Revista Latinoamericana de Investigación en Matemática Educativa
ISSN: 1665-2436
ISSN: 2007-6819
relime@clame.org.mx
Comité Latinoamericano de Matemática Educativa
Organismo Internacional

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Revista Latinoamericana de Investigación en Matemática Educativa, vol. 17, 4-1, 2014
Comité Latinoamericano de Matemática Educativa, Organismo Internacional
Disponible en: https://www.redalyc.org/articulo.oa?id=33553644007
DOI: https://doi.org/10.12802/relime.13.1747

Estructura y niveles de habilidad en estudiantes de escuela primaria sobre geometría transformacional

Primary school students' structure and levels of abilities in transformational geometry

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DOI: https://doi.org/10.12802/relime.13.1747
Redalyc: https://www.redalyc.org/articulo.oa?id=33553644007

Recepción: 14 Enero 2013
Aprobación: 02 Septiembre 2013

Resumen:

Este trabajo utilizó análisis factorial confirmatorio para investigar los factores y la estructura de la habilidad para los conceptos de geometría transformacional. Los resultados sugieren que las tres transformaciones geométricas (traslación, reflexión, rotación) consisten de cuatro factores y tienen estructuras similares. Se utilizó el análisis de RASCH para crear una escala de los ítems de factores, la cual se interpretó a la luz del marco teórico del espacio de trabajo geométrico. Se identificaron cinco niveles de habilidades de visualización en la geometría transformacional. Este trabajo sugiere que el desarrollo de la comprensión en la geometría transformacional puede explicarse con base en el proceso de visualización del espacio de trabajo geométrico personal de los estudiantes.

Palabras clave: Geometría transformacional, Espacio de Trabajo Geométrico, Visualización, Deconstrucción dimensional.

Abstract:

This paper used CFA analyses to investigate the factors and structure of transformational geometry concepts ability. The results suggest that the three geometric transformations (translation, reflection and rotation) consist of four factors and have similar structures. RASCH analysis was used to create a scale of the factor items, which was interpreted in light of the theoretical framework of geometrical working space. Five levels of visualization abilities in transformational geometry were identified. This paper suggests that the development of understanding in transformational geometry can be explained based on the visualization process of the students' personal geometrical working space.

Keywords: Transformational geometry, Geometrical working space, Visualization, Dimensional deconstruction.

Resumo:

Este trabalho utilizou análise fatorial confirmatória para pesquisar os fatores e a estrutura da habilidade para os conceitos de geometria transformacional. Os resultados sugerem que as três transformações geométricas (translação, reflexão e rotação) consistem em quatro fatores e têm estruturas similares. Foi utilizada a análise de RASCH para criar uma escala dos itens de fatores, à qual foi interpretada à luz do modelo teórico do espaço de trabalho geométrico. Foram identificados cinco níveis de habilidades de visualização na geometria transformacional. Este trabalho sugere que o desenvolvimento da compreensão na geometria transformacional pode ser explicado com base no processo de visualização do espaço de trabalho geométrico pessoal dos estudantes.

Palavras-chave: Geometria transformacional, Espaço de Trabalho Geométrico, Visualização, Desconstrução dimensional.

Résumé:

Cet article utilise l'analyse factorielle confirmatoire pour étudier les facteurs et la structure de l'habilité des concepts de géométrie des transformations. Les résultats suggèrent que les trois transformations géométriques (translation, réflexion, rotation) sont constituées de quatre facteurs et ont des structures similaires. L'analyse de RASCH a été utilisée pour créer une échelle des composants de facteur, qui a été interprétée à la lumière du cadre théorique de l'Espace de Travail Géométrique. Cinq niveaux d'habilités de visualisation de la géométrie des transformations ont été identifiés. Cet article suggère que le développement de la compréhension de la géométrie des transformations peut être expliqué sur la base du processus de visualisation de l'espace de travail géométrique personnel des élèves.

Mots clés: Géométrie des transformations, Espace de Travail Géométrique, Visualisation, Déconstruction dimensionnelle.

1. Introduction

Transformational geometry (TG) refers to a mental or physical transformation of shapes. According to NCTM's *Principals and Standards for School Mathematics* (2002), "Instructional programs from kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" (p.41).

