \| w^2_{tg} \|_{0,\Gamma}. \tag{33b}$$

Proof. I. The proof is a direct application of the fundamental theorem of calculus on the smooth functions $C^\infty(\Omega_2)$, which is a dense subspace in $H(\partial_z, \Omega_2)$.

II. A direct application of equations (32) on each coordinate of w # $H(\partial_z, \Omega_2)$ delivers the result.

III. It follows from a direct application of (i) and (ii) on $w^2_{\hat{n}}, w^2_{tg}$ respectively.

Next we show that the sequence of solutions is globally bounded under the following hypothesis.

Hypothesis 3. In the following, it will be assumed that the sequences $(f^{2,\epsilon} : \epsilon > 0) \subseteq L^2(\Omega_2)$ and $(h^{1,\epsilon} : \epsilon > 0) \subseteq L^2(\Omega_1)$ are bounded, i.e., there exists C > 0 such that

$$\|f^{2,\epsilon}\|_{0,\Omega_2} \leq C, \qquad \|h^{1,\epsilon}\|_{0,\Omega_1} \leq C, \qquad \text{for all } \epsilon > 0. \tag{34}$$

Theorem 3.3 (Global a-priori Estimate). *Let* $([v^\epsilon, p^\epsilon] : e > 0)$ # X x Y *be the sequence of solutions to the* ϵ*-Problems* (30). *There exists a constant* $K > 0$ *such that*

$$\|v^{1,\epsilon}\|^2_{0,\Omega_1} + \| D^\epsilon(\epsilon\, v^{2,\epsilon})\|^2_{0,\Omega_2} + \|\partial_z v^{2,\epsilon}\|^2_{0,\Omega_2} + \|v^{2,\epsilon}_{\hat{n}}\|^2_{0,\Gamma} + \|\epsilon\, v^{2,\epsilon}_{tg}\|^2_{0,\Gamma} \leq K, \quad \text{for all } \epsilon > 0. \tag{35}$$

Proof. Set w = v^ϵ in (30a), $\varphi = p^\epsilon$ in (30b) and add them together. (Observe that the mixed terms were canceled out on the diagonal.) Next, apply the Cauchy-Bunyakowsky-Schwartz inequality to the right hand side and recall the Hypothesis 2; this gives

$$\begin{aligned} &\| v^{1,\epsilon} \|^2_{0,\Omega_1} + \epsilon^2 \int_{\Omega_2} D^\epsilon v^{2,\epsilon} : D^\epsilon v^{2,\epsilon} + \| \partial_z v^{2,\epsilon} \|^2_{0,\Omega_2} + \| v^{1,\epsilon} \cdot \hat{n} \|^2_{0,\Gamma} + \| \epsilon\, v^{2,\epsilon}_{tg} \|^2_{0,\Gamma} \\ &\quad \leq \frac{1}{k}\Big(\|f^{2,\epsilon}_{\tau}\|_{0,\Omega_2} \|(\epsilon\, v^{2,\epsilon}_{tg})\|_{0,\Omega_2} + \|f^{2,\epsilon}_{\hat{n}}\|_{0,\Omega_2} \|(\epsilon\, v^{2,\epsilon}_{\hat{n}})\|_{0,\Omega_2} + \int_{\Omega_1} h^{1,\epsilon} p^{1,\epsilon}\, dx \Big). \end{aligned} \tag{36}$$

We continue focusing on the last summand of the right hand side in the expression above, i.e.,

$$\begin{aligned} \int_{\Omega_1} h^{1,\epsilon} p^{1,\epsilon}\, dx &\leq \| p^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \\ &\leq C \| \nabla p^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \\ &= \| \mathcal{Q}\, v^{1,\epsilon} \|_{0,\Omega_1} \| h^{1,\epsilon} \|_{0,\Omega_1} \leq \widetilde{C} \| v^{1,\epsilon} \|_{0,\Omega_1}. \end{aligned} \tag{37}$$

The second inequality holds due to Poincaré's inequality, given that $p^{1,\epsilon} = 0$ on $\partial\Omega_1 - \Gamma$, as stated in Equation (27a). The equality holds due to (26b). The third inequality holds because the tensor $\mathcal{Q}$ and the family of sources $(h^{1,\epsilon} : \epsilon > 0) \subset L^2(\Omega_1)$ are bounded as stated in Hypothesis 2 and Hypothesis 3 (Equation (34)), respectively. Next, we control the $L^2(\Omega_2)$-norm of $v^{2,\epsilon}$. Since $v^{2,\epsilon}$ # $H^1(\Omega_2)$ # $H(\partial_z, \Omega_2)$, the estimates (33) apply;

combining them with (37) and the bound (34) (from Hypothesis 3) in Inequality (36) we have

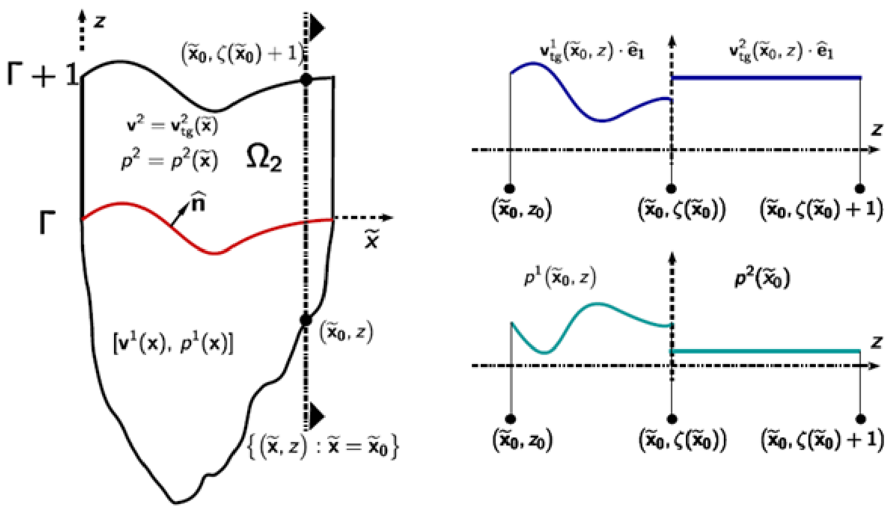
$$\begin{aligned}
&\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1}^2 + \epsilon^2 \int_{\Omega_2} D^\epsilon \mathbf{v}^{2,\epsilon} : D^\epsilon \mathbf{v}^{2,\epsilon} + \|\partial_z \mathbf{v}^{2,\epsilon}\|_{0,\Omega_2}^2 + \|\mathbf{v}^{1,\epsilon} \cdot \widehat{\boldsymbol{n}}\|_{0,\Gamma}^2 + \|\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon}\|_{0,\Gamma}^2 \\
&\quad \leq C\Big(\| \partial_z \left(\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon}\right) \|_{0,\Omega_2} + 2\| \left(\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon}\right) \|_{0,\Gamma} + \| \partial_z \left(\epsilon\, \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon}\right) \|_{0,\Omega_2} \\
&\qquad + 2\| \left(\epsilon\, \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon}\right) \|_{0,\Gamma} + \widetilde{C}\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \Big) \\
&\quad \leq C\Big(\| \partial_z \left(\epsilon\, \mathbf{v}^{2,\epsilon}\right) \|_{0,\Omega_2} + 2\| \left(\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon}\right) \|_{0,\Gamma} + 2\| \left(\epsilon\, \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon}\right) \|_{0,\Gamma} + \widetilde{C}\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \Big).
\end{aligned}$$

Here, the last inequality is due to the equality $\|\partial_z \left(\epsilon\, \mathbf{v}^{2,\epsilon}\right)\|_{0,\Omega_2}^2 = \|\partial_z \left(\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon}\right)\|_{0,\Omega_2}^2 + \|\partial_z \left(\epsilon\, \mathbf{v}_{N}^{2,\epsilon}\right)\|_{0,\Omega_2}^2$. Next, using the equivalence of norms $\| \cdot \|_1, \| \cdot \|_2$ for 4-D vectors in the previous expression yields

$$\begin{aligned}
&\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1}^2 + \epsilon^2 \int_{\Omega_2} D^\epsilon \mathbf{v}^{2,\epsilon} : D^\epsilon \mathbf{v}^{2,\epsilon} + \| \partial_z \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2}^2 + \| \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Gamma}^2 + \| \epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon} \|_{0,\Gamma}^2 \\
&\leq C\Big\{\|\partial_z (\epsilon\, \mathbf{v}^{2,\epsilon})\|_{0,\Omega_2}^2 + \|(\epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon})\|_{0,\Gamma}^2 + \|(\epsilon\, \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon})\|_{0,\Gamma}^2 + \widetilde{C}\|\mathbf{v}^{1,\epsilon}\|_{0,\Omega_1}^2\Big\}^{1/2} \\
&\leq C\Big\{\| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1}^2 + \| D^\epsilon (\epsilon\, \mathbf{v}^{2,\epsilon}) \|_{0,\Omega_2}^2 + \| \partial_z \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2}^2 + \| \mathbf{v}_{\widehat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Gamma}^2 + \| \epsilon\, \mathbf{v}_{\text{tg}}^{2,\epsilon} \|_{0,\Gamma}^2\Big\}^{1/2}.
\end{aligned} \tag{38}$$

From the expression above, the global Estimate (35) follows.

In the next subsections we use weak convergence arguments to derive the functional setting of the limiting problem (see Figure 2), for the structure of the limiting functions.



(a) Limit solutions in the reference domain

(b) Velocity and Pressure Schematic Traces for the solution on the hyperplane $\{(\widetilde{\mathbf{x}}, z) : \widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}_0\}$.

Figure 2

Figure (a) depicts the dependence of the limit solution [v, p], for both regions Ω_1, Ω_2 and a generic hyperplane $\{(\widetilde{\mathbf{x}}, z) : \widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}_0\}$. Figure (b) shows plausible schematics for traces of the velocity and pressure restricted to the hyperplane $\{(\widetilde{\mathbf{x}}, z) : \widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}_0\}$, depicted in Figure (a).

Corollary 3.4 (Convergence of the Velocities). *Let* $([\mathrm{v}^\epsilon, \mathrm{p}^\epsilon] : \epsilon > 0)$ # X x Y *be the sequence of solutions to the* e-*Problems* (30). *There exists a subsequence, still denoted* $(\mathrm{v}^\epsilon : \epsilon > 0)$, *for which the following holds:*

I. *There exist* $\mathrm{v}^1 \in \mathrm{H}_{\mathrm{div}}(\Omega_1)$ *such that*

$$\mathbf{v}^{1,\epsilon} \to \mathbf{v}^1 \quad \textit{weakly in} \quad \mathbf{H}_{\mathrm{div}}(\Omega_1), \tag{39a}$$

$$\nabla \cdot \mathbf{v}^{1,\epsilon} = h^1. \tag{39b}$$

II. *There exist* $\chi \in \mathbf{L}^2(\Omega_2)$ *and* v^2 # $H^1(\Omega_2)$ such that

$$\partial_z \mathbf{v}^{2,\epsilon} \to \chi \ \textit{weakly in } \mathbf{L}^2(\Omega_2), \qquad \partial_z(\epsilon\, \mathbf{v}^{2,\epsilon}) \to \mathbf{0} \ \textit{strongly in } \mathbf{L}^2(\Omega_2), \tag{40a}$$

$$\epsilon\, \mathbf{v}^{2,\epsilon} \to \mathbf{v}^2 \ \textit{weakly in } \mathbf{H}^1(\Omega_2), \ \textit{strongly in } \mathbf{L}^2(\Omega_2); \tag{40b}$$

moreover, $\mathbf{v}^2$ *satisfies*

$$\mathbf{v}^2 = \mathbf{v}^2(\widetilde{\mathbf{x}}). \tag{40c}$$

III. *There exists* ξ # $H(\partial_z, \Omega_2)$ *such that*

$$\mathbf{v}^{2,\epsilon}_{\widehat{\mathbf{n}}} \to \xi \ \textit{weakly in } H(\partial_z, \Omega_2), \qquad (\epsilon\, \mathbf{v}^{2,\epsilon}_{\widehat{\mathbf{n}}}) \to 0 \ \textit{strongly in } H(\partial_z, \Omega_2); \tag{41a}$$

furthermore, ξ *satisfies the interface and boundary conditions*

$$\xi\big|_{\Gamma} = \mathbf{v}^1 \cdot \widehat{n}\big|_{\Gamma}, \qquad \xi\big|_{\Gamma+1} = 0. \tag{41b}$$

IV. *The following properties hold:*

$$\mathbf{v}^2 \cdot \widehat{n} = 0, \qquad \chi \cdot \widehat{n} = \partial_z \xi. \tag{42}$$

Proof. I. (The proof is identical to part (i) of Corollary 11 in [14]; we write it here for the sake of completeness.) Due to the global a-priori estimate (35), there must exist a weakly convergent subsequence and a limit v^1 # $H_{div}(\Omega_1)$ such that (39a) holds only in the weak $L^2(\Omega_1)$-sense. Because of the hypothesis 3 and the equation (26c), the sequence $(\boldsymbol{\nabla} \cdot \mathbf{v}^{1,\epsilon} : \epsilon > 0) \subset L^2(\Omega_1)$ is bounded. Then, there must exist yet another subsequence, still denoted the same, such that (39a) holds in the weak $H_{div}(\Omega_1)$-sense. Now, recalling that the divergence operator is linear and continuous with respect to the H_{div}-norm, the identity (39b) follows.

