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GARCH-type volatility in the multiplicative quadrinomial tree method: An application to real options

Volatilidad tipo GARCH en el método de árboles cuatrinomiales multiplicativos: una aplicación para opciones reales

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Abstract

This article applies the multiplicative quadrinomial tree numerical method with non-constant volatility to assess a real option of abandonment, based on an estimate of the conditional volatility for WTI oil commodity prices and their respective equivalence in a GARCH-diffusion model. The methodology refers to the use of an estimate of type GARCH (1,1) and the numerical method through a quadrinomial tree. There are two main findings: 1) when employing the quadrinomial method, the value of the real option turned out to be greater than the value estimated through the traditional multiplicative binomial method, due to underestimation of the real value of volatility that occurs in a specific period according to the latter method; and 2) a methodological contribution that demonstrates plainly way the presence of non-constant conditional volatility as well as being able to value all types of options using stochastic volatility.

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Keywords: GARCH; Time series; GARCH-diffusion model; Quadrinomial trees; Option pricing

Resumen

Este artículo utiliza el método de árboles cuatrinomiales multiplicativos con volatilidad no constante para valorar una opción real de abandono, a partir de la estimación de la volatilidad condicional para la serie de precios del commoditie crudo tipo WTI y su respectiva equivalencia con un modelo de difusión GARCH. La metodología propuesta refiere el uso una estimación tipo GARCH (1,1) y el uso del método numérico por arboles cuatrinomiales. Los dos principales hallazgos son: 1) cuando se emplea el método cuatrinomial, el valor de la opción tiende a ser mayor que el estimado por el método tradicional de árboles binomiales multiplicativos, debido a una subestimación del valor real de la volatilidad para el ultimo método, para un periodo de tiempo específico; y 2) la contribución metodológica propuesta puede ser utilizada de una forma relativamente sencilla cuando existe presencia de volatilidad condicional no constante y permite la valoración de todo tipo de opciones utilizando volatilidad estocástica.

Código JEL: C19, C32, C65, G13, G32
Palabras clave: GARCH; Series de tiempo; Modelo de difusión GARCH; Arboles cuatrinomiales; Valoración de opciones

Introduction

Among academics, the discounted cash flow (DCF) is the most used method for valuing capital assets, but it is also criticized because it does not include important elements such as the occurrence of contingent events, the present risk in cash flows and volatility (Trigeorgis, 1996). The real options approach (ROA) serves as a complementary methodology to the DCF method, allowing to include volatility as a fundamental parameter to quantify risk and to collect some elements associated with uncertainty (Keswani & Shackleton, 2006). Trigeorgis (1996), Mun (2006) and Brandão, Dyer and Hahn (2012) have argued the difficulty of its estimation but have also indicated the importance of doing so appropriately to be able to perform an adequate valuation. Additionally, to assume that the volatility parameter is constant and unconditional, such as the one used in the ROA method, is considered impractical to model the price returns of financial series, capital assets and commodities, given the presence of a series of empirical characteristics such as leptokurtosis, heavy tail distributions, clustering volatility, and conditional variance that changes randomly over time (Grajales Correa & Pérez Ramírez, 2007).

According to this, it is necessary to make a good estimate of the real option for the ROA method, value that depends largely on the way that volatility is described and estimated. In other words, the option is very sensitive to the value that is calibrated for this factor, as demonstrated by Trigeorgis (1990), who in an analytical and empirical way concluded that an increase in that volatility by 50% could result in an increase by 40% of the real option. Concurrently, Keswani and Shackleton (2006) presented variations of more than 210% in the value of the real option when the volatility increases from 10% to 30% (Brandão et al., 2012). Taking the above into account, it was possible to conclude that the theory used to estimate the parameter of volatility contained in the ROA method has only been focused towards the calculation of its structure and its unconditional and constant behavior, without exploring the advantages that its conditional and stochastic estimation has (Vasseur, Sanchez, & Escobar, 2019).