Research in TG during the last few years has focused on the development of knowledge and understanding of transformations (Yanik & Flores, 2009) and various theoretical frameworks have been used (Hollebrands, 2003; Molina, 1990; Soon, 1989). Kidder (1976) suggests that performing geometric transformations is a multi - faceted mental operation. However, the components that synthesize this ability appear not to have been clearly defined. This seems to be critical in order to study the development of knowledge and understanding of TG.

Based on the pilot results of a large scale project investigating the ability in TG, this paper aims to investigate the structure and development of the primary school students' ability in TG concepts. Hence, its aim is twofold: 1) to investigate the components that synthesize primary school students' TG ability, drawing on the findings of previous studies, and 2) to describe primary school students' levels of abilities in TG, drawing on theoretical frameworks for understanding geometry and interpreting figures. Specifically, we draw on the notions of geometric work space (GWS) (Kuzniak, 2006), and visualization process (Duval, 2005).

2. Literature Review

2.1. The development of knowledge and understanding of transformational geometry

One of the first debates in TG research was the development of learning TG concepts. The first studies focused on the order of learning translation, reflection and rotation (Moyer, 1978). Schultz and Austin (1983) suggest that there cannot be a specific order for understanding the three geometric transformations, since some configurations can influence the relative difficulty of rotations and reflections, such as the direction of the transformation. However, these studies used only one type of task, that of performing a transformation.

The first attempts that used a variety of tasks to study the development of TG ability were based on the van Hiele levels of geometric understanding (Molina, 1990; Soon, 1989). These studies used different tasks matched to each level, such as: performing, recognizing, understanding and relating the properties of transformations. However, the components that synthesize this ability have not been confirmed in literature. Moreover, these studies focused on the type of task, ignoring previous findings regarding the order of difficulty in learning transformations and the configurations that influence difficulty.

Edwards (2003) proposed another model which discriminates between two qualitatively different conceptions of geometric transformations: 1) motion, where the plane is conceived as a background and geometric figures are manipulated on top of it, and 2) mapping, where a transformation can be considered as a special function that maps all points in the plane to other points while preserving some properties and changing others.

Taking into consideration the various types of tasks and configurations found in previous studies, this paper emphasizes on visualization process and figure apprehension (Duval, 2011) to describe the levels of development that may qualitatively differentiate students' levels of knowledge and understanding of TG concepts. It is hypothesized that visualization process can explain the development of TG ability better than the type of geometric transformation or task.

2.2. The geometric work space

The GWS (Kuzniak, 2006, 2011) is the place that is organized to ensure the geometric work. From an epistemological view, it puts the following components in a network: *a)* the real and concrete objects, *b)* the artefacts, and *c)* a theoretical system of reference. Adapting Duval (1995), the cognitive processes for using these components in geometrical problem solving are: *a)* a visualization process with regard to space representation and material support, *b)* a construction process determined by the instruments and geometrical configurations, and *c)* a discursive reasoning process that conveys argumentation and proof. This paper will emphasize on the first process.

Three different levels can describe the diversity of GWS existing in the school context: *a)* the reference, *b)* the appropriate, and *c)* the personal GWS (Kuzniak, 2011). Geometry intended by the teacher/curriculum is described in the reference GWS, which must be fitted out in an appropriate GWS, to enable an actual implementation in a classroom where each student works within his / her personal GWS. This paper focuses on the personal GWS. When a problem is posed to an individual, it is handled within his/her personal GWS, which generally depends on the person's cognitive abilities (i.e. visualization). Houdement and Kuzniak (1999) describe the way in which three different paradigms could explain the different forms of geometry. A paradigm is composed of a theory that guides observation, activity and judgment, and permits new knowledge production. In primary school, all GWS levels can be described within *Geometry I* (GI). GI finds its validation in the material and tangible world. When students pass to secondary education, they are expected to start working within *Geometry II* (GII). GII is built on a model that approaches reality without being fused with it (Kuzniak & Rauscher, 2011). Both Geometries have a close link to the real world, but in different ways (Kuzniak, 2012).