II. From the estimate (35), it follows that $(\partial_z\, v^{2,\epsilon}\colon \epsilon > 0)$ is bounded in $L^2(\Omega_2)$. Then, there exists a subsequence (still denoted the same) and $\chi \in \mathbf{L}^2(\Omega_2)$ such that $(\partial_z\, v^{2,\epsilon}\colon \epsilon > 0)$ and $(\partial_z(\epsilon\, v^{2,\epsilon}) : \epsilon > 0)$ satisfy the statement (40a). Also from (35) the trace on the interface $(\epsilon\, \mathbf{v}^{2,\epsilon}|_{\Gamma} : \epsilon > 0)$ is bounded in $L^2(\Gamma)$. Applying the inequality (32b) for vector functions, we conclude that $(\epsilon\, v^{2,\epsilon}\colon \epsilon > 0)$ is bounded in $L^2(\Omega_2)$ and consequently in $H(\partial_z, \Omega_2)$. Then, there must exist v^2 # $H(\partial_z, \Omega_2)$ such that

$$\epsilon\, \mathbf{v}^{2,\epsilon} \to \mathbf{v}^2 \quad \text{weakly} \quad \text{in} \quad \mathbf{H}(\partial_z, \Omega_2). \tag{43}$$

Also, from the strong convergence in the statement (40a), it follows that v^2 is independent from z, i.e., (40c) holds.

Again, from (35) we know that the sequence $(\epsilon\, D^{\epsilon}\, v^{2,\epsilon}\colon \epsilon > 0)$ is bounded in $L^2(\Omega_2)$. Recalling the identity (15) we have that the expression

$$\epsilon\, \boldsymbol{D}^{\epsilon}\, \mathbf{v}^{2,\epsilon} = \boldsymbol{\nabla}_T \left(\epsilon\, \mathbf{v}^{2,\epsilon}\right) + (\epsilon - 1)\partial_z\, \mathbf{v}^{2,\epsilon} \boldsymbol{\nabla}^t_T\, \zeta$$

is bounded. In the equation above, the left hand side and the second summand of the right hand side are bounded in $L^2(\Omega_2)$; then we conclude that the first summand of the right hand side is also bounded. Hence, we have $(\epsilon\, \boldsymbol{\nabla} \mathbf{v}^{2,\epsilon} : \epsilon > 0)$ is bounded in $L^2(\Omega_2)$, and therefore the sequence $(\epsilon\, v^{2,\epsilon}\colon \epsilon > 0)$ is bounded in $H^1(\Omega_2)$; consequently, the statement (40b) holds.

III. Since $(\partial_z \mathbf{v}^{2,\epsilon} : \epsilon > 0) \subset \mathbf{L}^2(\Omega_2)$ is bounded, in particular $(\partial_z \mathbf{v}^{2,\epsilon} \cdot \widehat{n} : \epsilon > 0) \subset L^2(\Omega_2)$ is also bounded. From (35), we know that $(\mathbf{v}^{2,\epsilon} \cdot \widehat{n}|_\Gamma : \epsilon > 0) \subset L^2(\Gamma)$ is bounded and again, due to Inequality (32b), we conclude that $(\mathbf{v}^{2,\epsilon} \cdot \widehat{n} : \epsilon > 0) \subset L^2(\Omega_2)$ is bounded. Then, the sequence $(\mathbf{v}^{2,\epsilon} \cdot \widehat{n} : \epsilon > 0)$ is bounded in $H(\partial_z, \Omega_2)$; consequently, there must exist a subsequence (still denoted the same) and a limit ξ # $H(\partial_z, \Omega_2)$, such that $(\mathbf{v}^{2,\epsilon}_{\widehat{n}} : \epsilon > 0)$ and $(\epsilon \mathbf{v}^{2,\epsilon}_{\widehat{n}} : \epsilon > 0)$ satisfy the statement (41a). From here it is immediate to conclude the relations (41b).

IV. Since $(\epsilon \mathbf{v}^{2,\epsilon} \cdot \widehat{n}) \to 0$, and due to (43), we conclude that $\mathbf{v}^2 \cdot \widehat{n} = 0$. Finally, due to (40), we have that $\chi \cdot \widehat{n} = \partial_z \xi$, and the proof is complete.

Theorem 3.5 (Convergence of Pressures). *Let* $([v^\epsilon, p^\epsilon] : \epsilon > 0)$ # X x Y *be the sequence of solutions to the e-Problems* (30). *There exists a subsequence, still denoted* $(p^\epsilon : \epsilon > 0)$, *verifying the following:*

I. *There exists* p^1 # $H^1(\Omega_1)$ *such that*

$$p^{1,\epsilon} \to p^1 \ \textit{weakly in}\ H^1(\Omega_1)\ \textit{and strongly in}\ L^2(\Omega_1), \tag{44a}$$

$$Q\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0 \ \textit{in}\ \Omega_1, \qquad p^1 = 0 \ \textit{on}\ \partial\Omega_1 - \Gamma, \tag{44b}$$

where $\mathbf{v}^1$ *is the weak limit of Statement* (39a).

II. *There exists* p^2 # $L^2(\Omega_2)$ *such that*

$$p^{2,\epsilon} \to p^2 \quad \textit{weakly in}\ L^2(\Omega_2). \tag{45}$$

III. *The pressure* $p = [p^1, p^2]$ *belongs to* $L^2(\Omega)$.

Proof. I. (The proof is identical to part (i) Lemma 11 in [14]; we write it here for the sake of completeness.) Due to (16b) and (36) it follows that

$$\left\|\boldsymbol{\nabla} p^{1,\epsilon}\right\|_{0,\Omega_1} = \left\|\sqrt{Q}\,\mathbf{v}^{1,\epsilon}\right\|_{0,\Omega_1} \leq C,$$

where C is an adequate positive constant. From (11a), the Poincaré inequality gives the existence of a constant $\widetilde{C} > 0$ satisfying

$$\left\|p^{1,\epsilon}\right\|_{1,\Omega_1} \leq \widetilde{C}\left\|\boldsymbol{\nabla} p^{1,\epsilon}\right\|_{0,\Omega_1}, \qquad \text{for all } \epsilon > 0. \tag{46}$$

Therefore, the sequence $(p^{1,\epsilon} : \epsilon > 0)$ is bounded in H and the convergence statement (44a) follows directly. Again, given that $p^{1,\epsilon}$ satisfies the Darcy equation (16b) and that the gradient # is linear and continuous in $H^1(\Omega_1)$, the equality $Q\mathbf{v}^1 + \boldsymbol{\nabla} p^1 = 0$ in (44b) follows. Finally, since $p^{1,\epsilon}|_{\Omega_1 - \Gamma} = 0$ for every element of the weakly convergent subsequence, and the trace map $\varphi \mapsto \varphi|_\Gamma$ is linear, it follows that p^1 satisfies the boundary condition in (44b).

II. In order to show that the sequence $(p^{2,\epsilon} : \epsilon > 0)$ is bounded in $L^2(\Omega_2)$, take any $\phi \in C_0^\infty(\Omega_2)$ and define the auxiliary function

$$\varpi(\widetilde{\mathrm{x}}, z) \overset{\text{def}}{=} \int_z^{\zeta(\widetilde{\mathrm{x}})+1} \phi(\widetilde{\mathrm{x}}, t)\, dt, \qquad \zeta(\widetilde{\mathrm{x}}) \leq z \leq \zeta(\widetilde{\mathrm{x}}) + 1. \tag{47}$$

Since ζ # $C^2(G)$, it is clear that $\varpi \in H^1(\Omega_2)$ and $\|\varpi\|_{1,\Omega_2} \leq C\|\phi\|_{0,\Omega_2}$. Hence, the function $\mathbf{w}^2 \overset{\text{def}}{=} (0_T, \varpi) = \varpi\,\widehat{e}_N$ belongs to X_2; moreover,

$$\begin{aligned} &\| \mathbf{w}^2 \|_{0,\Omega_2} + \| \partial_z \mathbf{w}^2 \|_{0,\Omega_2} \leq \widetilde{C} \| \phi \|_{0,\Omega_2}, \\ &\| \mathbf{w}^2(\mathrm{tg})\big|_\Gamma \|_{0,\Gamma} = \| \mathbf{w}^2_{\mathrm{tg}}\big|_\Gamma \|_{0,\Gamma} \leq \widetilde{C} \| \phi \|_{0,\Omega_2}. \end{aligned} \tag{48}$$

Here, the second inequality follows from the first one and due to the estimate (31a). Next, observe that $\varpi\,\widehat{e}_N \cdot \widehat{n}(\cdot)\,\mathbb{1}_\Gamma \in L^2(\Gamma) \subseteq H^{-1/2}(\partial\Omega_1)$; then, Lemma 1.1 gives the existence of a function w^1 # $\mathrm{H}_{\mathrm{div}}\,(\Omega_1)$ such that

$$\begin{aligned} \mathbf{w}^1 \cdot \widehat{n} = \mathbf{w}^2 \cdot \widehat{n} = \varpi(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}))\widehat{e}_N \cdot \widehat{n}(\widetilde{\mathbf{x}}) = \int_{\zeta(\widetilde{\mathbf{x}})}^{\zeta(\widetilde{\mathbf{x}})+1} \phi\,(\widetilde{\mathbf{x}}, t)\, dt \text{ on } \Gamma, \\ \mathbf{w}^1 \cdot \widehat{n} = 0 \text{ on } \partial\Omega_1 - \Gamma, \\ \|\mathbf{w}^1\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq \|\varpi\, \widehat{e}_N \cdot \widehat{n}(\cdot)\|_{0,\Gamma} \leq C\|\phi\|_{0,\Omega_2}. \end{aligned} \tag{49}$$

Here, the last inequality holds because $\sup\{\widehat{n}(\widetilde{\mathbf{x}}) \cdot \widehat{e}_N : \widetilde{\mathbf{x}} \in \Gamma\} < \infty$. Hence, the function $\mathrm{w} \overset{\mathrm{def}}{=} [\mathrm{w}^1, \mathrm{w}^2]$ belongs to the space X. Testing (30a) with w yields

$$\begin{aligned} &\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1 - \int_{\Omega_1} p^{1,\epsilon}\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 \\ &+ \int_{\Omega_2} p^{2,\epsilon} \phi + \epsilon^2 \int_{\Omega_2} \mu\, D^\epsilon\, \mathbf{v}^{2,\epsilon} : D^\epsilon\, \mathbf{w}^2 - \int_{\Omega_2} \mu\, \partial_z\, \mathbf{v}_N^{2,\epsilon}\, \phi \\ &+ \alpha \int_\Gamma (\mathbf{v}^{1,\epsilon} \cdot \widehat{n})(\mathbf{w}^1 \cdot \widehat{n}) dS + \epsilon^2 \int_\Gamma \gamma \sqrt{\mathcal{Q}}\, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\mathrm{tg}}^2 dS = \epsilon \int_{\Omega_2} \mathbf{f}_N^{2,\epsilon}\, \varpi. \end{aligned} \tag{50}$$

Applying the Cauchy-Bunyakowsky-Schwarz inequality to the integrals, and reordering, we get

$$\begin{aligned} \Big|\int_{\Omega_2} p^{2,\epsilon} \phi\Big| \leq\, & C_1 \| \mathbf{v}^{1,\epsilon} \|_{0,\Omega_1} \| \mathbf{w}^1 \|_{0,\Omega_1} + \| p^{1,\epsilon} \|_{0,\Omega_1} \| \boldsymbol{\nabla} \cdot \mathbf{w}^1 \|_{0,\Omega_1} \\ & + C_2 \| \epsilon D^\epsilon \mathbf{v}^{2,\epsilon} \|_{0,\Omega_2} \| \epsilon D^\epsilon \mathbf{w}^2 \|_{0,\Omega_2} + C_3 \| \partial_z \mathbf{v}_N^{2,\epsilon} \|_{0,\Omega_2} \| \phi \|_{0,\Omega_2} \\ & + C_4 \| \mathbf{v}^{1,\epsilon} \cdot \widehat{n} \|_{0,\Gamma} \| \mathbf{w}^2 \cdot \widehat{n} \|_{0,\Gamma} \\ & + C_5\, \epsilon \|\epsilon\, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon}\|_{0,\Gamma} \|\mathbf{w}_{\mathrm{tg}}^2\|_{0,\Gamma} + \| \epsilon\, \mathbf{f}_N^{2,\epsilon} \|_{0,\Omega_2} \| \varpi \|_{0,\Omega_2}. \end{aligned}$$