In the last three decades, several studies have focused on volatility estimation, given its importance to economic agents and its use in financial and economic applications and because volatility plays an important role in decisions that involve financial risk (Posedel, 2005). It is common to find financial time series that exhibit some stylized facts, effects only began to be collected after the appearance of the most used classical models corresponding to the non-linear time series of autoregressive conditional heteroskedasticity (ARCH). This model was developed in the 1980s in the seminal works of Bollerslev (1986) and Engle (1982), who focused their efforts on including a functional relation between current and past conditional volatility as well as on describing the statistical distribution of the errors in detail (Argáez Sosa, Batún Cutz, Guerrero Lara, Kantún Chim, Medina Peralta, & Pantí Trejo, 2014). The previously mentioned models included the inertial behavior of volatility as well as the autocorrelation effect in all financial time series (Novales, 1993); that is, fundamentally, it is necessary to consider all the past information of a variable in predicting its current and future behavior (De Arce, 1998).

The ARCH generalization process to estimate conditional volatility was presented by Bollerslev (1986). His model of generalized autoregressive conditionally heteroskedasticity (GARCH) allowed multiple developments and extensions (Hansen & Lunde, 2005). The basis and simplicity of the functional structure of GARCH, specifically the type (1,1), has been considered to be the starting point for several financial applications (Preminger & Storti, 2017).

Recently, some models include stochastic volatility and were created, fundamentally, to overcome the underlying problems when considering volatility in terms of the evaluation time horizon. Motivated by this empirical evidence, several authors, such as Chesney and Scott (1989), Heston (1993), Hull and White (1987), Scott (1987), Stein and Stein (1991),

and Wiggins (1987), proposed models with stochastic volatility as a parsimonious extension of the Black–Scholes model (Black & Scholes, 1973); among these models, the GARCH-diffusion type proposed by Drost and Werker (1996) and Duan (1996, 1997) excels. The novelty of this model is that it was the first approximation between a GARCH process and a stochastic volatility model. Additionally, multiple research developments of this model exist, both theoretical (Barone-Adesi, Rasmussen, & Ravanelli, 2005; Chourdakis & Dotsis, 2011; Christoffersen, Jacobs, & Mimouni, 2010; Ritchken & Trevor, 1999) and empirical (Figà-Talamanca, 2009; Plienpanich, Sattayatham, & Thao, 2009; Wu, Ma, & Wang, 2012; Wu, Yang, Ma, & Zhao, 2014).

This article is organized as follows. Section 2 describes the GARCH (1,1) type and the GARCH-diffusion model. Then, in section 3, the equivalence between the variance of these models is formally presented. Section 4 summarizes the multiplicative quadrinomial method to assess options, both real and financial. In section 5, some concepts to assess real options are summarized. Section 6 describes a counterfactual case as an example. Finally, section 7 includes discussion, conclusions, and suggestions for future research.

Volatility models

GARCH model

Volatility is a characteristic of any financial time series and, although it is considered non-constant, it is usually defined as homoscedastic and long-term estimates to be made. As it is unobservable, it is important to analyze and collect the oscillation that occurs in short-term, this is known as conditional volatility of the underlying asset (Casas Monsegny & Cepeda Cuervo, 2008). Based on this fact, Bollerslev (1986) and Engle (1982) developed models to analyze short-term conditional variations. Specifically, Bollerslev (1986) proposed the GARCH (p, q) model from the ARCH (q) model to overcome the weakness of estimating, in certain cases, many parameters, which complicate its estimation. Avoiding a possible loss of precision, a restricted alternative parameterization was defined, dependent on a reduced number of parameters. This model has the following structural form (Tsay, 2005):

$$\varepsilon_t \mid \Omega_{t-1} \sim N\left(0, h_t^2\right) \text{ with } \varepsilon_t = h_t \eta_t \text{ and } \eta_t \quad N(0,1)$$
$$h_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}^2 \tag{1.1}$$

where ε_t is the innovation variable or shock variable in the period t; h_t is the standard deviation in the period t; η_t is the normal random variable (iid) with mean 0 and variance 1 in period t; h_t^2 is the conditional variance in period t; α_i is the alpha coefficient; β_j is the beta coefficient; and ε_{t-i}^2 is the square of the shock's lagged value. Equation (1.1) is the conditioned variance depends on the previous perturbations as well as the lagged variances. The best-known and simplest model in the literature is the GARCH (1,1) model, which corresponds to an exponential smoothed model of variance and is considered to be a simple but useful model to estimate conditional volatility. Its functional structure is as follows:

$$h_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}^2 \tag{1.2}$$

with $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\beta_1 \geq 0$ where $\alpha_1 + \beta_1 \leq 1$ as a sufficient condition to be a stationary process.