2.3. Visualization in geometry

There are two ways of looking at figures and recognizing what they stand for: the natural perceptive and the mathematical (Duval, 2011). One important issue in the learning of geometry is to identify the figural units which can be discriminated in any constructed figure. According to Duval (2011), visualization ability in geometry is closely related to the ability of recognizing all figural units that can be mathematically relevant. He argues that becoming aware of the different ways of looking at figures is prior to the knowledge of the classical, basic figures.

According to Duval (1995), there are four apprehensions for a geometrical figure: perceptual, operative, discursive, and sequential. Visualization process is related to the first two types. Specifically, the perceptive way of visual recognition focuses exclusively on the most global shape or closed outline, and so the recognition of other possible reconfigurations is excluded. The perceptive way is activated and reinforced when figures are used as objects that can be empirically observed and it can either help or inhibit the heuristic recognition (Duval, 2011). Operative apprehension is a form of visual processing that concerns geometrical figures and relies on the different ways of modifying a certain figure. One way is dimensional deconstruction.

Dimensional deconstruction describes the transition of a drawing seen as a tangible object to the figure conceived as a generic and abstract object (Duval, 2005). For example, a figure can be seen as a 2D-object (a triangle as an area), a set of 1D-objects (sides) or 0D-objects (vertices). Another example is when one recognizes embedded 2D figures within a 2Dobject (e.g., a triangle inscribed in a circle). While the natural way perception focuses exclusively on 1D, 2D or 3D/3D figural units, just like material object, the mathematical way requires the dimensional deconstruction of any shape into figural units of 2D, 1D or 0D/2D. According to Bulf (2009), dimensional deconstruction is a possible strategy for 12-13 year old French students to solve symmetry tasks.

This paper demonstrates how Duval's theory (2005) about the role of figures in geometric reasoning can be used to describe the personal GWS of students at different levels of abilities in TG. Specifically, it focuses on the real and local space of the GWS which is related to visualization process and figure perception of primary school students.

3. Methodology

3.1. Participants

The participants were 166 primary school students. In order to study a wider spectrum of primary school students' development of TG abilities, the participants were selected from three successive grades. Specifically, there were 52 fourth - graders, 53 fifth - graders and 61 sixth - graders.

3.2. Instrument and procedure

The instrument of the study was a TG test, developed especially for the purpose of the project. The test had three sections: one on translation, one on reflection and one on rotation. Each section had four different types of tasks (see Appendix): 1) recognising the image of a translation / reflection / rotation among other choices, 2) recognising a translation / reflection / rotation among other choices, 3) defining the parameters of a given translation / reflection / rotation, and 4) constructing the image for a given translation / reflection / rotation. For each type, at least three tasks were given: one in a horizontal direction, one in a vertical direction and one in a diagonal direction. In types 3 and 4, there was an additional task with an overlapping image and in type 4 an additional task with an unfamiliar shape in horizontal direction. The tasks were split and administered to all students in two equally difficult parts. Each item's difficulty within each section was estimated based on previous research findings of Schultz and Austin (1983). Students were given 40 minutes to complete each part. To avoid practice effects, half of the students received one part of the test first, while the rest of the students received the other. Since instruction in geometric transformations in Cyprus was not emphasized in the curriculum and they mainly focused on the concept of reflection through symmetry, operational definitions and examples were given to the students before completing the tests, using visual aids of paper cards and drawings on the whiteboard for illustration. Moreover, written mathematical definitions and illustrations of each transformation were included in the students' printed tests.

Based on the theoretical frameworks described above, an a priori analysis of the tasks suggests that tasks of recognition (Types 1 and 2) can be solved efficiently with the use of a motion approach, as defined by Edward's (2003). Therefore, a student can solve a recognition task by visualising the figures changing positions as tangible 3D or 2D objects of the real world (GI) and without requiring dimensional deconstruction to 1D or to 0D. For example, in the recognising translation task (see Appendix), one is expected to imagine one or every shape sliding over the grid and onto the other shapes, in order to decide which one is matching. Tasks of Type 3 and 4, even though they can be approached using a motion strategy of visualising figures as tangible objects, it is expected that such an approach will often result to partially correct responses with errors in measures or image orientation. It is expected that the most efficient way to approach such tasks is mapping, as described by Edwards (2003). Thus, a student who is able to visualize the dimensional deconstruction of the figure to 1D or even 0D elements is more likely to succeed in such tasks. For example, in the case of constructing the reflection of a triangle over a vertical line accurately, one would have to focus on the points of the triangle's vertices, find their position on the other side of the line, and then reconstruct the triangle. However, it is expected that students at this level would still operate within GI, since they treat segments and points as material objects. Students who are able to understand the symbolic nature of points and their relations in space may be considered to be at a transitional stage to GII.

