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Inequalities for *D#* Synchronous Functions and Related Functionals

Artículo Original

Inequalities for $D-$ Synchronous Functions and Related Functionals

Desigualdades para funciones $D-$ sincrónicas y funciones relacionadas

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Abstract:

We introduce in this paper the concept of quadruple D−synchronous functions which generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and we also provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.

Keywords: Synchronous Functions, Lipschitzian functions, Chebyshev inequality, Cauchy-Bunyakovsky-Schwarz inequality.

Resumen:

Introducimos en este artículo el concepto de funciones D−sincrónicas cuádruples, que generaliza el concepto de un par de funciones sincrónicas; estableceremos una desigualdad similar a la desigualdad de Chebyshev y también presentamos algunas desigualdades de tipo Cauchy-Bunyakovsky-Schwarz para un funcional asociado con este cuádruple. Se dan algunas aplicaciones para funciones univariadas de la variable real. También se indican desigualdades discretas.

Palabras clave: Funciones D−sincrónicas, funciones Lipschitzianas, desigualdad de Chebyshev, desigualdad de Cauchy-Bunyakovsky-Schwarz.

1. Introduction

Let (Ω, A, ν) be a measurable space consisting of a set Ω, a σ-algebra A of subsets of Ω and a countably additive and *positive measure* ν on A with values in $[0, +\infty]$. For a ν-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x) |f(x)| d\nu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\nu$ instead of $\int_\Omega w(x) d\nu(x)$. Assume also that $\int_\Omega wd\nu = 1$.

We say that the pair of measurable functions (f, g) are *synchronous* on Ω if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{1}$$

for ν-a.e. $x, y \in \Omega$. If the inequality reverses in (1), the functions are called *asynchronous* on Ω.

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If (f, g) are synchronous on Ω and $f, g, fg \in L_w(\Omega, \nu)$, then the following inequality, that is known in the literature as *Chebyshev's Inequality*, holds:

$$\int_\Omega wfgd\nu \geq \int_\Omega wfd\nu \int_\Omega wgd\nu, \tag{2}$$

where $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$ and $\int_\Omega wd\nu = 1$.

If $f, g : \Omega \to \mathbb{R}$ are ν-measurable functions and $f, g, fg \in L_w(\Omega, \nu)$, then we may consider the *Chebyshev functional*

$$T_w(f, g) := \int_\Omega wfgd\nu - \int_\Omega wfd\nu \int_\Omega wgd\nu.$$

The following result is known in the literature as the *Grüss inequality*:

$$|T_w(f, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{3}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \qquad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{4}$$

for ν-a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If $f \in L_w(\Omega, \nu)$, then we may define

$$D_w(f) := \int_\Omega w(x) \left| f(x) - \int_\Omega w(y) f(y) d\nu(y) \right| d\nu(x). \tag{5}$$

The following refinement of Grüss inequality in the general setting of measure spaces is due to Cerone & Dragomir [1]:

Theorem 1.1. *Let $w, f, g : \Omega \to \mathbb{R}$ be ν-measurable functions with $w \geq 0$ ν-a.e. on Ω and $\int_\Omega wd\nu = 1$. If $f, g, fg \in L_w(\Omega, \nu)$ and there exist constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad for \quad \nu\text{-a.e. } x \in \Omega, \tag{6}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(\Delta-\delta) D_w(f). \tag{7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Motivated by the above results, we introduce in this paper the concept of quadruple $D-$synchronous functions that generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and also provide some CauchyBunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.

2. D–Synchronous functions

Let (Ω, A, ν) be a measurable space and $f, g, h, \ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω.

Definition 2.1. The quadruple (f, g, h, ℓ) is called $D-$Synchronous ($D-$Asynchronous) on Ω if

$$\det\begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det\begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \geq (\leq) 0 \tag{8}$$

for ν-a.e. (almost every) $x, y \in \Omega$.

This concept is a generalization of synchronous functions, since for $g = 1$, $\ell = 1$ the quadruple (f, g, h, ℓ) is $D-$Synchronous if, and only if, (f, h) is synchronous on Ω.