We pursue estimates in terms of $\|\phi\|_{0,\Omega_2}$; to that end we first apply the fact that all the terms involving the sol|ution on the right hand side, i.e., $\mathrm{v}^{1,\epsilon}$, $p^{1,\epsilon}$, $\epsilon D^\epsilon \mathrm{v}^{2,\epsilon}$, $\partial_z \mathrm{v}_N^{2,\epsilon}$, $\mathrm{v}^{1,\epsilon} \cdot \widehat{n}$ and $\mathrm{v}_{\mathrm{tg}}^{2,\epsilon}|_\Gamma$ are bounded. In addition, the forcing term $\epsilon\, \mathrm{f}_N^{2,\epsilon}$ is bounded. Replacing the norms of the aforementioned terms by a generic constant on the right hand side we have

$$\begin{aligned} \Big|\int_{\Omega_2} p^{2,\epsilon} \phi\Big| \leq C \Big(& \| \mathbf{w}^1 \|_{0,\Omega_1} + \| \boldsymbol{\nabla} \cdot \mathbf{w}^1 \|_{0,\Omega_1} + \| \epsilon D^\epsilon \mathbf{w}^2 \|_{0,\Omega_2} \\ & + \| \phi \|_{0,\Omega_2} + \| \mathbf{w}^2 \cdot \widehat{n} \|_{0,\Gamma} + \epsilon \|\mathbf{w}_{\mathrm{tg}}^2\|_{0,\Gamma} + \| \varpi \|_{0,\Omega_2}\Big). \end{aligned} \tag{51}$$

In the expression above the first summand of the second line needs further analysis. We have

$$\begin{aligned} \| \epsilon D^\epsilon \mathbf{w}^2 \|_{0,\Omega_2} &= \| \epsilon \boldsymbol{\nabla}_T \mathbf{w}^2 + (\epsilon - 1)\, \partial_z \mathbf{w}^2\, \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2} \\ &\leq \epsilon \| \boldsymbol{\nabla}_T \mathbf{w}^2 + \partial_z \mathbf{w}^2\, \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2} + \| \partial_z \mathbf{w}^2\, \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2} \\ &\leq \epsilon \| \boldsymbol{\nabla}_T \mathbf{w}^2 + \partial_z \mathbf{w}^2\, \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2} + \| \partial_z \mathbf{w}^2 \|_{0,\Omega_2} \| \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2}. \end{aligned}$$

Combining (48) with the expression above, we conclude

$$\| \epsilon D^\epsilon \mathbf{w}^2 \|_{0,\Omega_2} \leq \epsilon \| \boldsymbol{\nabla}_T \mathbf{w}^2 + \partial_z \mathbf{w}^2 \boldsymbol{\nabla}_T^t \zeta \|_{0,\Omega_2} + C \|\phi\|_{0,\Omega_2}. \tag{52}$$

Introducing the latter estimate in the inequality (51), the first two summands on the right hand side of the first line are bounded by a multiple of $\|\phi\|_{0,\Omega_2}$ due to (49). The second and third summands on the second line are trace terms which are also controlled by a multiple of $\|\phi\|_{0,\Omega_2}$, due to (48). The fourth summand on the second line is trivially controlled by $\|\phi\|_{0,\Omega_2}$ because of its construction. Combining all these observations with (52), we get

$$\Big|\int_{\Omega_2} p^{2,\epsilon}\,\phi\Big| \le \epsilon \,\|\,\boldsymbol{\nabla}_T\,\mathbf{w}^2 + \partial_z\,\mathbf{w}^2\,\boldsymbol{\nabla}_T^t\zeta\,\|_{0,\Omega_2} + C\,\|\,\phi\,\|_{0,\Omega_2},$$

where C > 0 is a new generic constant. Taking upper limit as $\epsilon \to 0$ in the previous expression gives

$$\limsup_{\epsilon\downarrow 0}\Big|\int_{\Omega_2} p^{2,\epsilon}\,\phi\Big| \le C\,\|\phi\|_{0,\Omega_2}. \tag{53}$$

The above holds for any $\phi \in C_0^\infty(\Omega_2)$, then, the sequence ($p^{2,\epsilon}$: ϵ > 0) # $L^2(\Omega_2)$ is bounded and, consequently, the convergence statement (45) follows.

III. From the previous part, it is clear that the sequence ([$p^{1,\epsilon}$, $p^{2,\epsilon}$] : ϵ > 0) is bounded in $L^2(\Omega)$; therefore, p also belongs to $L^2\Omega$), which completes the proof.

Remark 3.6. Notice that the upwards normal vector $\widehat{n}$ orthogonal to the surface Γ is given by the expression

$$\widehat{n} = \frac{1}{|(-\boldsymbol{\nabla}_T\zeta, 1)|}\begin{Bmatrix} -\boldsymbol{\nabla}_T^t\zeta \\ 1 \end{Bmatrix}, \tag{54a}$$

and the normal derivative satisfies

$$|(-\boldsymbol{\nabla}_T\zeta,1)|\,\partial_{\widehat{\mathbf{n}}} = |(-\boldsymbol{\nabla}_T\zeta,1)|\frac{\partial}{\partial\widehat{n}} = \widehat{n}\cdot\boldsymbol{\nabla}, \quad \text{on } \Gamma. \tag{54b}$$

We use the identities above to identify the dependence of x, ξ and p^2 (see Figure 2 above).

Theorem 3.7. *Let x, ξ be the higher order limiting terms in Corollary 3.4 (ii) and (iii), respectively. Let* p^2 *be the limit pressure in* Ω_2 *in Lemma 3.5 (ii). Then,*

$$\partial_z\,\xi = -\mathbf{v}^1\cdot\widehat{n}\big|_\Gamma, \quad \xi(\widetilde{\mathbf{x}}, z) = \mathbf{v}^1\cdot\widehat{n}(\widetilde{\mathbf{x}})(1-z), \quad \textit{for } \zeta(\widetilde{\mathbf{x}}) \le z \le \zeta(\widetilde{\mathbf{x}})+1. \tag{55a}$$

$$p^2 = p^2(\widetilde{\mathbf{x}}). \tag{55b}$$

$$\chi = \chi(\widetilde{\mathbf{x}}), \qquad \chi\cdot\widehat{n} = -\mathbf{v}^1\cdot\widehat{n} \textit{ on } \Gamma. \tag{55c}$$

In particular, $\partial_z\xi = \partial_z\xi(\widetilde{\mathbf{x}})$.

Proof. Take Φ = (0, φ^2) # Y, test (30b) and reorder the summands conveniently; we have

$$\begin{aligned}
0 =& \epsilon\int_{\Omega_2} \boldsymbol{\nabla}_T\cdot\mathbf{v}_T^{2,\epsilon}\,\varphi^2 + \epsilon\int_{\Omega_2}\partial_z\mathbf{v}_T^{2,\epsilon}\cdot\boldsymbol{\nabla}_T\zeta\,\varphi^2 - \int_{\Omega_2}\partial_z\mathbf{v}_T^{2,\epsilon}\cdot\boldsymbol{\nabla}_T\zeta\,\varphi^2 + \int_{\Omega_2}\partial_z\mathbf{v}_N^{2,\epsilon}\,\varphi^2 \\
&= \int_{\Omega_2}\boldsymbol{\nabla}_T\cdot(\epsilon\,\mathbf{v}_T^{2,\epsilon})\,\varphi^2 + \int_{\Omega_2}\partial_z(\epsilon\,\mathbf{v}_T^{2,\epsilon})\cdot\boldsymbol{\nabla}_T\zeta\,\varphi^2 + \int_{\Omega_2}\partial_z\mathbf{v}^{2,\epsilon}\cdot(-\boldsymbol{\nabla}_T\zeta,1)\,\varphi^2 \\
&= \int_{\Omega_2}\boldsymbol{\nabla}_T\cdot(\epsilon\,\mathbf{v}_T^{2,\epsilon})\,\varphi^2 + \int_{\Omega_2}\partial_z(\epsilon\,\mathbf{v}_T^{2,\epsilon})\cdot\boldsymbol{\nabla}_T\zeta\,\varphi^2 + \int_{\Omega_2}|(-\boldsymbol{\nabla}_T\zeta,1)|\,\partial_z(\mathbf{v}^{2,\epsilon}\cdot\widehat{n})\varphi^2.
\end{aligned}$$

Letting e # 0 in the expression above we get

$$\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^2\varphi^2 + \int_{\Omega_2}\partial_z\mathbf{v}^2\cdot\boldsymbol{\nabla}_T\zeta\,\varphi^2 + \int_{\Omega_2}|(-\boldsymbol{\nabla}_T\zeta,1)|\,\partial_z\,\xi\,\varphi^2 = 0.$$

Recalling Equation (40c) we have that $\partial_z \mathrm{v}^2 = 0$; hence,

$$\int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \mathbf{v}^2 \, \varphi^2 + \int_{\Omega_2} \big|(-\boldsymbol{\nabla}_T \zeta,\, 1)\big| \, \partial_z \xi \, \varphi^2 = 0.$$

Since the above holds for all $\varphi^2 \in L_0^2(\Omega_2)$, it follows that

$$\boldsymbol{\nabla}_T \cdot \mathbf{v}^2 + |(-\boldsymbol{\nabla}_T \zeta,\, 1)| \partial_z \, \xi = c,$$

where c is a constant. In the previous expression we observe that two out of three terms are independent from z; then it follows that the third term is also independent from z. Since the vector $(-\boldsymbol{\nabla}_T \zeta,\, 1)$ is independent from z, we conclude that $\partial_z \xi = \partial_z \xi(\widetilde{\mathrm{x}})$. This, together with the boundary conditions (41b), yield the second equality in (55a).

Take $\Psi = (\phi^1, \ldots, \phi^N) \in (C_0^\infty(\Omega_2))^N$, for each i =1, 2,..., N; build the "antiderivative" ϖ^i of ϕ^i using the rule (47), and define $\mathrm{w}^2 = (\varpi^1, \ldots, \varpi^N)$. Use Lemma 1.2 to construct w^1 # $\mathrm{H}_{\mathrm{div}}(\Omega_1)$ such that $\mathrm{w}^1 \cdot \widehat{n} = \mathrm{w}^2 \cdot \widehat{n}$ on Γ, $\mathrm{w}^1 \cdot \widehat{n} = 0$ on $\partial\Omega_1 - \Gamma$ and

$$\left\|\mathbf{w}^1\right\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq C \left\|\Psi\right\|_{\mathbf{L}^2(\Omega_2)}; \tag{56}$$

therefore, $\mathrm{w} \overset{\mathrm{def}}{=} (\mathrm{w}^1, \mathrm{w}^2) \in \mathrm{X}_2$. Test (30a) with w and regroup the higher order terms; we have

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^{1,\epsilon} \cdot \mathbf{w}^1 - \int_{\Omega_1} p^{1,\epsilon} \, \boldsymbol{\nabla} \cdot \mathbf{w}^1 - \epsilon \int_{\Omega_2} p^{2,\epsilon} \, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 \\
&- \epsilon \int_{\Omega_2} p^{2,\epsilon} \, \partial_z \, \mathbf{w}_T^2 \cdot \boldsymbol{\nabla}_T \zeta + \int_{\Omega_2} p^{2,\epsilon} \, \partial_z \, \mathbf{w}_T^2 \cdot \boldsymbol{\nabla}_T \zeta - \int_{\Omega_2} p^{2,\epsilon} \, \partial_z \mathbf{w}_N^2 \\
&+ \epsilon^2 \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \boldsymbol{D}^\epsilon \mathbf{w}^2 + \int_{\Omega_2} \mu \, \partial_z \, \mathbf{v}^{2,\epsilon} \cdot \partial_z \, \mathbf{w}^2 \\
&+ \alpha \int_\Gamma (\mathbf{v}^{1,\epsilon} \cdot \widehat{n}) \, (\mathbf{w}^1 \cdot \widehat{n}) \, dS + \epsilon^2 \int_\Gamma \gamma \sqrt{\mathcal{Q}} \, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\mathrm{tg}}^2 \, dS = \epsilon \int_{\Omega_2} \mathbf{f}^{2,\epsilon} \cdot \mathbf{w}^2.
\end{aligned} \tag{57}$$

The limit of all the terms in the expression above when e # 0 is clear except for one summand, which we discuss independently; i.e.,

$$\begin{aligned}
\epsilon^2 \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \boldsymbol{D}^\epsilon \mathbf{w}^2 &= \epsilon^2 \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \Big(\boldsymbol{\nabla}_T \, \mathbf{w}^2 + \partial_z \, \mathbf{w}^2 \boldsymbol{\nabla}_T^t \zeta - \frac{1}{\epsilon} \partial_z \, \mathbf{w}^2 \boldsymbol{\nabla}_T^t \zeta\Big) \\
&= \epsilon^2 \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : (\boldsymbol{\nabla}_T \, \mathbf{w}^2 + \partial_z \, \mathbf{w}^2 \boldsymbol{\nabla}_T^t \zeta) \\
&+ \epsilon \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta).
\end{aligned}$$

In the latter expression, the first summand clearly tends to zero when e # 0. Therefore, we focus on the second summand:

$$\begin{aligned}
\epsilon \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta) &= \int_{\Omega_2} \mu \, \boldsymbol{\nabla}_T \, (\epsilon \, \mathbf{v}^{2,\epsilon}) : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta) \\
&+ \int_{\Omega_2} \mu \, \partial_z (\epsilon \, \mathbf{v}^{2,\epsilon}) \boldsymbol{\nabla}_T^t \zeta : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta) + \int_{\Omega_2} \mu \, \partial_z \mathbf{v}^{2,\epsilon} (-\boldsymbol{\nabla}_T^t \zeta) : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta).
\end{aligned}$$