After completing the test, the students' responses were graded. Types of tasks 1 and 2 were multiple choice, and were graded with 1mark for correct and 0 for incorrect responses. In tasks of type 3 and 4, 0 marks were given for incorrect responses and 1 for correct responses. Partial credit was given for responses with some correct elements. Items with no response received 0 marks.

3.3. Statistical procedures

For testing the fit of the theoretical model regarding the structure of TG ability, MPLUS was used with Maximum Likelihood (ML) estimator. More than one fit indices were used to evaluate the extent to which the data fit the theoretical model. The fit indices and their optimal values were: a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96 since a significant chi-square indicates lack of satisfactory model fit, b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2004).

RASCH analysis was used for investigating the development of knowledge and understanding of TG ability. This method is based on the assumption that the difference between item difficulty and person ability should govern the probability of any person being successful on any particular item, and ranks both the persons and the items on the same scale, based on these probabilities. The fit indices are: *a)* the infit (weighted) mean square statistic, and *b)* the outfit (unweighted) mean square statistic. The normalized statistics (using the Wilson – Hilferty transformation), infit t and outfit t, have a mean near zero and a standard deviation near one when the data conform to the measurement model. No items or persons should have a zero as a score neither should they have a perfect score. This study used the dichotomous model of RASCH, which predicts the conditional probability of a binary outcome (correct / incorrect). Therefore, for this analysis, the data were recoded as 1 mark for correct and 0 for incorrect or partially correct responses.

4. Results

The first aim of this study was to investigate the components that synthesize primary school students' TG ability and its structure. For this aim, confirmatory factor analyses (CFA), with subsequent model tests, were performed. The model presented in Figure 1 seems to have the best fit for all three TG concepts. As expected, there are four first-order factors for each geometric transformation: 1) "recognize the image", 2) "recognize the transformation" (translation, reflection, or rotation), 3) "define the parameters", and 4) "construct the image". Two of the expected factors, "recognise the image" and "recognise the transformation", seem to constitute a second order factor, which contributes significantly to the ability at primary school. This factor was named "recognise properties", since the common characteristic shared by these tasks is the recognition of each transformations' properties regarding the preservation or change of the orientation, position, and/or size of the figure. Further CFA with students' mean scores for each of the twelve factors (4 for each transformation) confirm that "translation ability", "reflection ability", and "rotation ability" all load in a higher order factor, which is considered to be "TG ability" (CFI = .96, x^2 = 71.90, df = 52, x^2 $/df$ = 1.39, RMSEA = .05). The factor loadings and their interpreted dispersion (r^2) are .95 (.91) for "translation ability", .98 (.97) for "reflection ability", and .90 (.81) for "rotation ability".

The second aim was to describe primary school students' levels of abilities in TG. A RASCH dichotomous analysis was performed. The analysis suggests that the data fit the model well (X = .00, SD = 1.81, Infit Mean Square = .99, Outfit Mean Square = 1.01, Infit t= −.06, Outfit t= .12, Reliability of Estimates = .98). Figure 2 presents the scale that resulted from this analysis.