If $g, \ell \neq 0$ ν-a.e on Ω, then

$$\begin{aligned} &\det\begin{pmatrix} f(x) & f(y) \\ g(x) & g(y) \end{pmatrix} \det\begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y) \end{pmatrix} \\ &= (f(x)g(y) - g(x)f(y))(h(x)\ell(y) - \ell(x)h(y)) \\ &= g(x)\ell(x)g(y)\ell(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)\left(\frac{h(x)}{\ell(x)} - \frac{h(y)}{\ell(y)}\right) \end{aligned} \tag{9}$$

for ν-a.e. $x, y \in \Omega$. So, if $g\ell > 0$ ν-a.e on Ω the quadruple (f, g, h, ℓ) is $D-$Synchronous if, and only if, $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ is synchronous on Ω.

Theorem 2.2. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f, g, h, ℓ) is D-Synchronous ($D-$Asynchronous), $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and $fh, g\ell, gh, f\ell \in L_w(\Omega, \nu)$. Then,*

$$\det\begin{pmatrix} \int_\Omega wfh d\nu & \int_\Omega wgh d\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu \end{pmatrix} \geq (\leq) 0. \tag{10}$$

Proof. Since the quadruple (f, g, h, ℓ) is $D-$Synchronous, then

$$\begin{aligned} 0 \leq & \left(f(x) g(y) - g(x) f(y)\right)\left(h(x) \ell(y) - \ell(x) h(y)\right) \\ = & f(x) h(x) g(y) \ell(y) + g(x) \ell(x) f(y) h(y) \\ & - f(x) \ell(x) g(y) h(y) - g(x) h(x) f(y) \ell(y) \end{aligned} \tag{11}$$

for ν-a.e. $x, y \in \Omega$.
This is equivalent to

$$\begin{aligned} & f(x) h(x) g(y) \ell(y) + g(x) \ell(x) f(y) h(y) \\ & \qquad \geq f(x) \ell(x) g(y) h(y) + g(x) h(x) f(y) \ell(y) \end{aligned} \tag{12}$$

for ν-a.e. $x, y \in \Omega$.
Multiply (12) by $w(x)\, w(y) \geq 0$ to get

$$\begin{aligned} & w(x) f(x) h(x) w(y) g(y) \ell(y) + w(x) g(x) \ell(x) w(y) f(y) h(y) \\ & \qquad \geq w(x) f(x) \ell(x) w(y) g(y) h(y) + w(x) g(x) h(x) w(y) f(y) \ell(y) \end{aligned} \tag{13}$$

for ν-a.e. $x, y \in \Omega$.
If we integrate the inequality (13) over $x \in \Omega$, then we get

$$\begin{aligned} & w(y) g(y) \ell(y) \int_\Omega w f h d\nu + w(y) f(y) h(y) \int_\Omega w g \ell d\nu \\ & \qquad \geq w(y) g(y) h(y) \int_\Omega w f \ell d\nu + w(y) f(y) \ell(y) \int_\Omega w g h d\nu \end{aligned} \tag{14}$$

for ν-a.e. $y \in \Omega$.
Finally, if we integrate the inequality (14) over y $\in \Omega$, then we get

$$\begin{aligned} & \int_\Omega w f h d\nu \int_\Omega w g \ell d\nu + \int_\Omega w g \ell d\nu \int_\Omega w f h d\nu \\ & \qquad \geq \int_\Omega w f \ell d\nu \int_\Omega w g h d\nu + \int_\Omega w g h d\nu \int_\Omega w f \ell d\nu, \end{aligned}$$

which is equivalent to the desired result (10).

Corollary 2.3. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and such that $g\ell > 0$ ν-a.e on Ω, $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ is synchronous (asynchronous) on Ω, $w \geq 0$ a.e. on Ω with* $\int_\Omega$ *$w d\nu = 1$* and fh, *$g\ell, gh, f\ell \in L_w(\Omega, \nu)$; then the inequality (10) is valid.*

Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω , $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and $fh, g\ell, gh, f\ell \in L_w(\Omega, \nu)$; then we can consider the functionals

$$\mathcal{D}(f,g,h,\ell;w,\Omega) := \det\begin{pmatrix} \int_\Omega wfhd\nu & \int_\Omega wghd\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu \end{pmatrix} = \int_\Omega wfhd\nu \int_\Omega wg\ell d\nu - \int_\Omega wf\ell d\nu \int_\Omega wghd\nu, \tag{15}$$

and, for $(f, g) = (h, \ell)$,

$$\mathcal{D}(f,g;w,\Omega) := \mathcal{D}(f,g,f,g;w,\Omega) = \det\begin{pmatrix} \int_\Omega wf^2 d\nu & \int_\Omega wfgd\nu \\ \int_\Omega wfgd\nu & \int_\Omega wg^2 d\nu \end{pmatrix} = \int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfgd\nu\right)^2, \tag{16}$$

provided $f^2, g^2 \in L_w(\Omega, \nu)$.