All the terms in the right hand side can pass to the limit. Recalling the statement (40a), we conclude that

$$\begin{aligned}
\epsilon^2 \int_{\Omega_2} \mu \, \boldsymbol{D}^\epsilon \mathbf{v}^{2,\epsilon} : \boldsymbol{D}^\epsilon \mathbf{w}^2 \to &\int_{\Omega_2} \mu \, \boldsymbol{\nabla}_T \, \mathbf{v}^2 : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta) \\
&+ \int_{\Omega_2} \mu \, \chi (-\boldsymbol{\nabla}_T^t \zeta) : \partial_z \, \mathbf{w}^2 (-\boldsymbol{\nabla}_T^t \zeta).
\end{aligned}$$

Letting e # 0 in (57), and considering the equality above, we get

$$\begin{aligned}\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1 - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1 + \int_{\Omega_2} p^2\,(\partial_z\,\mathbf{w}^2_T, \partial_z\,\mathbf{w}^2_N)\cdot(\boldsymbol{\nabla}_T\zeta, -1)\\ + \int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\,\mathbf{v}^2 : \partial_z\,\mathbf{w}^2(-\boldsymbol{\nabla}_T^t\zeta) + \int_{\Omega_2}\mu\,\chi(-\boldsymbol{\nabla}_T^t\zeta) : \partial_z\,\mathbf{w}^2(-\boldsymbol{\nabla}_T^t\zeta) \qquad (58)\\ + \int_{\Omega_2}\mu\,\chi\cdot\partial_z\,\mathbf{w}^2 + \alpha\int_\Gamma(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS = 0.\end{aligned}$$

We develop a simpler expression for the sum of the fourth, fifth and sixth terms:

$$\begin{aligned}&\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\,\mathbf{v}^2 : \partial_z\,\mathbf{w}^2(-\boldsymbol{\nabla}_T^t\zeta) + \int_{\Omega_2}\mu\,\chi(-\boldsymbol{\nabla}_T^t\zeta) : \partial_z\,\mathbf{w}^2(-\boldsymbol{\nabla}_T^t\zeta) + \int_{\Omega_2}\mu\,\chi\cdot\partial_z\,\mathbf{w}^2\\ &= \int_{\Omega_2}\mu\,(\boldsymbol{\nabla}\mathbf{v}^2\cdot(-\boldsymbol{\nabla}_T\zeta, 1))\cdot\partial_z\,\mathbf{w}^2 + \int_{\Omega_2}\mu\,\chi\,(-\boldsymbol{\nabla}_T\zeta, 1)^t : \partial_z\,\mathbf{w}^2(-\boldsymbol{\nabla}_T\,\zeta, 1)^t\\ &= \int_{\Omega_2}\mu|(-\boldsymbol{\nabla}_T\zeta, 1)|\,\partial_{\widehat{\boldsymbol{n}}}\,\mathbf{v}^2\cdot\partial_z\,\mathbf{w}^2 + \int_{\Omega_2}\mu\,\chi\cdot\partial_z\mathbf{w}^2\,(-\boldsymbol{\nabla}_T\zeta, 1)\cdot(-\boldsymbol{\nabla}_T\zeta, 1).\end{aligned}$$

Here $\partial_{\widehat{\boldsymbol{n}}}$ is the normal derivative defined in the identity (54b). We introduce the previous equality in (58); this yields

$$\begin{aligned}\int_{\Omega_1}\mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1 - \int_{\Omega_1}p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1 - \int_{\Omega_2}|(-\boldsymbol{\nabla}_T\zeta, 1)|p^2\,\partial_z\,(\mathbf{w}^2\cdot\widehat{\boldsymbol{n}})\\ + \int_{\Omega_2}\mu\,|(-\boldsymbol{\nabla}_T\zeta, 1)|\partial_{\widehat{\boldsymbol{n}}}\mathbf{v}^2\cdot\partial_z\,\mathbf{w}^2 + \int_{\Omega_2}\mu\,\chi\cdot\partial_z\,\mathbf{w}^2|(-\boldsymbol{\nabla}_T\zeta, 1)|^2\\ + \alpha\int_\Gamma(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS = 0. \quad (59)\end{aligned}$$

Next, we integrate by parts the second summand in the first line, add it to the first summand and recall that $\partial_z\mathbf{w}^2 = -\Psi$ by construction. Hence,

$$\begin{aligned}&-\int_\Gamma p^1\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS + \int_{\Omega_2}|(-\boldsymbol{\nabla}_T\zeta, 1)|p^2\,\widehat{\boldsymbol{n}}\cdot\Psi\\ &-\int_{\Omega_2}\mu|(-\boldsymbol{\nabla}_T\zeta, 1)|\partial_{\widehat{\boldsymbol{n}}}\mathbf{v}^2\cdot\Psi - \int_{\Omega_2}\mu\,\chi\cdot\Psi\,|(-\boldsymbol{\nabla}_T\zeta, 1)|^2 + \alpha\int_\Gamma(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS = 0.\\ &\hfill(60)\end{aligned}$$

In the expression above we develop the surface integrals as integrals over the projection G of Γ on #[N-1]; this gives

$$\begin{aligned}-\int_\Gamma p^1\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS + \alpha\int_\Gamma(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS\\ = \int_G\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right](\mathbf{w}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\,d\widetilde{\mathbf{x}}.\end{aligned}$$

Recalling that $\mathbf{w}^1\cdot\widehat{\boldsymbol{n}} = \int_{\zeta(\widetilde{\mathbf{x}})}^{\zeta(\widetilde{\mathbf{x}})+1}\Psi(\widetilde{\mathbf{x}}, z)\,dz\cdot\widehat{\boldsymbol{n}}$ on Γ, the latter equality becomes in

$$\begin{aligned}&\int_G\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right]\int_{\zeta(\widetilde{\mathbf{x}})}^{\zeta(\widetilde{\mathbf{x}})+1}\Psi(\widetilde{\mathbf{x}}, z)\,dz\cdot\widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}})\,d\widetilde{\mathbf{x}}\\ &= \int_G\int_{\zeta(\widetilde{\mathbf{x}})}^{\zeta(\widetilde{\mathbf{x}})+1}\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right](\widetilde{\mathbf{x}})\,\widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}})\cdot\Psi(\widetilde{\mathbf{x}}, z)\,dz\,d\widetilde{\mathbf{x}}\\ &= \int_{\Omega_2}\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right]\widehat{\boldsymbol{n}}\cdot\Psi\,dz\,d\widetilde{\mathbf{x}}.\end{aligned}$$

Introducing the previous in (60), we have

$$\begin{aligned}\int_{\Omega_2}\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right]\widehat{\boldsymbol{n}}\cdot\Psi\,dz\,d\widetilde{\mathbf{x}} + \int_{\Omega_2}|(-\boldsymbol{\nabla}_T\zeta, 1)|p^2\,\widehat{\boldsymbol{n}}\cdot\Psi\,d\widetilde{\mathbf{x}}\,dz\\ - \int_{\Omega_2}\mu\,|(-\boldsymbol{\nabla}_T\zeta, 1)|\partial_{\widehat{\boldsymbol{n}}}\mathbf{v}^2\cdot\Psi\,d\widetilde{\mathbf{x}}\,dz - \int_{\Omega_2}\mu\,|(-\boldsymbol{\nabla}_T\zeta, 1)|^2\chi\cdot\Psi\,d\widetilde{\mathbf{x}}\,dz = 0.\end{aligned}$$

Since the above holds for all $\Psi\in(C_0^\infty(\Omega_2))^N$, it follows that

$$\begin{aligned}\frac{1}{\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{e}}_N}\left[-p^1|_\Gamma + \alpha\,(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}\,|_\Gamma)\right]\widehat{\boldsymbol{n}} + |(-\boldsymbol{\nabla}_T\zeta, 1)|p^2\,\widehat{\boldsymbol{n}}\\ - \mu\,|(-\boldsymbol{\nabla}_T\zeta, 1)|\partial_{\widehat{\boldsymbol{n}}}\mathbf{v}^2 - \mu\,|(-\boldsymbol{\nabla}_T\zeta, 1)|^2\chi = 0, \text{ in } \mathbf{L}^2(\Omega_2). \quad (61)\end{aligned}$$

In order to get the normal balance on the interface w(e coul)d repeat the previous strategy, but with a quantifier $\Psi \in C_0^\infty(\Omega_2)^N$ satisfying $\Psi = (\Psi \cdot \widehat{n})\, \widehat{n}$, i.e., such that it is parallel to the normal direction. This would be equivalent to replace ψ by $(\Psi \cdot \widehat{n})\, \widehat{n}$, n in all the previous equations. Consequently, in order to get the normal balance, it suffices to apply $(\cdot \frac{\widehat{n}}{|(-\nabla_T \zeta, 1)|^2})$ Equation (61); such operation yields:

$$\frac{1}{\widehat{n} \cdot \widehat{e}_N} \frac{1}{|(-\nabla_T \zeta, 1)|^2} \left[- p^1|_\Gamma + \alpha \left(\mathbf{v}^1 \cdot \widehat{n}\, |_\Gamma \right) \right] + \frac{1}{|(-\nabla_T \zeta, 1)|} p^2 - \mu \frac{1}{|(-\nabla_T \zeta, 1)|} \partial_{\widehat{n}} \mathbf{v}^2 \cdot \widehat{n} - \mu\, \partial_z \xi = 0. \quad (62)$$

In the last expression the identity (42) has been used. Also notice that all the terms are independent from z, then the equation (55b) follows. Consequently, all the terms but the last in (61) are independent from z; therefore we conclude that X is independent from z. Recalling (42) and (55a), the second equality in (55c) follows and the proof is complete.

4. The limiting problem

In this section we derive the form of the limiting problem and characterize it as a Darcy-Brinkman coupled system, where the Brinkman equation takes place in a parametrized (N - 1)-dimensional manifold of $\#^N$. First, we need to introduce some extra hypotheses to complete the analysis.

Hypothesis 4. In the following, it will be assumed that the sequence of forcing terms $(\mathbf{f}^{2,\epsilon} : \epsilon > 0) \subseteq \mathbf{L}^2(\Omega_2)$ and $(h^{1,\epsilon} : \epsilon > 0) \subseteq L^2(\Omega_1)$ are weakly convergent, i.e., there exist f^2 # $L^2(\Omega 2)$ and h^1 # $L^2(\Omega_1)$ such that

$$\mathbf{f}^{2,\epsilon} \rightharpoonup \mathbf{f}^2, \qquad h^{1,\epsilon} \rightharpoonup h^1 . \quad (63)$$

4.1. The tangential behavior of the limiting problem

Recalling (40c) and (42), clearly the lower order limiting velocity has the structure

$$\begin{Bmatrix} \mathbf{v}_{\mathrm{tg}}^2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{v}_{\mathrm{tg}}^2(\widetilde{\mathbf{x}}) \\ 0 \end{Bmatrix} . \quad (64)$$

The above motivates the following definition.

Definition 4.1. Let $\widetilde{\mathbf{x}} \mapsto U(\widetilde{\mathbf{x}})$ be the matrix-valued map introduced in Definition 2.5. Define the space X_{tg} # X_2 by

$$\mathbf{X}_{\mathrm{tg}} \overset{\mathrm{def}}{=} \left\{ \mathbf{w}^2 \in \mathbf{X}_2 : \mathbf{w}^2 = U(\widetilde{\mathbf{x}}) \begin{Bmatrix} \mathbf{w}_{\mathrm{tg}}^2(\widetilde{\mathbf{x}}) \\ 0 \end{Bmatrix} \right\}, \quad (65)$$

endowed with the $H^1(\Omega_2)$-norm.