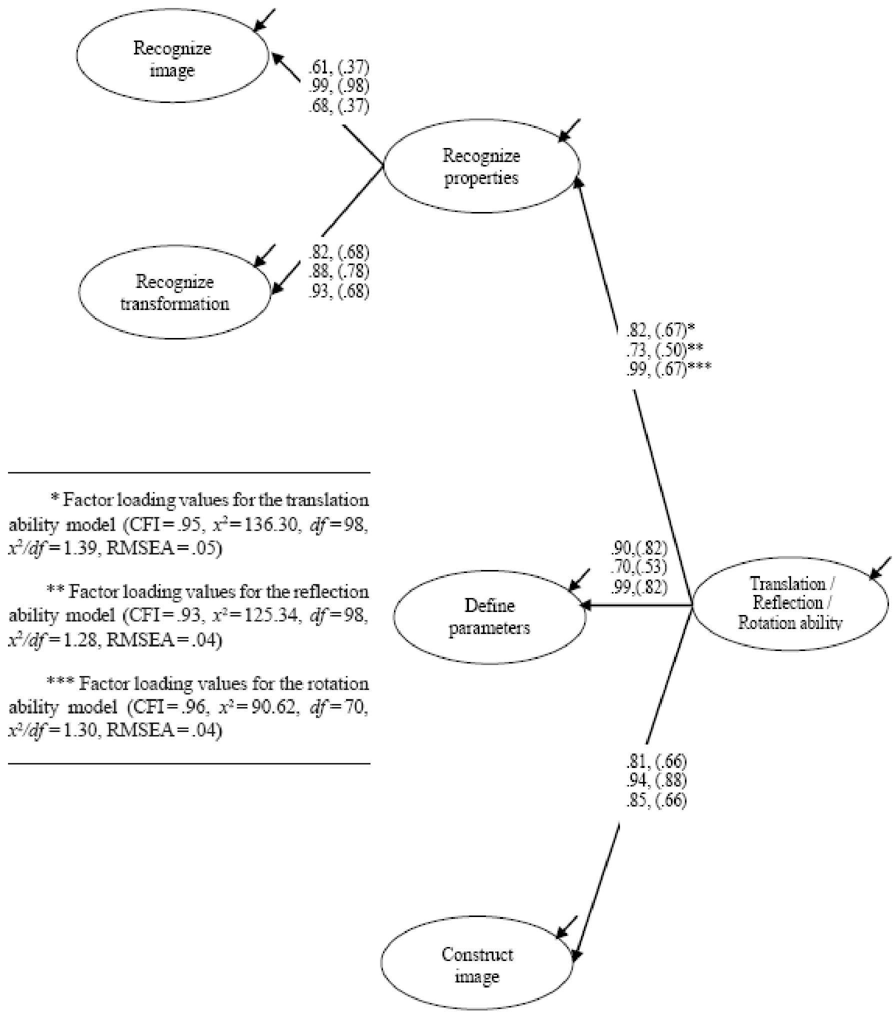

FIGURE 1
The proposed model of ability for the three geometric transformations

On the left side of the figure, the students are ranked according to their ability. Each X represents one student. On the right side of the figure, the items of the test are ranked according to their level of difficulty. More able students, i.e. those that correctly answered more items, are at the top of the scale while less able students are at the bottom. Similarly, items that were harder for the students are at the top of the scale while easier items are at the bottom. Each item is coded as a string of three symbols. The first symbol is a letter, which indicates the type of transformation: T for translation, F for Reflection and R for rotation; the second symbol is a number, which indicates the type of task, according to the factors described in the previous section: 1 for "recognize image", 2 for "recognize transformation", 3 for "define parameters", and 4 for "construct image". The last symbol is a number from 1 to 5, indicating the serial number of the item in the corresponding factor.

The dotted lines mark the different levels. There seem to be five levels of abilities: L1 (-5.0 to -2.5 logits), L2 (-2.49 to -0.9 logits), L3 (-0.89 to 0.89 logits), L4 (0.9 to 2.49 logits) and L5 (2.5 to 5.0 logits). After examining the assumptions suggested in literature that what forms the levels of abilities can be either the type of transformation or the type of task, (which did not give a clear picture of the qualitative differences between the levels), we decided to compare the levels in light of the personal GWS framework (Kuzniak, 2006) and specifically the visualization processes that we suppose are common requirements for solving the tasks that were grouped at the same level. Hence, we studied the similarities of the tasks that were grouped and we drew on the ideas of figure apprehension and dimensional deconstruction (Duval, 2005) to understand how students could have approached the task and how they visualised the figures. The naming of the levels was influenced by Edwards' (2003) terminology in the field of TG. Thus, they were named: 1) wholistic image, 2) motion of an object, 3) mapping of an object, 4) mapping of the plane, and 5) self-regulated mapping of the plane, for reasons that are explained further on.