This model can be used to emulate the conditional volatility, both the prices and the returns of financial assets as well as the risk value in portfolios and the exchange rates between currencies. For example, Casas Monsegny and Cepeda Cuervo (2008) used Gillette's prices and found that this model explains their behavior to a large degree. Gazda and Výrost (2003) used data from the Slovakia Stock Index (SAX) to estimate the respective returns, using three different models – GARCH, exponential GARCH (EGARCH), and threshold GARCH (TARCH) – and found that the adjustment of the conditional variance could be achieved with any of these methods. Engle and Patton (2007) also used it in an index, specifically the Dow Jones, and reached the conclusion that the GARCH-type models help to capture different stylized facts offered by the market. Additionally, Martens (2002) modeled and forecast the volatility of the S&P500 futures index, using high-frequency data. Engle (2001) used the model to estimate the risk value in a hypothetical portfolio composed by 50% of the Nasdaq index, 30% of the Dow Jones index, and long-term bonds. Its use was also extended to other contexts, for example, exchange rates. Alexander and Lazar (2006) used it to model the American dollar in regard to three important currencies: Pound Sterling, Euro, and Yen; their main finding indicated that the empirical evidence favors the use of the normal mixture in conjunction with a GARCH (1,1) model. Hansen and Lunde (2005) presented an important work, in which it was indicated that the model was not surpassed by 330 similar and more sophisticated ARCH-like models. They arrived at this conclusion by using data on the exchange rate between the German mark and the American dollar; however, they also indicated that, when using the returns of IBM's stock, the GARCH model apparently is not the most appropriate when a leverage effect is present.

GARCH-diffusion model

This model was introduced by Wong in 1964, but its popularity grew following the works of Nelson (1990a, 1990b). Drost and Werker (1996) and Duan (1996, 1997) were the first authors to propose an approximation between a GARCH process and the stochastic volatility model and to indicate that it is possible to exchange such processes as required, offering an important development for future research in such regard. Later, Ritchken and Trevor (1999) developed the same idea but expressed it in an algorithm to assess options, both European and American, based on the trinomial tree model. Subsequently, Barone-Adesi et al. (2005) analytically derived the first four moments of the model and obtained a closed solution to value an option; in addition, they analyzed the implicit volatility surfaces with this solution. An important condition was discovered by Christoffersen et al. (2010), who demonstrated empirically, through the use of realized volatilities, S&P500 returns, and an option data panel, that the Heston (1993) model was poorly specified, because in the diffusion model presented by the author, volatility was found in the square root instead of being considered linear. These conclusions were reaffirmed by Chourdakis and Dotsis (2011); although they also suggested that the model should consider a nonlinear drift against a linear one. Recent studies have indicated that this model gives a better description of the behavior and dynamics of financial series than other types of models, such as the well-known model of Heston (1993) (Aït-Sahalia & Kimmel, 2007; Jones, 2003; Wu, Zhou, & Wang, 2018). It has been used as a good model for adjusting financial option data (Chourdakis & Dotsis, 2011; Christoffersen et al., 2010; Kaeck & Alexander, 2012; Wu et al., 2012).

The most recent research on this model is considered to be empirical, and it has been used successfully in applications to different contexts; for example, Figà-Talamanca (2009) used multiple stock market indexes to compare the theoretical and empirical autocovariances, concluding that this model captures autocovariance observed in the data. At the same time, Plienpanich et al. (2009) integrated a disturbance through fractional noise into the diffusion model; their results showed that a better estimation of the stock price of a commercial bank was attained using this model rather than the traditional Black–Scholes model. Similar conclusions were obtained by Wu et al. (2012), who analyzed the Hang Seng index (HSI) and concluded that the GARCH-diffusion model offers better predictions of the price of warrants than the classical model. Years later, Wu et al. (2014) studied the Hong Kong stock market through American options and found the same advantage. Finally, Wu and Zhou (2016) used the Chinese volatility index (iVIX); their main findings indicated that the risk of volatility

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market values and the risk premium volatility were negative, which implied that investors in the Shanghai stock exchange are risk averse.