We can improve the inequality (10) as follows:

Theorem 2.4. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f, g, h, ℓ) is D−Synchronous, $w \geq 0$ a.e. on Ω with $\int_\Omega wd\nu = 1$ and $fh, g\ell, gh, f\ell \in L_w(\Omega, \nu)$; then,*

$$\mathcal{D}(f,g,h,\ell;w,\Omega) \geq \max\{|\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|, |\mathcal{D}(f,g,|h|,|\ell|;w,\Omega)|, |\mathcal{D}(|f|,|g|,|h|,|\ell|;w,\Omega)|\} \geq 0. \tag{17}$$

Proof. We use the continuity property of the modulus, namely

$$|a-b| \geq ||a|-|b||,\ a,b \in \mathbb{R}.$$

Since (f, g, h, ℓ) is D−Synchronous, then

$$(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y)) = |f(x)g(y)-g(x)f(y)||h(x)\ell(y)-\ell(x)h(y)| \geq \begin{cases} |(|f(x)||g(y)|-|g(x)||f(y)|)(h(x)\ell(y)-\ell(x)h(y))| \\ |(f(x)g(y)-g(x)f(y))(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \\ |(|f(x)||g(y)|-|g(x)||f(y)|)(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \end{cases} \tag{18}$$

for ν-a.e. $x, y \in \Omega$.
As in the proof of Theorem 2.2, we have the identity

$$\mathcal{D}(f,g,h,\ell;w,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y)) \times (h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y). \tag{19}$$

By using the identity (19) and the first branch in (18) we have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &\geq \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(|f(x)|\,|g(y)| - |g(x)|\,|f(y)|) \\ &\qquad \times (h(x)\,\ell(y) - \ell(x)\,h(y))|\,d\nu(x)\,d\nu(y) \\ &\geq \frac{1}{2}\left|\int_\Omega\int_\Omega w(x)\,w(y)\,(|f(x)|\,|g(y)| - |g(x)|\,|f(y)|)\right. \\ &\qquad \left. \times (h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right| \\ &= |\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|,\end{aligned}$$

which proves the first part of (17).
The second and third part of (17) can be proved in a similar way and details are omitted.

3. Further results for the functional D

We have the following Schwarz's type inequality for the functional D:

Theorem 3.1. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and $f^2, g^2, h^2, \ell^2 \in L_w(\Omega, \nu)$. Then,*

$$\mathcal{D}^2(f,g,h,\ell;w,\Omega) \leq \mathcal{D}(f,g;w,\Omega)\,\mathcal{D}(h,\ell;w,\Omega). \tag{20}$$

Proof. As in the proof of Theorem 2.4, we have the identities

$$\mathcal{D}(f,g,h,\ell;w,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y)) \times (h(x)\,\ell(y) - \ell(x)\,h(y))\,d\nu(x)\,d\nu(y),$$

$$\mathcal{D}(f,g;w,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y) - g(x)\,f(y))^2\,d\nu(x)\,d\nu(y)$$

and

$$\mathcal{D}(h,\ell;w,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,\ell(y) - \ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y).$$

By the Cauchy-Bunyakovsky-Schwarz double integral inequality we have

$$\left(\int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))\,(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right)^{2}$$
$$\leq \int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(h(x)\,g(y)-g(x)\,h(y))^{2}\,d\nu(x)\,d\nu(y)$$
$$\times \int_{\Omega}\int_{\Omega} w(x)\,w(y)\,(h(x)\,\ell(y)-\ell(x)\,h(y))^{2}\,d\nu(x)\,d\nu(y),$$

which produces the desired result (20).