In L1, "wholistic image", students seem to perceptually conceive simple relations of up - down and left - right within the figure, that exist in the real world, but without understanding neither the properties of the transformation nor of the geometrical figures represented. The focus is not on what the shapes represent, but on their positioning as part of a global figure. Students at this level have a personal GWS that allows them to visually process the figures only in a perceptual way. They seem to visualize figure of the plane and the objects as a whole image, as a realistic photograph in the physical world. They possibly cannot deconstruct this figure into units, and they see the grid which represents the plane and the images as a concrete part of it, without motion. In L2, "motion of an object", students begin to detach the shape from the figure of the plane and are able to visualize it moving on top of it. The emphasis is still on the shape as a tangible object, but the students can visualize the dimensional deconstruction of the representation to two separate 2D figures: the plane and the geometrical shape. Hence, they may be able to visualize a 2D/2D deconstruction. However, geometric reasoning in the personal GWS of students at this level still relies strongly on perceptual apprehension.

The tasks that were grouped in L3, "mapping of an object", suggest that students probably begin to dimensionally deconstruct the 2D shape into 1D sides, and focus on the sides and their mapping. Hence, the personal GWS of students at L3 begins to employ operative apprehension. Students seem to be able to visualize the mapping of a single side and use reconfigurations to reconstruct the image of the geometrical shape based on its definition and attributes (right angles, size etc). From a cognitive perspective, they begin to intuitively realize the transformation properties related to direction, orientation and distance, and they can apply them in simple situations such as constructing images in straight-line displacement and in recognizing circular displacement.