In general terms, this type of model is usually characterized by not having a closed solution and belongs to the class of non-affine models; in addition, solutions must be achieved through numerical methods, simulation, or the use of integrals in stochastic differential equations (Barone-Adesi et al., 2005). The system equations presented by this model have the following functional structure (Wu et al., 2012):

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \tag{1.3}$$

$$dV_t = \alpha\left(\theta - V_t\right)dt + \sigma V_t dW_t^{(2)} \tag{1.4}$$

where the parameters α, θ, and σ are constant and equivalent to the mean-reversion speed, the mean long-term volatility or tendency, and the volatility of volatility, respectively. For its part, $W_t^{(1)}$ and $W_t^{(2)}$ correspond Independent One-Dimensional Standard Wiener Motion processes.

Equivalence between conditional variance process of GARCH (1,1) and variance of GARCH-diffusion stochastic volatility model

Proposition 1. The functional structure GARCH (1,1) type presented in section 2.1, Equation (1.2) is equivalent to the differential equation proposed in (1.4), i.e. V_t is equivalent to h_t^2, with ,

$$\alpha = \frac{\left(1-\alpha_1-\beta_1\right)}{\Delta t},\ \theta = \frac{\alpha_0}{1-\alpha_1-\beta_1} \text{ and } \sigma = \frac{\alpha_1\sqrt{2}}{\sqrt{\Delta t}}.$$

Proof.

As $\varepsilon_t \mid \Omega_{t-1} \quad N\left(0, h_t^2\right)$, its moment generating function is given by $\varphi_\varepsilon^{(t)} = \exp\left(-\frac{1}{2}h_t^2 t^2\right)$ and, $E\left[\left(\varepsilon_t^n \mid \Omega_{t-1}\right)\right] = \frac{\partial^n \varphi_\varepsilon^{(t)}}{\partial t^n}\Big|_{t=0}$.

It is easy to see that $\varphi_\varepsilon^{(0)} = 1,\ E\left[\left(\varepsilon_t \mid \Omega_{t-1}\right)\right] = 0,\ E\left[\left(\varepsilon_t^3 \mid \Omega_{t-1}\right)\right] = 0.$ It can also be established that,

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$$\begin{cases} E\left[\left(\varepsilon_t^2 \mid \Omega_{t-1}\right)\right] = h_t^2 \\ E\left[\left(\varepsilon_t^4 \mid \Omega_{t-1}\right)\right] = 3h_t^4 \\ \text{var}\left[\left(\varepsilon_t^2 \mid \Omega_{t-1}\right)\right] = 2h_t^4 \end{cases} \tag{1.5}$$

Consider the variable $y_t = h_t^2 + \sqrt{2}h_t^2 e_t;\ e_t \quad N(0,1)$. Using the characteristic function of the normal distribution on y_t it follows that,

$$\begin{aligned} E\left[\left(\exp(ity_t) \mid \Omega_{t-1}\right)\right] &= E\left[\left(\exp\left(ith_t^2 + it\sqrt{2}h_t^2 e_t\right) \mid \Omega_{t-1}\right)\right] \\ &= \exp\left(ith_t^2\right)\exp\left(-\frac{1}{2}t^2 2h_t^4\right) \end{aligned}$$

where, $y_t \mid \Omega_{t-1} \quad N\left(h_t^2, 2h_t^4\right)$ that is y_t as ε_t^2 they conserve the same two first moments conditioned to Ω_{t-1}.

From Equation (1.2) then,

$$\begin{aligned} h_t^2 - h_{t-1}^2 &= \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1 h_{t-1}^2 - h_{t-1}^2;\ \text{as } y_t \overset{d}{=} \varepsilon_t^2 \text{ (Hull, 2014, pp. 272-273)} \\ h_t^2 - h_{t-1}^2 &= \alpha_0 - \left(1-\alpha_1-\beta_1\right)h_{t-1}^2 + \alpha_1\sqrt{2}h_{t-1}^2 e_{t-1} \\ \Delta h_t^2 &= \frac{\left(1-\alpha_1-\beta_1\right)}{\Delta t}\left(\frac{\alpha_0}{1-\alpha_1-\beta_1} - h_{t-1}^2\right)\Delta t + \frac{\alpha_1\sqrt{2}}{\sqrt{\Delta t}}h_{t-1}^2\sqrt{\Delta t}e_{t-1} \\ \Delta h_t^2 &= \alpha\left(\theta - h_{t-1}^2\right)\Delta t + \sigma h_{t-1}^2 \Delta W_t;\ \Delta W_t = \sqrt{\Delta t}e_t \quad N(0,\Delta t). \end{aligned} \tag{1.6}$$

Thus, the limit process of (1.6) is given by the stochastic differential equation $dh_t^2 = \alpha\left(\theta - h_t^2\right)dt + \sigma h_t^2 dW_t;\ W_t$ it is a One-Dimensional Standard Wiener Motion.