We have the following result:

Lemma 4.2. *The space* X_{tg} # X_2 *is closed.*

Proof. Let $(\mathbf{w}^2(\ell) : \ell \in \mathbb{N}) \subset \mathbf{X}_{\mathrm{tg}}$ and w^2 # X_2 be such that $\|\mathbf{w}^2(\ell) - \mathbf{w}^2\|_{1,\Omega_2} \rightarrow 0.$ W(e) must show that w^2 # X_{tg}. First, notice that the convergence in X_2 implies $\|\mathbf{w}^2(\ell) - \mathbf{w}^2\|_{0,\Omega_2} \rightarrow 0.$ Recalling (20) and the fact that $U(\widetilde{\mathbf{x}})$) is orthogonal, we have

$$\begin{bmatrix} U^{T,\mathrm{tg}} & U^{T,\widehat{\boldsymbol{n}}} \\ U^{N,\mathrm{tg}} & U^{N,\widehat{\boldsymbol{n}}} \end{bmatrix}^t (\widetilde{\mathbf{x}}) \begin{Bmatrix} \mathbf{w}_T^2(\ell) \\ \mathbf{w}_N^2(\ell) \end{Bmatrix} (\widetilde{\mathbf{x}}) = \begin{Bmatrix} \mathbf{w}_{\mathrm{tg}}^2(\ell) \\ 0 \end{Bmatrix} (\widetilde{\mathbf{x}}).$$

In the identity above, we observe that $\mathrm{w}_T^2(\ell), \mathrm{w}_T^2(\ell)$) are convergent in the H^1-norm and that the orthonormal matrix U has differentiability and boundedness properties. Therefore, we conclude that $\mathrm{w}_{\mathrm{tg}}^2(\ell)$ is convergent in the H_1-norm, and denote the limit by $\mathrm{w}_{\mathrm{tg}}^2 = \mathrm{w}_{\mathrm{tg}}^2(\widetilde{\mathrm{x}}, z)$. Now take the limit in the expression above in the L^2-sense; given that there are no derivatives involved, we have

$$\begin{bmatrix} U^{T,\mathrm{tg}} & U^{T,\widehat{\boldsymbol{n}}} \\ U^{N,\mathrm{tg}} & U^{N,\widehat{\boldsymbol{n}}} \end{bmatrix}^t (\widetilde{\mathbf{x}}) \begin{Bmatrix} \mathbf{w}_T^2 \\ \mathbf{w}_N^2 \end{Bmatrix} (\widetilde{\mathbf{x}}) = \begin{Bmatrix} \mathbf{w}_{\mathrm{tg}}^2(\widetilde{\mathbf{x}}, z) \\ 0 \end{Bmatrix}.$$

Observe that the latter expression implicitly states that $\mathrm{w}_{\mathrm{tg}}^2 = \mathrm{w}_{\mathrm{tg}}^2(\widetilde{\mathrm{x}})$. Finally, applying once more the inverse matrix, we have

$$\mathbf{w}^2 = \begin{Bmatrix} \mathbf{w}_T^2 \\ \mathbf{w}_N^2 \end{Bmatrix} (\widetilde{\mathbf{x}}) = \begin{bmatrix} U^{T,\mathrm{tg}} & U^{T,\widehat{\boldsymbol{n}}} \\ U^{N,\mathrm{tg}} & U^{N,\widehat{\boldsymbol{n}}} \end{bmatrix} (\widetilde{\mathbf{x}}) \begin{Bmatrix} \mathbf{w}_{\mathrm{tg}}^2 \\ 0 \end{Bmatrix} (\widetilde{\mathbf{x}}).$$

Here the equality is in the L^2-sense. However, we know that $\mathrm{w}_{\mathrm{tg}}^2 \in [H^1(\Omega_2)]^{N-1}$ therefore the equality holds in the H^1 -sense too, i.e. X_{tg} is closed as desired.

Next we use space X_{tg} to determine the limiting problem in the tangential direction.

Lemma 4.3 (Limiting tangential behavior's variational statement). *Let* v^2 *be the limit found in Theorem 3.4* (*ii*). *Then, the following weak variational statement is satisfied:*

$$-\int_{\Omega_2} p^2 \, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 + \int_{\Omega_2} \mu \, \boldsymbol{\nabla}_T \mathbf{v}^2 : \boldsymbol{\nabla}_T \mathbf{w}^2 + \int_\Gamma \beta \sqrt{Q} \, \mathbf{v}_{\mathrm{tg}}^2 \cdot \mathbf{w}_{\mathrm{tg}}^2 \, dS = \int_{\Omega_2} \mathbf{f}_{\mathrm{tg}}^2 \cdot \mathbf{w}_{\mathrm{tg}}^2, \quad \textit{for all } \mathbf{w}^2 \in \mathbf{X}_{\mathrm{tg}}. \tag{66}$$

Proof. Let w^2 # X_{tg}; then w = (0, w^2) # X; test (30a) with w and get

$$-\epsilon \int_{\Omega_2} p^{2,\epsilon} \, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 + \epsilon^2 \int_{\Omega_2} \mu \, D^\epsilon \mathbf{v}^{2,\epsilon} : D^\epsilon \mathbf{w}^2 + \epsilon^2 \int_\Gamma \beta \sqrt{Q} \, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\mathrm{tg}}^2 \, dS = \epsilon \int_{\Omega_2} \mathbf{f}_{\mathrm{tg}}^{2,\epsilon}.$$

Divide the whole expression over e, expand the second summand according to the identity (15) and recall that $\partial_z w^2 = 0$; this gives

$$-\int_{\Omega_2} p^{2,\epsilon} \, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 + \int_{\Omega_2} \mu \left[\boldsymbol{\nabla}_T \left(\epsilon \, \mathbf{v}^{2,\epsilon} \right) + (\epsilon - 1) \partial_z \, \mathbf{v}^{2,\epsilon} \boldsymbol{\nabla}_T^t \zeta \right] : \boldsymbol{\nabla}_T \, \mathbf{w}^2 + \int_\Gamma \beta \sqrt{Q} \, \epsilon \, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\mathrm{tg}}^2 \, dS = \int_{\Omega_2} \mathbf{f}_{\mathrm{tg}}^{2,\epsilon} \cdot \mathbf{w}_{\mathrm{tg}}^2.$$

Letting e # 0, the limit v^2 meets the condition

$$-\int_{\Omega_2} p^2 \, \boldsymbol{\nabla}_T \cdot \mathbf{w}_T^2 + \int_{\Omega_2} \mu \left[\boldsymbol{\nabla}_T \mathbf{v}^2 - \chi \boldsymbol{\nabla}_T^t \zeta \right] : \boldsymbol{\nabla}_T \mathbf{w}^2 + \int_\Gamma \beta \sqrt{Q} \, \mathbf{v}_{\mathrm{tg}}^2 \cdot \mathbf{w}_{\mathrm{tg}}^2 \, dS = \int_{\Omega_2} \mathbf{f}_{\mathrm{tg}}^2 \cdot \mathbf{w}_{\mathrm{tg}}^2. \tag{67}$$

We modify the higher order term using the property $\partial_z w^2 = 0$:

$$-\int_{\Omega_2} \mu\,\chi\,\nabla_r^t\zeta : \nabla_r \mathbf{w}^2 = \int_{\Omega_2} \mu\,\chi(-\nabla_r^t\zeta, 1) : \nabla \mathbf{w}^2 = \int_{\Omega_2} \mu\,|(-\nabla_r\zeta, 1)|\chi \cdot (\nabla \mathbf{w}^2 \cdot \widehat{n})\,.$$

Recall that $\mathbf{w}^2 \cdot \widehat{n} = 0,$, because w^2 # X_{tg}; then $\partial_{\widehat{n}} \mathbf{w}^2 = \nabla \mathbf{w}^2 \cdot \widehat{n} = 0.$ Replacing the above expression in (67), the statement (66) follows because all the previous reasoning is valid for w^2 # X_{tg} arbitrary.

4.2. The higher order effects and the limiting problem

The higher order effects of the e-problem have to be modeled in the adequate space; to that end we use the information attained. We know the higher order term x satisfies the condition (55c) and it belongs to $L^2(\Omega_2)$. This motivates the following definition:

Definition 4.4. Define

I. The subspace

$$\mathbf{W}_{\widehat{n}} \overset{\text{def}}{=} \{[\mathbf{w}^1, \eta] \in \mathbf{X} : \eta_{\text{tg}} = 0_T,\ \eta \cdot \widehat{n} = \mathbf{w}^1 \cdot \widehat{n}\big|_\Gamma(\widetilde{\mathbf{x}})(1-z)\}, \qquad (68)$$

endowed with its natural norm.

II. The space of limit normal effects in the following way:

$$\mathbf{X}^0_{\widehat{n}} \overset{\text{def}}{=} \{[\mathbf{w}^1, \eta] \in \mathbf{H}_{\text{div}}(\Omega_1) \times \mathbf{H}(\partial_z, \Omega_2) : \\ \eta_{\text{tg}} = 0_T,\ \partial_z \eta = 0,\ \eta \cdot \widehat{n} = -\mathbf{w}^1 \cdot \widehat{n}\big|_\Gamma(\widetilde{\mathbf{x}})(1-z)\}, \qquad (69a)$$

endowed with its natural norm

$$\big\|[\mathbf{w}^1, \eta]\big\|^2_{\mathbf{X}^0_{\widehat{n}}} \overset{\text{def}}{=} \|\mathbf{w}^1\|^2_{\mathbf{H}_{\text{div}}(\Omega_1)} + \|\eta\|^2_{\mathbf{H}(\partial_z, \Omega_2)}. \qquad (69b)$$

Remark 4.5. I. It is direct to prove that $\mathbf{X}^0_{\widehat{n}}$ is closed.

II. Observe that, due to its structure, the component η of an element in $\mathbf{X}^0_{\widehat{n}}$ can be completely described by its normal trace on Γ, i.e., the norm

$$\big\|[\mathbf{w}^1, \eta]\big\|^2_{\mathbf{X}^0_{\widehat{n}}} \overset{\text{def}}{=} \|\mathbf{w}^1\|^2_{\mathbf{H}_{\text{div}}(\Omega_1)} + \|\eta \cdot \widehat{n}\|^2_{0,\Gamma} \qquad (70)$$

is equivalent to the norm (69b). This feature will permit the dimensional reduction of the limiting problem formulation later on (see Section 5.2).

III. Let v^1 and ξ be the limits found in the statements (39a) and (41a), respectively. The function $[v^1, \xi]$ belongs to $\mathbf{X}^0_{\widehat{n}}$, with

$$\boldsymbol{\xi} \overset{\text{def}}{=} U \begin{Bmatrix} 0_T \\ \xi \end{Bmatrix}. \qquad (71)$$

This was one of the motivations behind the definition of the space $\mathbf{X}^0_{\widehat{n}}$ above.

iv. The information about the higher order term x is complete only in its normal direction $\chi(\widehat{n})$. Furthermore, the facts that x depends only on $\widetilde{\mathbf{x}}$ (see Equation (55c)) and that $\chi \cdot \widehat{n} = \partial_z \xi = -\mathbf{v}^1 \cdot \widehat{n}|_\Gamma$, show that only information corresponding to the normal component of x will be preserved by the modeling space $\mathbf{X}^0_{\widehat{n}}$, while the tangential component of the higher order term x(tg) will be given away for good. It is also observed that most of the terms involving the presence of x require only its normal component, e.g. $\chi \cdot \partial_{\hat{n}} \mathrm{w}^2 = \chi(\hat{n}) \cdot \partial_{\hat{n}} \mathrm{w}^2$ in the third summand of the variational statement (66). This was the reason why the space $\mathrm{X}^0_{\hat{\mathbf{n}}}$ excludes tangential effects of the higher order term.

Before characterizing the asymptotic behavior of the normal flux we need a technical lemma.

Lemma 4.6. *The subspace* $\mathbf{W}_{\hat{\mathbf{n}}} \subseteq \mathbf{X}$ *is dense in* $\mathrm{X}^0_{\hat{\mathbf{n}}}$.

Proof. Consider an element $\mathrm{w} = (\mathrm{w}^1, \eta) \in \mathrm{X}^0_{\hat{\mathbf{n}}}$; then η $_{tg}$ = 0_T, and $\eta \cdot \hat{n} \in H(\partial_z, \Omega_2)$ is completely defined by its trace on the interface Γ. Given e > 0, take $\varpi \in H^1_0(\Gamma)$ such that $\|\varpi - \eta \cdot \hat{n}|_\Gamma\|_{L^2(\Gamma)} \leq \epsilon$. Now extend the function to the whole domain using the rule $\varrho(\tilde{\mathrm{x}}, z) \overset{\mathrm{def}}{=} \varpi(\tilde{\mathrm{x}})(1-z)$; then $\|\varrho - \eta \cdot \hat{n}\|_{H(\partial_z, \Omega_2)} \leq \epsilon$. From the construction of ϱ we know that $\|\varrho|_\Gamma - \eta \cdot \hat{n}|_\Gamma\|_{0,\Gamma} = \|\varpi - \eta \cdot \hat{n}|_\Gamma\|_{0,\Gamma} \leq \epsilon$. Define $g = \varrho|_\Gamma - \eta \cdot \hat{n}|_\Gamma \in L^2(\Gamma)$; due to Lemma 1 there exists u # $H_{\text{div}}(\Omega_1)$ such that $\mathbf{u} \cdot \hat{n} = g$ on Γ, $\mathbf{u} \cdot \hat{n} = 0$ on $\partial\Omega_1$ - Γ and $\|\mathbf{u}\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq C_1 \|g\|_{0,\Gamma}$, with C_1 depending only on Ω_1. Then, the function w^1 + u is such that $(\mathbf{w}^1 + \mathbf{u}) \cdot \hat{n} = \mathbf{w}^1 \cdot \hat{n} + \varpi - \eta \cdot \hat{n} = \varpi$ and $\|\mathrm{w}^1 + \mathrm{u} - \mathrm{w}^1\|_{\mathbf{H}_{\text{div}}(\Omega_1)} = \|\mathbf{u}\|_{\mathbf{H}_{\text{div}}(\Omega_1)} \leq C_1 \|g\|_{0,\Gamma} \leq C_1 \epsilon$. Moreover, defining

$$\mathbf{w}^2 \overset{\mathbf{def}}{=} U \begin{Bmatrix} 0_T \\ \varrho \end{Bmatrix},$$

we notice that the function (w^1 +u, w^2) belongs to $\mathbf{W}_{\hat{\mathbf{n}}}$. Due to the previous observations we have

$$\left\|\mathbf{w} - (\mathbf{w}^1 + \mathbf{u}, \mathbf{w}^2)\right\|_{\mathbf{X}^0} = \left\|(\mathbf{w}^1, \eta) - (\mathbf{w}^1 + \mathbf{u}, \mathbf{w}^2)\right\|_{\mathbf{X}^0_{\hat{\mathbf{n}}}} \leq \sqrt{C_1 + 1}\,\epsilon.$$

Given that the constants depend only on the domains Q_i and Q_2, it follows that $\mathbf{W}_{\hat{\mathbf{n}}}$ is dense in $\mathrm{X}^0_{\hat{\mathbf{n}}}$.