Corollary 3.2. *Let $f, g, h, \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω with $g^2, \ell^2 \in L_w(\Omega,\nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$, and $a, A, b, B \in \mathbb{R}$ such that $A > a$, $B > b$,*

$$ag \leq f \leq Ag \quad and \quad b\ell \leq h \leq B\ell \tag{21}$$

ν-a.e. on Ω. Then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{4}(A-a)(B-b)\int_{\Omega} wg^{2}d\nu \int_{\Omega} w\ell^{2}d\nu. \tag{22}$$

Proof. In [2] (see also [4, p. 8]) we proved the following reverse of Cauchy-BunyakovskySchwarz integral inequality

$$\int_{\Omega} wf^{2}d\nu \int_{\Omega} wg^{2}d\nu - \left(\int_{\Omega} wfgd\nu\right)^{2} \leq \frac{1}{4}(A-a)^{2}\left(\int_{\Omega} wg^{2}d\nu\right)^{2}$$

provided that $ag \leq f \leq Ag$ ν-a.e. on Ω and $g^2 \in L_w(\Omega,\nu)$.

Since, we also have

$$\int_{\Omega} wh^{2}d\nu \int_{\Omega} w\ell^{2}d\nu - \left(\int_{\Omega} wh\ell d\nu\right)^{2} \leq \frac{1}{4}(B-b)^{2}\left(\int_{\Omega} w\ell^{2}d\nu\right)^{2},$$

provided that $b\ell \leq h \leq B\ell$ ν-a.e. on Ω and $\ell^2 \in L_w(\Omega,\nu)$. Then, by (20) we have

$$\mathcal{D}^{2}(f,g,h,\ell;w,\Omega) \leq \frac{1}{16}(A-a)^{2}(B-b)^{2}\left(\int_{\Omega} wg^{2}d\nu\right)^{2}\left(\int_{\Omega} w\ell^{2}d\nu\right)^{2}$$

that is equivalent to the desired result (22).

For positive margins we also have:

Corollary 3.3. *Let $f, g, h, \ell:\Omega\to\mathbb{R}$ be four ν-measurable functions on Ω with $g^2, \ell^2 \in L_w(\Omega,\nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$, and $a, A, b, B > 0$ such that $A > a$, $B > b$,*

$$ag \leq f \leq Ag \ and \ b\ell \leq h \leq B\ell \tag{23}$$

ν-a.e. on Ω. Then we have

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\int_\Omega wfgd\nu\int_\Omega wh\ell d\nu. \tag{24}$$

Proof. In [3] (see also [4, p. 16]) we proved the following reverse of Cauchy-BunyakovskySchwarz integral inequality:

$$\int_\Omega wf^2d\nu\int_\Omega wg^2d\nu - \left(\int_\Omega wfgd\nu\right)^2 \leq \frac{(A-a)^2}{4aA}\left(\int_\Omega wfgd\nu\right)^2,$$

whenever $ag \leq f \leq Ag$ ν-a.e. on Ω.

Since

$$\int_\Omega wh^2d\nu\int_\Omega w\ell^2d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{(B-b)^2}{4bB}\left(\int_\Omega wh\ell d\nu\right)^2,$$

provided $b\ell \leq h \leq B\ell$ ν-a.e. on Ω, then by (20) we get the desired result (24).

If bounds for the sum and difference are available, then we have:

Corollary 3.4. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω with $g^2, \ell^2 \in L_w(\Omega,\nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega wd\nu = 1$. Assume that there exists the constants P_1, Q_1, P_2, Q_2 such that*

$$|g-f| \leq P_1, \quad |g+f| \leq Q_1, \quad |h-\ell| \leq P_2, \quad |h+\ell| \leq Q_2 \tag{25}$$

a.e. on Ω; then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{4}P_1Q_1P_2Q_2. \tag{26}$$

Proof. In the recent paper [5] we obtained amongst other the following reverse of CauchyBunyakovsky-Schwarz integral inequality:

$$\int_\Omega wf^2d\nu\int_\Omega wg^2d\nu - \left(\int_\Omega wfgd\nu\right)^2 \leq \frac{1}{4}P_1^2Q_1^2,$$

provided $|g-f| \leq P_1$, $|g+f| \leq Q_1$ a.e. on Ω.

Since

$$\int_\Omega wh^2d\nu\int_\Omega w\ell^2d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{1}{4}P_2^2Q_2^2,$$

if $|h-\ell| \leq P_2$, $|h+\ell| \leq Q_2$ a.e. on Ω, then by (20) we get the desired result (26).