Figure 1 below shows a numerical experiment, graphically demonstrating the equivalence between the variance of these two processes.

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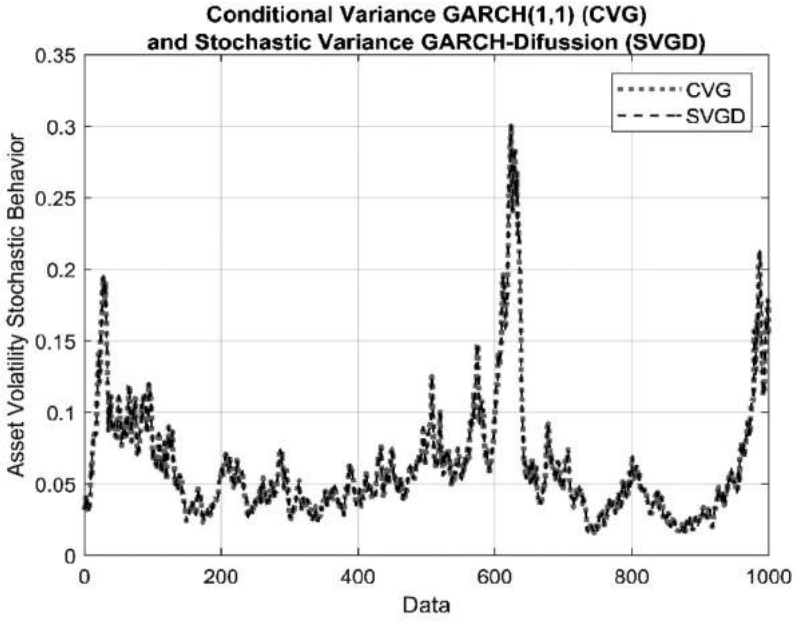

Figure 1. Equivalence between the conditional variance of the GARCH (1,1) process with estimated parameters $\omega = 0.001,\ \alpha = 0.08,\ \beta = 0.91$ and the stochastic variance of the GARCH-Diffusion model with , with $\alpha = 0.01$, $\theta = 0.1$ and $\sigma = 0.1131$ initial volatility for two processes, $\sigma_0^2 = V_0 = 0.032$ and $N = 1\ 000$ .Source: Prepared by the authors with simulated data using MATLAB

As shown in Figure 1, the dynamics of the conditional and stochastic volatility generated for each are apparently the same, equivalent values being found among these processes.

Multiplicative quadrinomial tree model from the GARCH-diffusion model

Once the parameters have been estimated by the GARCH (1,1) model and its equivalents defined by the GARCH-diffusion model, the numerical method is derived using the multiplicative quadrinomial tree to describe the discrete behavior of the price of the underlying asset. The dynamic factors and transition probabilities are presented below.

Considering the proposed differential equation system given by (1.3) and (1.4), over the time interval $[t_i, t_k]$, where $\mu \in$, $\alpha \geq 0$, $\theta > 0$, and $\sigma > 0$ are constant, while $\left\{W_t^{(1)}\right\}_{t\geq 0}$ and $\left\{W_t^{(2)}\right\}_{t\geq 0}$ are Independent One-Dimensional Standard Brownian Motions, supposing further

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that $S_t = S_i$ and $V_t = V_i$, the probabilities of the transition and growth factors for the processes V_t and S_t, respectively, are defined as (Pareja-Vasseur & Marin-Sánchez, 2019):

$$u_j^{(i)} = \exp\left(\sigma\sqrt{\Delta t}\right),\ d_j^{(i)} = \frac{1}{u_j^{(i)}},\ A_{jV}^{(i)} = \exp\left(\alpha\left(\frac{\theta}{V_j^{(i)}} - 1\right)\Delta t\right) \text{ and } p_j^{(i)} = \frac{1}{2} + \frac{\left(\frac{\alpha\theta}{V_j^{(i)}} - \alpha - \frac{1}{2}\sigma^2\right)\sqrt{\Delta t}}{2\sigma}$$

$$h_j^{(i)} = \exp\left(\sqrt{V_j^{(i)}}\sqrt{\Delta t}\right),\ l_j^{(i)} = \frac{1}{h_j^{(i)}},\ A_S = \exp\left(\mu\Delta t\right) \text{ and } q_j^{(i)} = \frac{1}{2} + \frac{\left(\mu - \frac{V_j^{(i)}}{2}\right)\sqrt{\Delta t}}{2\sqrt{V_j^{(i)}}}$$