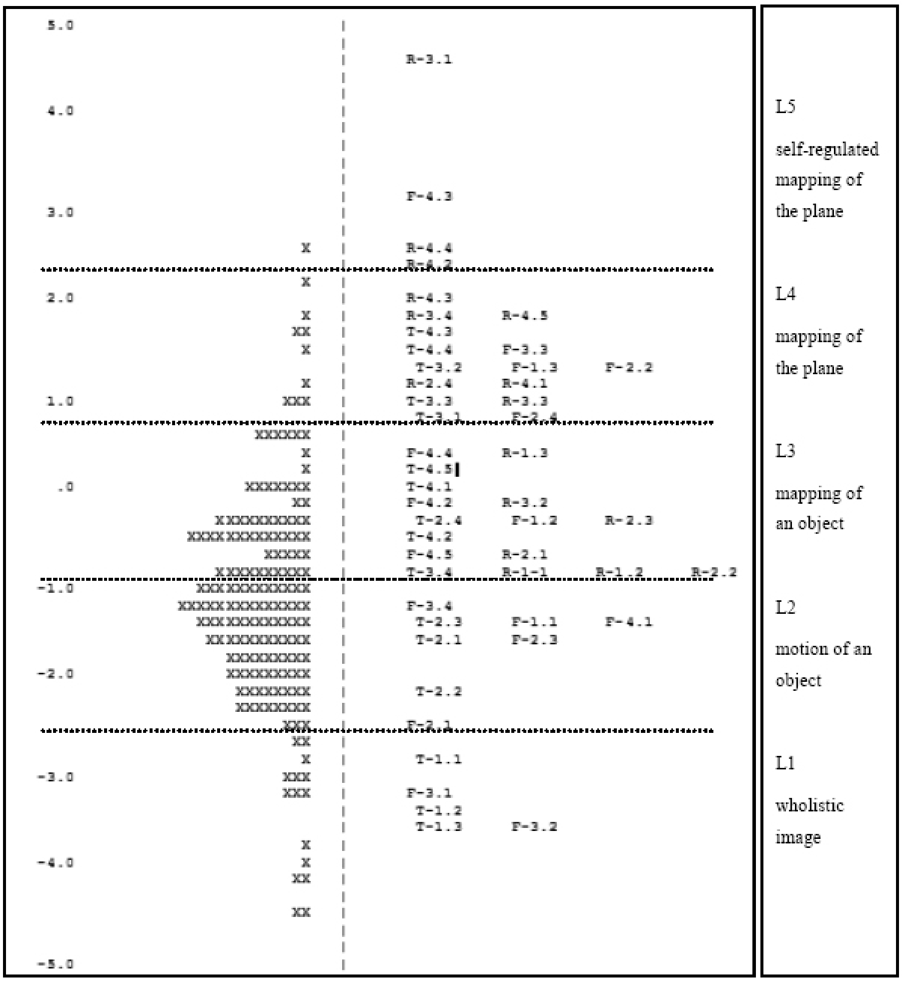

FIGURE 2
The scale of abilities in the transformational geometry concepts

Students at L4, “mapping of the plane”, seem to have a strong operative apprehension and ability to deconstruct the 2D geometrical shape into both 1D and 0D points and they understand the mapping of all the points, based on the properties of TG. The shapes are not anymore perceived only as global figures, but they still visualize the plane figure as an object. They begin to discover and apply the transformations’ properties to all the points of the shape, even in complex circular displacements. Note that students at this level were all sixth graders and their personal GWS at this age has probably been influenced by a more formal instruction on geometrical concepts that can be relevant to geometric transformations (i.e. shape properties, angles, circles). At L5, “self - regulated mapping of the plane”, there is only one sixth - grade student. It is our belief that what differentiates this student from L4 students is the ability for a more flexible visualization of figures and the representation of space. This student seems to be able to dimensionally deconstruct both the geometrical shape and the plane into 0D points. He/she can visualize and perform the mapping of all the points in various routes and direction (straight and circular displacements), and realizes that transformation affects all points of the plane. According to Duval (2005), a deadlock in the teaching of geometry is that a perceptive recognition of some figural units excludes the recognition of the others possibilities, and therefore goes against any possible transformation of a given figure into another. This student seems to have some flexibility in manipulating and controlling his / her mental images and can flexibly change between figural units of visualization and visual strategies. This could be evidence of “representational flexibility” (Gagatsis, Deliyianni, Elia & Panaoura, 2011) in the sense of flexibility to visualize and manipulate the mental representations of the different reconfigurations of a figure. Although we are not aware of many details regarding this student's cognitive profile, it is possible that his/her cognitive abilities may enable him/her to a personal GWS that is different from other students, perhaps with characteristics closer to GII. Hence, this student may reflect an "attempted transition to GII" (Bulf, 2009), a passage that remains blocked because the reference GWS in primary school is strongly rooted in GI. For most students, this negotiation between GI and GII appears and continues during secondary school.

5. Discussion

The aim of this paper was to describe primary school students' structure and levels of abilities in TG. In this section, we discuss the conclusions of our findings.

Regarding the first aim, our findings confirm Kidder's (1976) position that TG ability is multifaceted. It seems that the three geometric transformations are composed by similar factors, namely: recognition of image, recognition of transformation, identification of parameters, and construction of image.

Moreover, they seem to have a similar structure, with recognition of image and recognition of transformation factors forming a higher order factor – recognition of properties. This higher order factor and the two factors of identification of parameters and construction of image comprise ability in translation, reflection, and rotation respectively. The three factors of ability in each geometric transformation load on a higher order factor, which is TG ability.

For the second aim, we adopted the theoretical framework of GWS (Kuzniak, 2006) to interpret students' levels of ability. Five levels were found in relation to visualization process. Our findings suggest that the personal GWS of students at primary school level operates within GI and is strongly influenced by the natural world, even in their mental images. However, it seems that not all students that think within the same paradigm are at the same level of abilities nor share the same visualization process of geometrical reasoning. What seems to differentiate these levels may be some cognitive developmental abilities that form students' personal GWS, since the reference GWS does not emphasize instruction in geometric transformations and these differences cannot be attributed to teaching. Hence, even though they are not expected by the system to be working within GII for TG concepts, some students at the higher levels may have developed such a personal GWS regarding their visualization process of figures, which would make it possible for them to begin their transition to GII from primary school, given the appropriate GWS. This could suggest that some sixth grade students may be ready to be introduced to a more formal instruction on geometric transformations within GII. This should be taken into consideration by curriculum formers in the design of geometry curricula. However, this does not mean that students at the higher levels should not still approach the easie tasks within GI. Further research with qualitative data of students' arguments is required to describe the GWS paradigms primary school students at different levels of abilities when solving TG tasks. Further studies of students'cognitive profile may reveal reasons for the differentiation between levels. Such studies could focus on spatial ability, which is highly related to TG ability (Kirby & Boulter, 1999).