If bounds for each function are available, then we have:

Corollary 3.5. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants ai, A_i, b_i and B_i with $i \in \{1,2\}$ such that*

$$0 < a_1 \leq f \leq A_1 < \infty, \qquad 0 < a_2 \leq g \leq A_2 < \infty, \tag{27}$$

and

$$0 < b_1 \leq h \leq B_1 < \infty, \qquad 0 < b_2 \leq \ell \leq B_2 < \infty, \tag{28}$$

a.e. on Ω; then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{3}(A_1A_2 - a_1a_2)(B_1B_2 - b_1b_2). \tag{29}$$

Proof. We use the following Ozeki's type inequality obtained in [6]:

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \leq \frac{1}{3}(A_1A_2 - a_1a_2)^2,$$

provided $0 < a_1 \leq f \leq A_1 < \infty$, $0 < a_2 \leq g \leq A_2 < \infty$ a.e. on Ω.

Since

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{1}{3}(B_1B_2 - b_1b_2)^2,$$

when $0 < b_1 \leq h \leq B_1 < \infty$, $0 < b_2 \leq \ell \leq B_2 < \infty$ a.e. on Ω, then by (20) we get the desired result (29).

4. Results for univariate functions

Let $\Omega = [a,b]$ be an interval of real numbers, and assume that $f, g, h, \ell : [a,b] \to \mathbb{R}$ are measurable D −Synchronous (D−Aynchronous), $w \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$ and $fh, g\ell, gh, f\ell \in L_w([a,b])$; then,

$$\begin{aligned}\int_a^b w(t) f(t) h(t)\,dt \int_a^b w(t) g(t) \ell(t)\,dt \\ \geq (\leq) \int_a^b w(t) g(t) h(t)\,dt \int_a^b w(t) f(t) \ell(t)\,dt.\end{aligned} \tag{30}$$

Now, assume that $[a, b]$ # $(0, \infty)$ and take $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$ with $p, q, r, s \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = t^{p-q} \quad \text{and} \quad \frac{h(t)}{\ell(t)} = t^{r-s}.$$

If $(p-q)(r-s) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a, b]$ while if $(p-q)(r-s) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a, b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$ that

$$\int_a^b w(t)\, t^{p+r} dt \int_a^b w(t)\, t^{q+s} dt \geq (\leq) \int_a^b w(t)\, t^{q+r} dt \int_a^b w(t)\, t^{p+s} dt, \tag{31}$$

provided $(p-q)(r-s) > (<)\ 0$.

Assume that $[a, b]$ # $(0, \infty)$ and take $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$, $h(t) = \exp(\gamma t)$ and $\ell(t) = \exp(\delta t)$, with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = \exp[(\alpha-\beta)t] \quad \text{and} \quad \frac{h(t)}{\ell(t)} = \exp[(\gamma-\delta)t].$$

If $(\alpha-\beta)(\gamma-\delta) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a, b]$, while if $(\alpha-\beta)(\gamma-\delta) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a, b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$ that

$$\begin{aligned}&\int_a^b w(t) \exp[(\alpha+\gamma)t]\, dt \int_a^b w(t) \exp[(\beta+\delta)t]\, dt \\ &\qquad \geq (\leq) \int_a^b w(t) \exp[(\beta+\gamma)t]\, dt \int_a^b w(t) \exp[(\alpha+\delta)t]\, dt,\end{aligned} \tag{32}$$

provided $(\alpha-\beta)(\gamma-\delta) > (<)\ 0$.

Consider the functional

$$\begin{aligned}\mathcal{D}_{p,q,r,s}(w) := &\int_a^b w(t)\, t^{p+r} dt \int_a^b w(t)\, t^{q+s} dt \\ &- \int_a^b w(t)\, t^{q+r} dt \int_a^b w(t)\, t^{p+s} dt,\end{aligned} \tag{33}$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, and $p, q, r, s \in \mathbb{R}$.