To consider the recombination of two binomial trees $S^{(i)} = \left(S_j^{(i)}\right),\ j \in J^{(i)}$ and $V^{(i)} = \left(V_k^{(i)}\right),\ k \in K^{(i)}$, which possess the same number of nodes along their time axes, that is, $J^{(i)} = K^{(i)}$ for all i, $q_j^{(i)}$, denotes the transition probabilities of $S^{(i)}$ with increases and decreases defined by $h_j^{(i)}$ and $l_j^{(i)}$, and for $V^{(i)}$ they are $p_k^{(i)}$, $u_k^{(i)}$, and $d_k^{(i)}$, respectively (Marín Sánchez, 2010). The direct product $S^{(i)} \times V^{(i)}$ is defined by a tree $T^{(i)}$ with a node described by $T_{j,k}^{(i)} = \left(S_j^{(i)}, V_k^{(i)}\right)$ at the time i. In the next step, $T_{j,k}^{(i)}$ generates four nodes – $T_{j+1,k+1}^{(i+1)} = \left(S_j^{(i)}h_j^{(i)}, V_k^{(i)}u_k^{(i)}\right)$, $T_{j,k+1}^{(i+1)} = \left(S_j^{(i)}l_j^{(i)}, V_k^{(i)}u_k^{(i)}\right)$, $T_{j+1,k}^{(i+1)} = \left(S_j^{(i)}h_j^{(i)}, V_k^{(i)}d_k^{(i)}\right)$, and $T_{j,k}^{(i+1)} = \left(S_j^{(i)}l_j^{(i)}, V_k^{(i)}d_k^{(i)}\right)$– the respective probabilities of which are $q_j^{(i)}p_k^{(i)}$, $\left(1-q_j^{(i)}\right)p_k^{(i)}$, $q_j^{(i)}\left(1-p_k^{(i)}\right)$ and $\left(1-q_j^{(i)}\right)\left(1-p_k^{(i)}\right)$, respectively (Lari-Lavassani, Simchi, & Ware, 2001; Pareja-Vasseur & Marin-Sánchez, 2019).

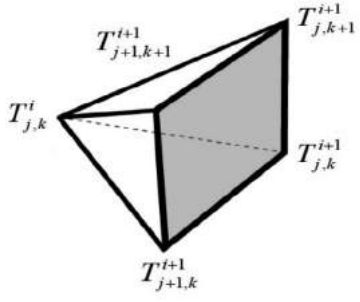

Figure 2a. The first step of the proposed multiplicative quadrinomial tree model of the stochastic asset price and volatility.
Source: Prepared by the authors using Photoshop

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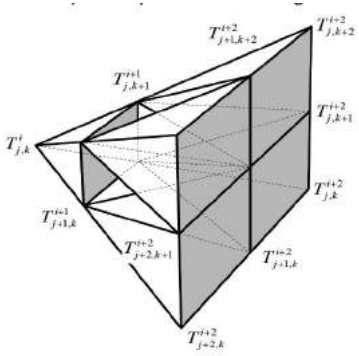

Figure 2b. Recombination of the second step of the proposed multiplicative quadrinomial tree model.
Source: Prepared by the authors using Photoshop

After all the corresponding parameters have been defined for V_t and for S_t, it is possible to construct a quadrinomial tree that emulates the behavior of price S_t, which, in each discrete step, has a value of branches equal to 4^n, where n corresponds to the number of steps, but, when the respective recombination is performed, the number of branches decreases to n^2, as can be seen in Figure 2b. Below we show the value of each position for the first step of the tree, as shown in Figure 2a (Pareja-Vasseur & Marin-Sánchez, 2019).

$$T^{(2)}_{2,2} = T^{(1)}_{1,1} uh^{(1)}_1 \quad , \quad T^{(2)}_{1,2} = T^{(1)}_{1,1} ul^{(1)}_1 \quad , \quad T^{(2)}_{2,1} = T^{(1)}_{1,1} dh^{(1)}_1 \quad , \quad T^{(2)}_{1,1} = T^{(1)}_{1,1} dl^{(1)}_1$$