Definition 4.7. Let μ be the shear viscosity of the fluid, and define its average in the z-direction by

$$\bar{\mu} \overset{\mathbf{def}}{=} \int_{\zeta(\tilde{\mathbf{x}})}^{\zeta(\tilde{\mathbf{x}})+1} \mu \, dz. \tag{72}$$

Lemma 4.8 (Limiting normal behavior's variational statement). *Let* v^1 , v^2 *be the limits found in Corollary 3.4, and let* p^1 , p^2 *be the limits found in Theorem 3.5. Then, the following variational statement is satisfied:*

$$\int_{\Omega_1} Q\,\mathbf{v}^1 \cdot \mathbf{w}^1\, d\mathbf{x} - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1\, d\mathbf{x} + \int_\Gamma p^2 |(-\boldsymbol{\nabla}_T \zeta, 1)| (\mathbf{w}^1 \cdot \hat{n}|_\Gamma)\, dS \\ + \int_\Gamma (\alpha + \bar{\mu})\,(\mathbf{v}^1 \cdot \hat{n})(\mathbf{w}^1 \cdot \hat{n})\, dS = 0, \quad \textit{for all } \mathbf{w}^1 \in \mathbf{X}^0_{\hat{\mathbf{n}}}. \tag{73}$$

Here, $\bar{\mu}$ *is the averaged viscosity introduced in Definition 4-7.*

Proof. Test (30a) with $[\mathrm{w}^1, \eta] \in \mathbf{W}_{\hat{n}}$ and let $\epsilon \to 0$; this gives

$$\int_{\Omega_1} Q\,\mathbf{v}^1 \cdot \mathbf{w}^1\, d\mathbf{x} - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1\, d\mathbf{x} + \int_{\Omega_2} p^2\, \partial_z \eta_T \cdot \boldsymbol{\nabla}_T \zeta\, d\tilde{\mathbf{x}}\, dz - \int_{\Omega_2} p^2\, \partial_z \eta_N\, d\tilde{\mathbf{x}}\, dz \\ + \int_{\Omega_2} \mu\, \chi \cdot \partial_z\, \eta\, d\tilde{\mathbf{x}}\, dz + \alpha \int_\Gamma (\mathbf{v}^1 \cdot \hat{n})\,(\mathbf{w}^1 \cdot \hat{n})\, dS = 0. \tag{74}$$

Notice that the third and fourth summands in the expression above can be written as

$$\begin{aligned}\int_{\Omega_2} p^2\, \partial_z\, \boldsymbol{\eta}_{T} \cdot \boldsymbol{\nabla}_{T}\zeta \, d\widetilde{\mathrm{x}}\, dz - \int_{\Omega_2} p^2\, \partial_z \eta_{N}\, d\widetilde{\mathrm{x}}\, dz &= - \int_{\Omega_2} p^2 \partial_z \boldsymbol{\eta} \cdot \begin{Bmatrix} -\boldsymbol{\nabla}_{T}^{t}\zeta \\ 1 \end{Bmatrix} \\ &= - \int_{\Omega_2} p^2 |(-\boldsymbol{\nabla}_{T}\zeta, 1)|\, \partial_z \boldsymbol{\eta} \cdot \widehat{\boldsymbol{n}} \\ &= - \int_{\Omega_2} p^2 |(-\boldsymbol{\nabla}_{T}\zeta, 1)| (-\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma}) \\ &= - \int_{\Gamma} p^2 |(-\boldsymbol{\nabla}_{T}\zeta, 1)| (-\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma})\, dS,\end{aligned}$$

where the second equality holds by the definition of $\mathbf{W}_{\widehat{\boldsymbol{n}}}$ and the last equality holds since p^2 is independent from z (see Equation (55b)). Next, recalling the identities (42), (55a) and (55c), observe that

$$\begin{aligned}\int_{\Omega_2} \mu\, \boldsymbol{\chi} \cdot \partial_z \boldsymbol{\eta} = \int_{\Omega_2} \mu\, (\boldsymbol{\chi} \cdot \widehat{\boldsymbol{n}})\, \partial_z (\boldsymbol{\eta} \cdot \widehat{\boldsymbol{n}}) &= \int_{\Omega_2} \mu\, \partial_z \xi\, (-\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma}) \\ &= \int_{\Omega_2} \mu\, (-\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma})\, (-\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma}) \\ &= \int_{\Gamma} \bar{\mu}\, (-\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma})\, (-\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma})\, dS.\end{aligned}$$

Replacing the last two identities in (74), we conclude that the variational statement (73) holds for every test function in $\mathbf{W}_{\widehat{\boldsymbol{n}}}$. Since the bilinear form of the statement is continuous with respect to the norm $\|\cdot\|_{\mathbf{X}^0_{\widehat{\boldsymbol{n}}}}$ and $\mathbf{W}_{\widehat{\boldsymbol{n}}}$ is dense in $\mathbf{X}^0_{\widehat{\boldsymbol{n}}}$, it follows that the statement holds for every element w # $\mathbf{X}^0_{\widehat{\boldsymbol{n}}}$.

4.3. Variational formulation of the limit problem

In this section we give a variational formulation of the limiting problem and prove it is well-posed. We begin characterizing the limit form of the conservation laws.

Lemma 4.9 (Mass conservation in the limit problem). *Let v^{1} , v^2 be the limits found in Theorem 3.4; then,*

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = h^1, \qquad \textit{in } \Omega_1; \tag{75a}$$

$$\int_{\Omega_2} \boldsymbol{\nabla}_{T} \cdot \mathbf{v}^2\, \varphi^2 - \int_{\Gamma} |(- \boldsymbol{\nabla}_{T}\zeta, 1)| (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \varphi^2\, dS = 0, \qquad \textit{for all } \varphi^2 \in L^2(\Omega_2),\ \varphi^2 = \varphi^2(\widetilde{\mathrm{x}}). \tag{75b}$$

Proof. Take $\Phi = (\varphi^{1},0)$ # Y, test (30b) and let ϵ # 0; we have

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1 \varphi^1 = \int_{\Omega_1} h^1 \varphi^1, \qquad \text{for all } \varphi^1 \in L^2(\Omega_1).$$

The statement above implies (75a).

For the variational statement (75b), first recall the dependence of the limit velocity given in equation (55b). Hence, consider $\Phi = (0, \varphi^{2})$ G Y such that $\varphi^2 = \varphi^2(\widetilde{\mathrm{x}})$, test (30b) and regroup terms using (54a). The previous yields

$$\int_{\Omega_2} \boldsymbol{\nabla}_{T} \cdot (\epsilon\, \mathbf{v}_{T}^{2,\epsilon})\, \varphi^2 + \int_{\Omega_2} \partial_z (\epsilon\, \mathbf{v}_{T}^{2,\epsilon}) \cdot \boldsymbol{\nabla}_{T}\zeta\, \varphi^2 + \int_{\Omega_2} |(-\boldsymbol{\nabla}_{T}\zeta, 1)| \partial_z (\mathbf{v}^{2,\epsilon} \cdot \widehat{\boldsymbol{n}}) \varphi^2 = 0.$$

Next, let ϵ # 0 and get

$$\int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \mathbf{v}^2\, \varphi^2 + \int_{\Omega_2} \partial_z \mathbf{v}^2 \cdot \boldsymbol{\nabla}_T \zeta\, \varphi^2 + \int_{\Omega_2} |(-\boldsymbol{\nabla}_T \zeta, 1)| \partial_z \xi\, \varphi^2\, d\widetilde{x}\, dz = 0.$$

In the expression above, recall that $\partial_z \mathbf{v}^2 = 0$, $\varphi^2 = \varphi_2(\widetilde{\mathrm{x}})$ and the identity (55a); then, the statement (75b) follows.

Next, we introduce the function spaces of the limiting problem:

Definition 4.10. Define the space of velocities by

$$\mathbf{X}^0 \stackrel{\text{def}}{=} \{\mathbf{w} + \mathbf{u} : \mathbf{w} \in \mathbf{X}^0_{\widehat{\mathbf{n}}},\ \mathbf{u} \in \mathbf{X}_{\mathrm{tg}}\}, \tag{76a}$$

endowed with the natural norm of the space $\mathrm{X}^0_{\widehat{\mathbf{n}}} \bigoplus \mathrm{X}_{\mathrm{tg}}$. Define the space of pressures by

$$\mathbf{Y}^0 \stackrel{\text{def}}{=} \{\Phi = (\varphi^1, \varphi^2) \in \mathbf{Y} : \varphi^2 = \varphi^2(\widetilde{\mathrm{x}})\}, \tag{76b}$$

endowed with its natural norm.

Theorem 4.11 (Limiting problem variational formulation). *Let v^1, v^2 be the limits found in Corollary 3.4, and let p^1, p^2 be the limits found in Theorem 3.5. Then, they satisfy the following variational problem:*

$$[\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0 :$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 - \int_{\Omega_2} p^2\, \boldsymbol{\nabla}_T \cdot \mathbf{w}^2_T + \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \mathbf{v}^2 : \boldsymbol{\nabla}_T \mathbf{w}^2 \\
&+ \int_\Gamma \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^2_{\mathrm{tg}} \cdot \mathbf{w}^2_{\mathrm{tg}}\, dS + \int_\Gamma (\alpha + \bar{\mu})(\mathbf{v}^1 \cdot \widehat{n})(\mathbf{w}^1 \cdot \widehat{n})\, dS + \int_\Gamma |(-\boldsymbol{\nabla}_T \zeta, 1)| p^2 (\mathbf{w}^1 \cdot \widehat{n}) dS \\
&\qquad = \int_{\Omega_2} \mathbf{f}^2_{\mathrm{tg}} \cdot \mathbf{w}^2_{\mathrm{tg}},
\end{aligned} \tag{77a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1 \varphi^1 + \int_{\Omega_2} \boldsymbol{\nabla}_T \cdot \mathbf{v}^2\, \varphi^2 - \int_\Gamma |(-\boldsymbol{\nabla}_T \zeta, 1)| (\mathbf{v}^1 \cdot \widehat{n}) \varphi^2\, dS = \int_{\Omega_1} h^1\, \varphi^1, \tag{77b}$$

$$\textit{for all } [\mathbf{w}, \Phi] \in \mathbf{X}^0 \times \mathbf{Y}^0.$$

Moreover, the problem (77) is *well-posed.* (*Here, $\bar{\mu}$ is the averaged viscosity introduced in Definition 4.7.*)

Proof. Since [v, p] satisfies the variational statements (66), (73), (75a), (75b) as shown in Lemmas 4.3, 4.8 and 4.9, respectively, it follows that [v, p] satisfies the problem (77) above.

In order to show that the problem is well-posed w(e prove)continuous depen(dence)of the solution with respect to the data. Test (77a) with v^1, v^2 and (77b) with $(\mathrm{p}^1, \mathrm{p}^2)$, add them together and get

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{v}^1 + \int_{\Omega_2} \mu\, \boldsymbol{\nabla}_T \mathbf{v}^2 : \boldsymbol{\nabla}_T \mathbf{v}^2 \\
&\quad + \int_\Gamma \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^2_{\mathrm{tg}} \cdot \mathbf{v}^2_{\mathrm{tg}}\, dS + \int_\Gamma (\alpha + \bar{\mu})(\mathbf{v}^1 \cdot \widehat{n})(\mathbf{v}^1 \cdot \widehat{n})\, dS \\
&\qquad = \int_{\Omega_2} \mathbf{f}^2_{\mathrm{tg}} \cdot \mathbf{v}^2_{\mathrm{tg}} + \int_{\Omega_1} h^1 p^1.
\end{aligned} \tag{78}$$

Applying the Cauchy-Bunyakowsky-Schwarz inequality to the right hand side of the expression above, and recalling that $\mathrm{v}^2_{\mathrm{tg}}$ is constant in the z-direction, we get

$$\begin{aligned}
\int_{\Omega_2} \mathbf{f}^2_{\mathrm{tg}} \cdot \mathbf{v}^2_{\mathrm{tg}} + \int_{\Omega_1} h^1 p^1 &\leq \|\mathbf{f}^2_{\mathrm{tg}}\|_{0,\Omega_2} \|\mathbf{v}^2_{\mathrm{tg}}\|_{0,\Omega_2} + \|h^1\|_{0,\Omega_1} \|p^1\|_{0,\Omega_1} \\
&\leq \|\mathbf{f}^2_{\mathrm{tg}}\|_{0,\Omega_2} \|\mathbf{v}^2_{\mathrm{tg}}\|_{0,\Gamma} + \widetilde{C}\, \|h^1\|_{0,\Omega_1} \|\boldsymbol{\nabla} p^1\|_{0,\Omega_1} \\
&\leq \|\mathbf{f}^2_{\mathrm{tg}}\|_{0,\Omega_2} \|\mathbf{v}^2_{\mathrm{tg}}\|_{0,\Gamma} + C\, \|h^1\|_{0,\Omega_1} \|\mathcal{Q} \mathbf{v}^1\|_{0,\Omega_1} \\
&\leq \widetilde{C} \Big[\|\mathbf{f}^2_{\mathrm{tg}}\|^2_{0,\Omega_2} + \|h^1\|^2_{0,\Omega_1} \Big]^{1/2} \Big[\|\mathbf{v}^2_{\mathrm{tg}}\|^2_{0,\Gamma} + \|\mathbf{v}^1\|^2_{0,\Omega_1} \Big]^{1/2}.
\end{aligned} \tag{79}$$