We observe that for $t \in [a, b]$ # $(0, \infty)$ we have

$$k_{p,q}(a,b) := \begin{cases} a^{p-q}, \text{ if } p \geq q, \\ b^{p-q}, \text{ if } p < q, \end{cases} \leq \frac{f(t)}{g(t)} = t^{p-q}$$
$$\leq K_{p,q}(a,b) := \begin{cases} b^{p-q}, \text{ if } p \geq q, \\ a^{p-q} \text{ if } p < q, \end{cases} \tag{34}$$

and, similarly,

$$k_{r,s}(a,b) \leq \frac{h(t)}{\ell(t)} = t^{r-s} \leq K_{r,s}(a,b).$$

Using the inequality (22) we have

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{4}[K_{p,q}(a,b) - k_{p,q}(a,b)][K_{r,s}(a,b) - k_{r,s}(a,b)]$$
$$\times \int_a^b w(t)\, t^{2q} dt \int_a^b w(t)\, t^{2s} dt, \tag{35}$$

while from (24) we have

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{4}\frac{[K_{p,q}(a,b) - k_{p,q}(a,b)][K_{r,s}(a,b) - k_{r,s}(a,b)]}{\sqrt{k_{p,q}(a,b)\, k_{r,s}(a,b)\, K_{p,q}(a,b)\, K_{r,s}(a,b)}}$$
$$\times \int_a^b w(t)\, t^{p+q} dt \int_a^b w(t)\, t^{r+s} dt. \tag{36}$$

We also have for $t \in [a,b]$ # $(0,\infty)$ that

$$u_p(a,b) := \begin{cases} a^p, \text{ if } p \geq 0, \\ b^p, \text{ if } p < 0, \end{cases} \leq f(t) = t^p$$
$$\leq U_p(a,b) := \begin{cases} b^p, \text{ if } p \geq 0, \\ a^p, \text{ if } p < 0, \end{cases}$$

and the corresponding bounds for $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$, with $p, q, r, s \in \mathbb{R}$. Making use of the inequality (29) we get

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{3}\left(U_p(a,b)\,U_q(a,b) - u_p(a,b)\,u_q(a,b)\right) \times \left(U_r(a,b)\,U_s(a,b) - u_r(a,b)\,u_s(a,b)\right). \tag{37}$$

Similar results may be stated for the functional

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(w) := \int_a^b w(t)\exp[(\alpha+\gamma)t]\,dt \int_a^b w(t)\exp[(\beta+\delta)t]\,dt - \int_a^b w(t)\exp[(\beta+\gamma)t]\,dt \int_a^b w(t)\exp[(\alpha+\delta)t]\,dt$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $[a, b]$ # $(0, \infty)$. Details are omitted.

We say that the function $\varphi : [a, b] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ if

$$|\varphi(t) - \varphi(s)| \leq L\,|t - s|$$

for any $t, s \in [a, b]$.

Define the functional

$$\mathcal{D}(f, g, h, \ell; w, [a,b]) := \int_a^b w(t)\,f(t)\,h(t)\,dt \int_a^b w(t)\,g(t)\,\ell(t)\,dt - \int_a^b w(t)\,g(t)\,h(t)\,dt \int_a^b w(t)\,f(t)\,\ell(t)\,dt.$$

In the next result we provided two upper bounds in terms of Lipschitzian constants:

Theorem 4.1. *Let $f, g, h, \ell : [a, b] \to \mathbb{R}$ be measurable functions and $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t)\,dt = 1$.*

(i) If $g(t), \ell(t) \neq 0$ for any $t \in [a, b]$, and $\frac{f}{g}$ is Lipschitzian with the constant $L > 0$, and is Lipschitzian with the constant $K_{\frac{h}{\ell}} > 0$, and $g\ell, g\ell e^2 \in L_w([a, b])$ with $e(t) = t$, $t \in [a, b]$, then

$$|\mathcal{D}(f, g, h, \ell; w, [a,b])| \leq LK\left[\int_a^b w(s)\,|g(s)|\,|\ell(s)|\,ds \int_a^b w(t)\,|\ell(t)|\,|g(t)|\,t^2 dt - \left(\int_a^b w(t)\,|g(t)|\,|\ell(t)|\,t dt\right)^2\right]. \tag{38}$$

(ii) If, in addition, we have $wg\ell \in L_\infty[a, b]$ and

$$\|wg\ell\|_\infty = \text{esssup}_{t\in[a,b]} |w(t)\,g(t)\,\ell(t)| < \infty,$$

then

$$|\mathcal{D}(f,g,h,\ell;w,[a,b])| \le \frac{1}{12}(b-a)^4 LK \|wg\ell\|_\infty^2 . \tag{39}$$