Our findings are important for the teaching of TG in primary school. They support the fact that the theory of GWS can be a useful epistemological tool for understanding development of TG, and guide teachers into adjusting their teaching methods to help students achieve higher levels of performance. Moreover, it provides evidence for the importance of practicing students' ability to identify figural units in educational frameworks for the teaching of geometry in general, which according to Duval (2011) is a fundamental principle in the learning of geometry. Hence, we should reflect about a new approach for introducing geometry in primary and secondary levels, whose principle would be that the awareness of the different ways of looking at figures is prior to the knowledge of the classical basic figures (Duval, 2011).

Agradecimientos

This work falls under the Cyprus Research Promotion Foundation's Framework Programme for Research, Technological Development and Innovation 2009 - 2010 (DESMI 2009-2010), co-funded by the Republic of Cyprus and the European Regional Development Fund (Grant: PENEK/0609/57).

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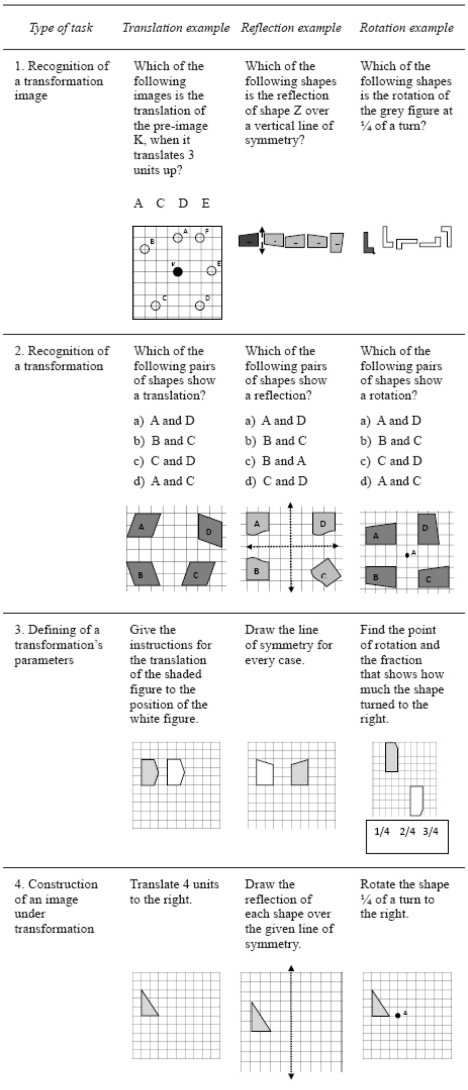
Appendix

Examples of tasks in the transformational geometry ability test



Type of task	*Translation example*	*Reflection example*	*Rotation example*
1. Recognition of a transformation image	Which of the following images is the translation of the pre-image K, when it translates 3 units up? A C D E	Which of the following shapes is the reflection of shape Z over a vertical line of symmetry?	Which of the following shapes is the rotation of the grey figure at ¼ of a turn?
2. Recognition of a transformation	Which of the following pairs of shapes show a translation? a) A and D b) B and C c) C and D d) A and C	Which of the following pairs of shapes show a reflection? a) A and D b) B and C c) B and A d) C and D	Which of the following pairs of shapes show a rotation? a) A and D b) B and C c) C and D d) A and C
3. Defining of a transformation's parameters	Give the instructions for the translation of the shaded figure to the position of the white figure.	Draw the line of symmetry for every case.	Find the point of rotation and the fraction that shows how much the shape turned to the right. 1/4 2/4 3/4
4. Construction of an image under transformation	Translate 4 units to the right.	Draw the reflection of each shape over the given line of symmetry.	Rotate the shape ¼ of a turn to the right.

Financiamiento

Fuente: Cyprus Research Promotion Foundation's Framework Programme for Research, Technological Development and Innovation 2009 - 2010 (DESMI 2009-2010)
Nº de contrato: PENEK/0609/57