Here, the second and third inequalities holds because p^1 satisfies respectively the drained boundary conditions (Poincaré's inequality applies) and the Darcy's equation as stated in (44a). Finally, the fourth inequality is a new application of the Cauchy-Bunyakowsky-Schwarz inequality for 2-D vectors. Introducing (79) in (78), and recalling Hypothesis 2 on the coefficients $\mathcal{Q}$, α, β and μ, we have

$$\Big[\|\mathbf{v}^1\|_{0,\Omega_1}^2 + \|\mathbf{v}^1\cdot\widehat{n}\|_{\Gamma}^2 + \|\nabla_T \mathbf{v}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|\mathbf{v}_{\mathrm{tg}}^2\|_{0,\Gamma}^2\Big]^{1/2} \leq \widetilde{C}\Big[\big\|\mathbf{f}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|h^1\|_{0,\Omega_1}^2\Big]^{1/2}. \tag{80}$$

Recalling (39b), the expression above implies that

$$\|\mathbf{v}^1\|_{\mathbf{H}_{\mathrm{div}}(\Omega_1)} \leq \widetilde{C}\Big[\big\|\mathbf{f}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|h^1\|_{0,\Omega_1}^2\Big]^{1/2}. \tag{81}$$

Next, given that $\mathbf{w}_{\mathrm{tg}}^2$ is independent from z (see (40c)), it follows that $\|\mathbf{v}_{\mathrm{tg}}^2\|_{0,\Gamma} = \|\mathbf{v}_{\mathrm{tg}}^2\|_{0,\Omega_2}$ and $\|\nabla \mathbf{v}_{\mathrm{tg}}^2\|_{0,\Omega_2} = \|\nabla_T \mathbf{v}_{\mathrm{tg}}^2\|_{0,\Omega_2}$. Therefore (80) yields

$$\|\mathbf{v}^2\|_{1,\Omega_2} \leq \widetilde{C}\Big[\big\|\mathbf{f}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|h^1\|_{0,\Omega_1}^2\Big]^{1/2}. \tag{82}$$

Again, recalling that p^1 satisfies the Darcy's equation and the drained boundary conditions (Poincaré's inequality applies) as stated in (44a), the estimate (81) implies $\|p^1\|_{1,\Omega_1} \leq \widetilde{C}\Big[\big\|\mathbf{f}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|h^1\|_{0,\Omega_1}^2\Big]^{1/2}.$ (83)

Next, in order to prove continuous dependence for p^2, recall (61), where it is observed that all the terms are already continuously dependent on the data; then it follows that

$$\|p^2\|_{0,\Omega_2} \leq C\Big[\|\mathbf{f}_{\mathrm{tg}}^2\|_{0,\Omega_2}^2 + \|h^1\|_{0,\Omega_1}^2\Big]^{1/2}. \tag{84}$$

Finally, in order to prove the uniqueness of the solution, assume there are two solutions, test the problem (77) with its difference and subtract them. We conclude that the difference of solutions must satisfy the problem (77) with null forcing terms. This implies, due to (81), (82) (83) and (84), that the difference of solutions is equal to zero, i.e. the solution is unique. Since (77) has a solution, which is unique and it continuously depends on the data, it follows that the problem is well-posed.

Corollary 4.12. *The weak convergence statements in Corollaries 3.4 and 3.5 hold for the whole sequence* $((v^\epsilon, p^\epsilon) : \epsilon > 0)$ *of solutions.*

Proof. It suffices to observe that, due to Hypothesis 4, the limiting problem (77) has unique forcing terms. Therefore, any subsequence of the solutions $((v^\epsilon, p^\epsilon) : \epsilon > 0)$ would have a weakly convergent subsequence, whose limit is the solution of problem (77) (v, p), which is also unique, due to Theorem 4.11. Hence, the result follows.

5. *Closing remarks*

We finish the paper highlighting some aspects that were meticulously addressed in [14].

5.1. A mixed formulation for the limiting problem

For an independent well-posedness proof of the problem (77), define the operators

$$A^0 : \mathbf{X}^0 \to (\mathbf{X}^0)', \qquad A^0 \overset{\text{def}}{=} \begin{bmatrix} \mathcal{Q} + \gamma'_{\widehat{\boldsymbol{n}}}[\alpha + \bar{\mu}]\,\gamma_{\widehat{\boldsymbol{n}}} & 0 \\ 0 & \gamma'_{\text{tg}}\beta\sqrt{\mathcal{Q}}\,\gamma_{\text{tg}} - \boldsymbol{\nabla}_{T}\cdot\mu\boldsymbol{\nabla}_{T} \end{bmatrix} \tag{85a}$$

And

$$B^0 : \mathbf{X}_0 \to (\mathbf{Y}^0)', \qquad B^0 \overset{\text{def}}{=} \begin{bmatrix} \boldsymbol{\nabla}\cdot & 0 \\ \gamma'_{\text{tg}}|(-\boldsymbol{\nabla}_{T}\zeta, 1)|\gamma_{\widehat{\boldsymbol{n}}} & \boldsymbol{\nabla}_{T}\cdot \end{bmatrix}. \tag{85b}$$

Then, the variational formulation of the problem (77) has the following mixed formulation:

$$\begin{aligned} [\mathbf{v}, p] \in \mathbf{X}^0 \times \mathbf{Y}^0 : A^0\,\mathbf{v} - (B^0)'\,p &= \mathbf{f}^2, \\ B^0\,\mathbf{v} &= h^1. \end{aligned} \tag{86}$$

The proof now follows showing that the hypotheses of Theorem 1.3 are satisfied. The strategy is completely analogous to that exposed in Lemma 17, Lemma 18 and Theorem 19 in [14].

5.2. Dimensional reduction of the limiting problem

It is direct to see that since X_{tg} and Y^0 do not change on the z-direction inside Ω_2, the integrals on this domain can be reduced to integrals on the interface Γ. This yields a problem coupled on Ω_1 x Γ equivalent to (77). To that end we introduce the space:

$$\mathbf{X}_{\widehat{\boldsymbol{n}}}^{00} \overset{\text{def}}{=} \{\mathbf{w}^1 \in \mathbf{H}_{\text{div}}(\Omega_1) : \mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}\big|_{\Gamma} \in L^2(\Gamma)\}, \tag{87a}$$

endowed with the norm (70), and the space

$$\mathbf{X}_{\text{tg}}^{00} \overset{\text{def}}{=} \{\mathbf{w}^2 \in \mathbf{H}^1(\Gamma) : \mathbf{w}^2(\widetilde{\mathbf{x}}) \cdot \widehat{n}(\widetilde{\mathbf{x}}) = 0 \text{ for all } \widetilde{\mathbf{x}} \in G, \mathbf{w}^2 = 0 \text{ on } \partial\Gamma\}, \tag{87b}$$

endowed with its natural norm.

Remark 5.1. Notice the following:

I. The space, $\mathbf{X}_{\widehat{\boldsymbol{n}}}^{00}$ is isomorphic to $\mathbf{X}_{\widehat{\boldsymbol{n}}}^{0}$ (69a).

II. Since r is a surface (a parametrized manifold in $\#^N$) as described by the identity (6), it is completely characterized by its global chart ζ : G → #. Therefore a function u : Γ → #, γ → u(γ), can be seen as $u_G : G \to \mathbb{R}$, $\widetilde{\mathbf{x}} \mapsto u(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}))$, with G being the orthogonal projection of the surface Γ into $\#^{N-1}$. Identifying u with u_G allows to well-define integrability and differentiability. Hence, the space $L^2(\Gamma)$ is characterized by the equality: $\int_{\Gamma} u^2 dS = \int_G u_G^2 |(\boldsymbol{\nabla}\zeta, 1)| d\widetilde{\mathbf{x}}$, where $d\widetilde{\mathbf{x}}$ is the Lebesgue measure in G # $\#^{N-1}$. In the same fashion, the space $H^1(\Gamma)$ is the closure of the $C^1(\Gamma)$ space in the natural norm $\|u\|_{0,\Gamma}^2 \overset{\text{def}}{=} \|u\|_{0,\Gamma}^2 + \|\boldsymbol{\nabla}_T u\|_{0,\Gamma}^2$. (Clearly, $\#_T$ suffices to store all the differential variation of a function u : Γ → #.)

With the definitions above, define the space of velocities

$$\mathbf{X}^{00} \overset{\text{def}}{=} \{\mathbf{w} + \mathbf{u} : \mathbf{w} \in \mathbf{X}^{00}_{\widehat{\boldsymbol{n}}},\, \mathbf{u} \in \mathbf{X}^{00}_{\text{tg}}\}, \tag{88a}$$

endowed with the natural norm of the space $\mathrm{x}^{0)}_{\widehat{n}} \oplus \mathrm{x}^{00}_{\text{tg}}$. Next, define the space of pressures by

$$\mathbf{Y}^{00} \overset{\text{def}}{=} L^2(\Omega_1) \times L^2(\Gamma), \tag{88b}$$

endowed with its natural norm. Therefore, the problem (77) is equivalent to

$$[\mathbf{v}, p] \in \mathbf{X}^{00} \times \mathbf{Y}^{00} :$$

$$\begin{aligned}
&\int_{\Omega_1} \mathcal{Q}\, \mathbf{v}^1 \cdot \mathbf{w}^1 - \int_{\Omega_1} p^1\, \boldsymbol{\nabla} \cdot \mathbf{w}^1 \\
&\qquad - \int_\Gamma p^2\, \boldsymbol{\nabla}_T \cdot \mathbf{w}^2_T + \int_\Gamma \bar{\mu}\, \boldsymbol{\nabla}_T \mathbf{v}^2 : \boldsymbol{\nabla}_T \mathbf{w}^2 + \int_\Gamma \beta \sqrt{\mathcal{Q}}\, \mathbf{v}^2 \cdot \mathbf{w}^2\, dS \\
&+ \int_\Gamma (\alpha + \bar{\mu})(\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}})\, dS + \int_\Gamma |(-\boldsymbol{\nabla}_T \zeta, 1)| p^2 (\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}}) dS = \int_\Gamma \bar{\mathbf{f}}^2 \cdot \mathbf{w}^2,
\end{aligned} \tag{89a}$$

$$\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}^1 \varphi^1 + \int_\Gamma \boldsymbol{\nabla}_T \cdot \mathbf{v}^2\, \varphi^2 - \int_\Gamma |(-\boldsymbol{\nabla}_T \zeta, 1)| (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \varphi^2\, dS = \int_{\Omega_1} h^1\, \varphi^1, \tag{89b}$$

$$\text{for all } [\mathbf{w}, \Phi] \in \mathbf{X}^{00} \times \mathbf{Y}^{00},$$

where $\bar{\mathrm{f}}^2(\widetilde{\mathrm{x}}) \overset{\text{def}}{=} \int_{\zeta(\widetilde{\mathrm{x}})}^{\zeta(\widetilde{\mathrm{x}})+1} \mathrm{f}^2 dz$.

Remark 5.2 (The Brinkman equation). Notice that in the equation (89a), the product $\mathbf{v}^2_{\text{tg}} \cdot \mathbf{w}^2_{\text{tg}}$ has been replaced by $\mathrm{v}^2 \cdot \mathrm{w}^2$ (for consistency $\bar{\mathrm{f}}^2_{\text{tg}} \cdot \mathrm{w}^2_{\text{tg}}$ was replaced by $\bar{\mathrm{f}}^2 \cdot \mathrm{w}^2$). This is done in order to attain a Brinkman-type form in the third, fourth and fifth summands of equation (89a). Also notice that although $\mathbf{v}^2 \cdot \widehat{\boldsymbol{n}} = 0$ and $\mathbf{w}^2 \cdot \widehat{\boldsymbol{n}} = 0$, the product $\boldsymbol{\nabla}_T \mathbf{v}^2 : \boldsymbol{\nabla}_T \mathbf{w}^2$ can not be replaced by $\boldsymbol{\nabla}_T \mathbf{v}^2_{\text{tg}} : \boldsymbol{\nabla}_T \mathbf{w}^2_{\text{tg}}$, due to the differential operators (the orthogonal matrix U depends on $\widetilde{\mathbf{x}}$). This is the reason why we give up expressing the activity on the interface Γ exclusively in terms of tangential vectors, as its is natural to look for.