Proof. We have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) &= \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(f(t)\,g(s)-g(t)\,f(s))\\ &\quad\times (h(t)\,\ell(s)-\ell(t)\,h(s))\,dtds\\ &= \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,g(t)\,g(s)\,\ell(t)\,\ell(s)\\ &\quad\times\left(\frac{f(t)}{g(t)}-\frac{f(s)}{g(s)}\right)\left(\frac{h(t)}{\ell(t)}-\frac{h(s)}{\ell(s)}\right)dtds.\end{aligned}$$

By taking modulus in this equality, we get

$$\begin{aligned}&|\mathcal{D}(f,g,h,\ell;w,[a,b])|\\ &\quad\le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\times\left|\frac{f(t)}{g(t)}-\frac{f(s)}{g(s)}\right|\left|\frac{h(t)}{\ell(t)}-\frac{h(s)}{\ell(s)}\right|dt\,ds\\ &\quad\le \frac{1}{2}LK\int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\,(t-s)^2\,dt\,ds.\end{aligned} \tag{40}$$

Now, observe that

$$\begin{aligned}&\int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\,(t-s)^2\,dtds\\ &\qquad= \int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\,(t^2-2ts+s^2)\,dtds\\ &\qquad= 2\left(\int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\,t^2dtds\right.\\ &\qquad\quad\left.-\int_a^b\int_a^b w(t)\,w(s)\,|g(t)|\,|g(s)|\,|\ell(t)|\,|\ell(s)|\,tsdtds\right)\\ &\qquad= 2\left[\int_a^b w(s)\,|g(s)|\,|\ell(s)|\,ds\int_a^b w(t)\,|g(t)|\,|\ell(t)|\,t^2dt\right.\\ &\qquad\quad\left.-\left(\int_a^b w(t)\,|g(t)|\,|\ell(t)|\,tdt\right)^2\right].\end{aligned}$$

(41)

On making use of (40) and (41) we get the desired result (38).

If $w g \ell \in L_{\infty}[a, b]$, then

$$\int_a^b \int_a^b w(t) w(s) |g(t)| |g(s)| |\ell(t)| |\ell(s)| (t-s)^2 dtds \leq \|wg\ell\|_{\infty}^2 \int_a^b \int_a^b (t-s)^2 dtds = \frac{1}{6}(b-a)^4 \|wg\ell\|_{\infty}^2 . \tag{42}$$

Therefore, by inequalities (40) and (42) we obtain the desired result (39).

5. Discrete inequalities

Consider the n-tuples of real numbers $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $z = (z_1, ..., z_n)$ and $u = (u_1, ..., u_n)$. We say that the quadruple (x, y, z, u) is $D-$*Synchronous* if

$$0 \leq \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \det \begin{pmatrix} z_i & z_j \\ u_i & u_j \end{pmatrix} = (x_i y_j - x_j y_i)(z_i u_j - z_j u_i) \tag{43}$$

for any $i, j \in \{1, ..., n\}$.

If $p = (p_1, ..., p_n)$ is a probability distribution, namely, $p_i \geq 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$, and the quadruple (x, y, z, u) is $D-$Synchronous, then by (10) we have:

$$\mathcal{D}_n(x, y, z, u; p) := \det \begin{pmatrix} \sum_{i=1}^n p_i x_i z_i & \sum_{i=1}^n p_i y_i z_i \\ \sum_{i=1}^n p_i x_i u_i & \sum_{i=1}^n p_i y_i u_i \end{pmatrix} = \sum_{i=1}^n p_i x_i z_i \sum_{i=1}^n p_i y_i u_i - \sum_{i=1}^n p_i x_i u_i \sum_{i=1}^n p_i y_i z_i \geq 0. \tag{44}$$

For an n-tuples of real numbers $x = (x_1, ..., x_n)$, we denote by $|x| := (|x_1|, ..., |x_n|)$. On making use of the inequality (17), then for any $D-$Synchronous quadruple (x, y, z, u) and for any probability distribution $p = (p_1, ..., p_n)$ we have

$$\mathcal{D}_n(x, y, z, u; p) \geq \max \{ |\mathcal{D}_n(|x|, y, z, u; p)|, |\mathcal{D}_n(x, |y|, z, u; p)|, |\mathcal{D}_n(|x|, |y|, z, u; p)| \} \geq 0. \tag{45}$$