Real option valuation

One of the techniques used to value the recent development of capital assets is known as ROA. This method, which is complementary to the DCF, seeks to introduce the volatility present in cash flows as well as the occurrence of contingent events. This methodology emerged from the theory of financial options but is applied when the valuation is performed for capital assets in real markets. This theoretical definition was coined by Stewart Myers, who indicated that many corporate assets could be seen as call options (Myers, 1977). This type of options, such as financial options, can be assessed using different techniques, of which the most appropriate corresponds to the numerical method with multiplicative binomial tree (Cox, Ross, & Rubinstein, 1979), because of its intuitive and simple handling. In this technique, the price in continuous time of an underlying asset approximates the Geometric Brownian Motion emulated through discrete time in the form of a tree, in which it is possible to analyze, graphically and numerically, the anticipated execution or otherwise of the option.

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The ROA technique estimates the strategic net present value (NPV) and this value is compared with the static NPV, which is estimated in the traditional way using the DCF method, finding the real option value (Maya Ochoa & Pareja Vasseur, 2014; Mun, 2002; Trigeorgis, 1996). According to this theory, it is common for use options to differ, contract, expand, or abandon, among others; for American call and put options modified or adapted to that context.

Algebraic expression

Assume that the life of an option (real or financial) over an asset that does not pay dividends, with an initial price S_0 and exercise price K, is divided into N sub-intervals, each with duration Δt. Define $f_{j,k}^{(i)}$ as the value of the option in the node (i,j,k). Based on Marín Sánchez (2010), the price of the asset has quadrinomial recombination in the node (i,j,k), which can be represented by the following expression:

$$T_{j,k}^{(i)} = S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{i-k} l_k^{i-w} * V_0 \prod_{m=1}^{j-1} u_m^m \prod_{m=1}^{i-j} d_j^{i-m} \tag{1.7}$$

with $T_0 = T_{1,1}^{(1)}$; $i = 2,\ldots,N$, $j = 1,2,\ldots,i$, and $k = 1,2,\ldots,i$. Keep in mind that, in this case, both u and d are constant, so Equation (1.7) is summarized as follows: $T_{j,k}^{(i)} = S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{i-k} l_k^{i-w} * V_0 u^{j-1} d^{i-j}$. Below there is a well-known algebraic approach based on the proposed methodology to assess the basic real options:

- In the case of an option to wait, the evaluation is performed in a similar way to an American financial call option; that is, the value on its maturity date is given by $\max(T_t - K)$; thus,

$$f_{j,k}^{(N)} = \max\left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j} - K, 0 \right),$$

for $j = 1,2,\ldots,i$ and $k = 1,2,\ldots,i$, while its discounted value is defined as,

$$f_{j,k}^{(i)} = \max\left[T_{j,k}^{(i)} - K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)}\right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)}\right) f_{j+1,k}^{(i+1)} \right.\right.$$
$$\left.\left. + \left(1 - q_j^{(i)}\right)\left(1 - p_k^{(i)}\right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i)^{-1}} A_S^{-1} \right]$$

- In the case of an expansion option, in which the expansion factor is defined by EF and K represents the additional investment for expansion, it is considered as a modified American call option, the value of which on its maturity date is given by $\max\left(T_t * EF - K, T_t\right)$; therefore,

$$f_{j,k}^{(N)} = \max\left(\left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}\right) * EF - K, S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}\right),$$

- for $j = 1, 2, \ldots, i$ and $k = 1, 2, \ldots, i$. In addition, its discounted value is defined as,

$$f_{j,k}^{(i)} = \max\left[T_{j,k}^{(i)} * EF - K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)}\right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)}\right) f_{j+1,k}^{(i+1)} + \left(1 - q_j^{(i)}\right)\left(1 - p_k^{(i)}\right) f_{j,k}^{(i+1)}\right) A_{jV}^{(i)^{-1}} A_S^{-1}\right]$$

- In the case of a contraction option, in which the contraction factor is defined by CF and, in this case, K represents the disinvestment or release of funds by contraction, it is considered as a modified American put option, the value of which on its maturity date is given by $\max\left(T_t, T_t * CF + K\right)$; hence,

$$f_{j,k}^{(N)} = \max\left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}, \left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}\right) * CF + K\right),$$

for $j = 1, 2, \ldots, i$ and $k = 1, 2, \ldots, i$. In addition, its discounted value is defined as,

$$f_{j,k}^{(i)} = \max\left[T_{j,k}^{(i)} * CF + K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)}\right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)}\right) f_{j+1,k}^{(i+1)} + \left(1 - q_j^{(i)}\right)\left(1 - p_k^{(i)}\right) f_{j,k}^{(i+1)}\right) A_{jV}^{(i)^{-1}} A_S^{-1}\right]$$