5.3. Strong convergence of the solutions

In contrast to the asymptotic analysis in [14], the strong convergence of the solutions can not be concluded. The main reason is the presence of the higher order term *x*, weak limit of the sequence $(\partial_z \mathbf{v}^{2,\epsilon} : \epsilon > 0)$. As it can be seen in the proof of Theorem 4.3, the higher order term x can be removed because the quantifier w^2 belongs to X_{tg}. However, when testing the problem (30) on the diagonal $[\mathrm{v}^\epsilon, p^\epsilon]$ and adding the equations to get rid of the mixed terms, the quantifier $\mathrm{v}^{2,\epsilon}$ does not belong to X_{tg}. As a consequence, the terms $\|\sqrt{\mu}\, \boldsymbol{D}^\epsilon(\epsilon \mathbf{v}^{2,\epsilon})\|^2_{0,\Omega_2} + \|\mu \partial_z \mathbf{v}^{2,\epsilon}\|_{0,\Omega_2}$ contain in its internal structure inner products of the type

$$\int_{\Omega_2} \mu \partial_z \mathbf{v}^{2,\epsilon} \begin{Bmatrix} -\boldsymbol{\nabla}\zeta \\ 1 \end{Bmatrix} : \boldsymbol{\nabla}(\epsilon\, \mathbf{v}^{2,\epsilon}) = \int_{\Omega_2} \mu |(-\boldsymbol{\nabla}\zeta, 1)| \partial_z \mathbf{v}^{2,\epsilon} \cdot \boldsymbol{\nabla}(\epsilon\, \mathbf{v}^{2,\epsilon}) \cdot \widehat{\boldsymbol{n}}, \tag{90}$$

which can not be combined/balanced with other terms present in the evaluation of the diagonal. The product above is not guaranteed to pass to the limit $\int_{\Omega_2} \mu |(-\boldsymbol{\nabla}\zeta, 1)| \mathcal{X} \cdot \boldsymbol{\nabla} \mathbf{v}^2 \cdot \widehat{\boldsymbol{n}}$, because both factors are known to converge weakly, but none has been proved to converge strongly. Such convergence would be ideal since v^2 # X_{tg}, therefore $\partial_{\hat{n}} v^2 = \nabla v^2 \cdot \hat{n} = 0$ and the term (90) would converge to zero. The latter would yield the strong convergence of the norms for $\|\nabla_T(\epsilon v^{2,\epsilon})\|_{0,\Omega_2}$ and $\|\partial_z v^{2,\epsilon}\|_{0,\Omega_2}$ and the desired strong convergence would follow.

More specifically, the surface geometry states that the normal $(\hat{n})$ and the tangential directions (tg) are the important ones, around which the information should be arranged. On the other hand, the estimates yield its information in terms of $\widetilde{\mathbf{x}}$ (T) and z (N). Such disagreement has the effect of keeping intertwined the higher order and lower order terms to the extent of allowing to conclude weak, but not strong convergence statements.

5.4. Ratio of velocities

The relationship of the velocity in the tangential direction with respect to the velocity in the normal direction is very high and tends to infinity as expected for most of the cases. We know that $(\| v_{\hat{n}}^{2,\epsilon} \|_{0,\Omega_2} : \epsilon > 0)$ is bounded, therefore $\| \epsilon v_{\hat{n}}^{2,\epsilon} \|_{0,\Omega_2} = \epsilon \| v_{\hat{n}}^{2,\epsilon} \|_{0,\Omega_2} \to 0.$ Suppose first that $v_{tg}^2 \neq 0,$ and consider the ratios

$$\frac{\| \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \|_{0,\Omega_2}}{\| \mathbf{v}_{\hat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Omega_2}} = \frac{\| \epsilon\, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \|_{0,\Omega_2}}{\| \epsilon\, \mathbf{v}_{\hat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Omega_2}} \geq \frac{\liminf \| \epsilon\, \mathbf{v}_{\mathrm{tg}}^{2,\epsilon} \|_{0,\Omega_2}}{\| \epsilon\, \mathbf{v}_{\hat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Omega_2}} > \frac{\| \mathbf{v}_{\mathrm{tg}}^{2} \|_{0,\Omega_2} - \delta}{\| \epsilon\, \mathbf{v}_{\hat{\boldsymbol{n}}}^{2,\epsilon} \|_{0,\Omega_2}} > 0.$$

The lower bound holds true for $\epsilon > 0$ small enough and adequate $\delta > 0$; then we conclude that the L^2-norms' ratio of the tangent component over the normal component blows-up to infinity, i.e., the tangential velocity is much faster than the normal one in the thin channel.

In contrast, if $v_{tg}^2 = 0$ nothing can be concluded, since it can not be claimed that $v^1 \cdot \hat{n} = 0$ on Γ unless $f^2 = 0$ is enforced, trivializing the activity on Ω_2. Therefore, it can only be concluded that $\|v_{tg}^{2,\epsilon}\|_{0,\Omega_2} \gg \|v_{\hat{n}}^{2,\epsilon}\|_{0,\Omega_2}$ for $\epsilon > 0$ small enough, when $v_{tg}^2 \neq 0,$ as discussed above.

5.5. Reduction to the flat horizontal case

In this section we show how the e-problems (30) and the limit problem (77) are corresponding generalizations of the systems (23) and (59) presented in [14]. We show this fact in several steps:

a. Recall that in [14] the interface r is flat horizontal and, for convenience, it was assumed that Γ # $\#^{N-1}$ x {0}. In our current scenario, this is attained by merely setting ζ = 0, which satisfies all the conditions of Hypothesis 1. Furthermore, the following differential operators verify

$$\nabla \zeta \equiv 0, \qquad \nabla_T \zeta \equiv 0, \qquad D^{\epsilon} \mathbf{w} \equiv \nabla_T \mathbf{w},$$

where $D^{\epsilon} w$ is defined in (15).

b. For ζ = 0, the stream line localizer of Definition 2.5 is the constant matrix valued function $\widetilde{\mathrm{x}} \mapsto U(\widetilde{\mathrm{x}}) = I,$ where I # $\#^{NxN}$ is the identity matrix. In particular $\hat{n} \equiv \hat{e}_N,$ which is independent from $\widetilde{\mathbf{x}}$.

c. Given that the stream line localizer is the identity matrix, the normal and tangential velocities introduced in the equations (19) satisfy

$$\mathbf{w}^2_{\widehat{\boldsymbol{n}}} = \mathbf{w}\cdot\widehat{\boldsymbol{n}} = \mathbf{w}^2_{N}, \qquad\qquad \mathbf{w}^2_{\mathbf{tg}} = \mathbf{w}^2_{T}.$$

Taking into account all the previous observations, the e-problems (30) reduce to

$$[\mathbf{v}^{\epsilon}, \mathbf{p}^{\epsilon}] \in \mathbf{X}\times\mathbf{Y}:$$

$$\begin{aligned}
\int_{\Omega_1} \mathcal{Q}\,\mathbf{v}^{1,\epsilon}\cdot\mathbf{w}^1\,d\mathbf{x} - \int_{\Omega_1} p^{1,\epsilon}\,\boldsymbol{\nabla}\cdot\mathbf{w}^1\,d\mathbf{x} - \epsilon\int_{\Omega_2} p^{2,\epsilon}\,\boldsymbol{\nabla}_T\cdot\mathbf{w}^2_T\,d\widetilde{\mathbf{x}}\,dz - \int_{\Omega_2} p^{2,\epsilon}\,\partial_z \mathbf{w}^2_N\,d\widetilde{\mathbf{x}}\,dz \\
+\epsilon^2\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^{2,\epsilon} : \boldsymbol{\nabla}_T\mathbf{w}^2\,d\widetilde{\mathbf{x}}\,dz + \int_{\Omega_2}\mu\,\partial_z\,\mathbf{v}^{2,\epsilon}\cdot\partial_z\,\mathbf{w}^2\,d\widetilde{\mathbf{x}}\,dz \\
+\alpha\int_{\Gamma}(\mathbf{v}^{1,\epsilon}\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS + \epsilon^2\int_{\Gamma}\beta\sqrt{\mathcal{Q}}\,\mathbf{v}^{2,\epsilon}_T\cdot\mathbf{w}^2_T\,dS \\
= \epsilon\int_{\Omega_2}\mathbf{f}^{2,\epsilon}\cdot\mathbf{w}^2\,d\widetilde{\mathbf{x}}\,dz,
\end{aligned} \tag{91a}$$

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^{1,\epsilon}\varphi^1\,d\mathbf{x} + \epsilon\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^{2,\epsilon}_T\,\varphi^2\,d\widetilde{\mathbf{x}}\,dz + \int_{\Omega_2}\partial_z\mathbf{v}^{2,\epsilon}_N\,\varphi^2\,d\widetilde{\mathbf{x}}\,dz = \int_{\Omega_1} h^{1,\epsilon}\,\varphi^1\,d\mathbf{x}, \tag{91b}$$

$$\text{for all } [\mathbf{w}, \Phi] \in \mathbf{X}\times\mathbf{Y}.$$

The summands of the second line in (91a) can be written in the following way:

$$\begin{aligned}
\epsilon^2\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^{\epsilon} : \boldsymbol{\nabla}_T\mathbf{w}^2\,d\widetilde{\mathbf{x}}\,dz &= \epsilon^2\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^{2,\epsilon}_T : \boldsymbol{\nabla}_T\mathbf{w}^2_T d\widetilde{\mathbf{x}}\,dz + \epsilon^2\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^{2,\epsilon}_N : \boldsymbol{\nabla}_T\mathbf{w}^2_N\,d\widetilde{\mathbf{x}}\,dz, \\
\int_{\Omega_2}\mu\,\partial_z\mathbf{v}^{2,\epsilon}\cdot\partial_z\mathbf{w}^2\,d\widetilde{\mathbf{x}}\,dz &= \int_{\Omega_2}\mu\,\partial_z\mathbf{v}^{2,\epsilon}_T\cdot\partial_z\mathbf{w}^2_T\,d\widetilde{\mathbf{x}}\,dz + \int_{\Omega_2}\mu\,\partial_z\mathbf{v}^{2,\epsilon}_N\cdot\partial_z\mathbf{w}^2_N\,d\widetilde{\mathbf{x}}\,dz.
\end{aligned}$$

Introducing the changes above in (91), the system (23) in [14] is attained.

Again, taking into account the simplifications corresponding to a flat horizontal interface ($\zeta = 0$) listed at the beginning of this section, the limit problem (77) reduces to

$$[\mathbf{v}, p] \in \mathbf{X}^0\times\mathbf{Y}^0:$$

$$\begin{aligned}
\int_{\Omega_1}\mathcal{Q}\,\mathbf{v}^1\cdot\mathbf{w}^1 - \int_{\Omega_1} p^1\,\boldsymbol{\nabla}\cdot\mathbf{w}^1 - \int_{\Omega_2} p^2\,\boldsymbol{\nabla}_T\cdot\mathbf{w}^2_T + \int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^2 : \boldsymbol{\nabla}_T\mathbf{w}^2 \\
+\int_{\Gamma}\beta\sqrt{\mathcal{Q}}\,\mathbf{v}^2_T\cdot\mathbf{w}^2_T\,dS + \int_{\Gamma}(\alpha+\bar{\mu})(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})\,dS + \int_{\Gamma} p^2(\mathbf{w}^1\cdot\widehat{\boldsymbol{n}})dS = \int_{\Omega_2}\mathbf{f}^2_T\cdot\mathbf{w}^2_T,
\end{aligned} \tag{92a}$$

$$\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^1\varphi^1 + \int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^2\,\varphi^2 - \int_{\Gamma}(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\varphi^2\,dS = \int_{\Omega_1} h^1\,\varphi^1, \tag{92b}$$

$$\text{for all } [\mathbf{w}, \Phi] \in \mathbf{X}^0\times\mathbf{Y}^0.$$

Notice that since $\zeta = 0$, the spaces X^{00}, Y^{00} in [14] are isomorphic to X^0 and Y^0 in (92), respectively. Finally, reordering the summands in the equalities above and writing

$$\begin{aligned}
\int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^2 : \boldsymbol{\nabla}_T\mathbf{w}^2 = \int_{\Gamma}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^2 : \boldsymbol{\nabla}_T\mathbf{w}^2 d\widetilde{\mathbf{x}} = \int_{\Omega_2}\mu\,\boldsymbol{\nabla}_T\mathbf{v}^2 : \boldsymbol{\nabla}_T\mathbf{w}^2 dS, \\
\int_{\Omega_2}\boldsymbol{\nabla}_T\cdot\mathbf{v}^2\,\varphi^2 = \int_{\Gamma}\boldsymbol{\nabla}_T\cdot\mathbf{v}^2\,\varphi^2 d\widetilde{\mathbf{x}} = \int_{\Gamma}\boldsymbol{\nabla}_T\cdot\mathbf{v}^2\,\varphi^2 dS,
\end{aligned}$$

we obtain the system (59) in [14].

The ϵ-problems (30) are isomorphic to the problems (23) in [14], and the limit problem (77) is isomorphic to (59) (Theorem 21) in [14]. In addition, the reasoning proving that (77) is the limit form of (30) stands for the case ζ = 0. Next, the strong convergence limitations discussed in Section 5.3 no longer hold, since the expression (90) reduces to

$$\int_{\Omega_2} \mu \partial_z \mathbf{v}^{2,\epsilon} \begin{Bmatrix} -\nabla\zeta \\ 1 \end{Bmatrix} : \nabla(\epsilon\, \mathbf{v}^{2,\epsilon}) = \int_{\Omega_2} \mu\, \partial_z \mathbf{v}^{2,\epsilon} \cdot \partial_z \mathbf{v}^{2,\epsilon}. \qquad (93)$$

From here, the same reasoning presented in Section 5 in [14] applies.

The previous observations, show that the present work entirely recovers the weak convergence results analogous to those presented in [14], but extending them to a considerable broader scenario. On the other hand, the strong convergence properties in [14] could not be generalized, and they should be treated on a case-wise basis, using particular features of the function Z, as it was done in the equality (93) above.

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Notes

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