Observe that if we consider

$$\mathcal{D}_n(x,y;p) := \mathcal{D}_n(x,y,x,y;p) = \sum_{i=1}^n p_i x_i^2 \sum_{i=1}^n p_i y_i^2 - \left(\sum_{i=1}^n p_i x_i y_i\right)^2,$$

then by (20) we have

$$|\mathcal{D}_n(x,y,z,u;p)|^2 \le \mathcal{D}_n(x,y;p)\,\mathcal{D}_n(z,u;p) \tag{46}$$

for any quadruple (x, y, z, u) and any probability distribution $p = (p_1, ..., p_n)$.
If $a, A, b, B \in \mathbb{R}$ and (x, y, z, u) are such that $A > a$, $B > b$,

$$ay_i \le x_i \le Ay_i \text{ and } bu_i \le z_i \le Bu_i \tag{47}$$

for any $i \in \{1, ..., n\}$, then by (22) we have

$$|\mathcal{D}_n(x,y,z,u;p)| \le \frac{1}{4}(A-a)(B-b)\sum_{i=1}^n p_i y_i^2 \sum_{i=1}^n p_i u_i^2. \tag{48}$$

If a, A, b, B > 0 and condition (47) is valid, then by (24) we have

$$|\mathcal{D}_n(x,y,z,u;p)| \le \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\sum_{i=1}^n p_i x_i y_i \sum_{i=1}^n p_i z_i u_i. \tag{49}$$

Now, if we use the *Klamkin-McLenaghan's inequality*

$$\sum_{i=1}^n p_i x_i^2 \sum_{i=1}^n p_i y_i^2 - \left(\sum_{i=1}^n p_i x_i y_i\right)^2 \le \left(\sqrt{A}-\sqrt{a}\right)^2 \sum_{i=1}^n p_i x_i y_i \sum_{i=1}^n p_i x_i^2$$

that holds for x, y satisfying the condition (47) with $A, a > 0$, then by (46) we get

$$\begin{aligned} &|\mathcal{D}_n(x,y,z,u;p)| \\ &\quad \le \left(\sqrt{A}-\sqrt{a}\right)\left(\sqrt{B}-\sqrt{b}\right) \\ &\qquad \times \left(\sum_{i=1}^n p_i x_i y_i\right)^{1/2}\left(\sum_{i=1}^n p_i x_i^2\right)^{1/2}\left(\sum_{i=1}^n p_i z_i u_i\right)^{1/2}\left(\sum_{i=1}^n p_i z_i^2\right)^{1/2}, \end{aligned}$$

(50)

provided (x, y, z, u) satisfy (47) with a, $A, b, B > 0$.
Now, assume that

$$0 < a_1 \le x_i \le A_1 < \infty, \qquad 0 < a_2 \le y_i \le A_2 < \infty, \tag{51}$$

and

$$0 < b_1 \le x_i \le B_1 < \infty, \qquad 0 < b_2 \le u_i \le B_2 < \infty, \tag{52}$$

for any $i \in \{1, ..., n\}$; then by (29) we get

$$|\mathcal{D}_n(x, y, z, u; p)| \le \frac{1}{3}(A_1A_2 - a_1a_2)(B_1B_2 - b_1b_2), \tag{53}$$

for any probability distribution $p = (p_1, ..., p_n)$.

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References

1 Cerone P. and Dragomir S.S., "A refinement of the Grüss inequality and applications", *TamkangJ. Math*. 38 (2007), No. 1, 37-49. doi: 10.5556/j.tkjm.38.2007.92.

2 Dragomir S.S., "A counterpart of Schwarz's inequality in inner product spaces". arXiv: math/0305373.

3 Dragomir S.S., "A generalization of the Cassels and Greub-Reinboldt inequalities in inner product spaces". arXiv: math/0306352.

4 Dragomir S.S., *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers Inc, New York, 2005.

5 Dragomir S.S., "Reverses of Schwarz inequality in inner product spaces with applications", Math. Nachr. 288 (2015), No. 7, 730-742. doi: 10.1002/mana.201300100.

6 Izumino S., Mori H. and Seo Y., "On Ozeki's inequality", *J. Inequal. Appl.* 2 (1998), No. 3, 235-253. doi: 10.1155/S1025583498000149.

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