- In the case of an abandonment option, in which the salvage value corresponds to K in this case, it is considered as a modified American put option, the value of which on its maturity date is given by max (T_t, K) ; thus,

$$f_{j,k}^{(N)} = \max\left(S_0 \prod_{w=1}^{k-1} h_w^w \prod_{w=1}^{N-k} l_k^{N-w} * V_0 u^{j-1} d^{N-j}, K\right),$$

for $j = 1, 2, \ldots, i$ and $k = 1, 2, \ldots, i$. In addition, its discounted value is defined as follows,

$$f_{j,k}^{(i)} = \max\left[K, \left(q_j^{(i)} p_k^{(i)} f_{j+1,k+1}^{(i+1)} + \left(1 - q_j^{(i)}\right) p_k^{(i)} f_{j,k+1}^{(i+1)} + q_j^{(i)} \left(1 - p_k^{(i)}\right) f_{j+1,k}^{(i+1)} \right.\right.$$
$$\left.\left. + \left(1 - q_j^{(i)}\right)\left(1 - p_k^{(i)}\right) f_{j,k}^{(i+1)} \right) A_{jV}^{(i)^{-1}} A_S^{-1} \right]$$

Application example

Methodology

The methodology used in this paper is based on deductive logic and it is classified as quantitative, descriptive and non-experimental, besides as longitudinal respect time series (Hernández, Fernández, & Baptista, 2010). The methodology here presented rest on the use of a deductive logic and it is possible to classified as theorical, quantitative, non-experimental and longitudinal through time series analysis. It is formalized via econometric, mathematic and computational tools. We built a quadrinomial recombination in a non-constant volatility environment, that implied for its construction a rigorous mathematic deduction, which one was applied to value the real option of an oil sector project. Accordingly, the whole investigation could be considerate as an illustrative case, based on Hayes, Kyer and Weber (2015) from a quantitative point of view.

Data collection

The data series was extracted from the Bloomberg platform and its WTI (West Texas Intermediate) oil futures quoted prices with ticket CL1, with an operating contract for 1 000 barrels. The commodity has been listed on Bloomberg since March 30, 1983, but the sample used was obtained from January 1, 2013 to August 14, 2018, resulting in a total of 1 466 observations. Due to the liquidity of this type of contract, it is common to use it as a point of reference for the most of crude international prices. Figure 3 shows the commodity prices.

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Figure 3. WTI oil commodity prices series used as the sample.
Source: Taken from the Bloomberg platform (2018).

Data analysis

Table 1 presents some of the most relevant news that affected the commodity price in the market, which suggested important variations in the behavior of this variable.

Table 1
Main events that influenced the WTI oil price variation in the sample

Date	Event	Price (USD)
April 18, 2013	The price was the lowest for that year, due to low gasoline production and low oil demand	86.68
September 5, 2013	The price was the maximum for that year, because President Barack Obama spoke about possible military intervention in Syria	108.37
January 9, 2014	The price fell and broke the USD 100 barrier, due to a rebound effect that resulted from high speculation	91.66
March 17, 2015	The market showed a downward trend since 2014, and the price reached the USD 40 level. The explanation was related to the weakness of the dollar value and the expectation of the policy report of the Federal Reserve	43.46
August 24, 2015	After a brief rise in the oil price since April of that year, it collapsed because investors were waiting for information about stockpiles in the US	38.24

February 11, 2016	This was the lowest price for the entire data series, mainly due to oversupply in the market and the level of the barrel inventory in Cushing, Oklahoma (the largest stockpile place in the US)	26.21
June 8, 2016	The price increased due to political conflicts in Nigeria that affected supply and reduced oil reserves in the US	51.23
August 2, 2016	The price fell below USD 40 again, due to excessive supply of crude oil and low dollar price	39.51
June 21, 2017	This was the lowest price in 2017; again, a direct effect of oversupply, due to high levels of stockpiles in the US, and from this moment, the price showed relative growth until the date on which the data were extracted	42.53

Source: Prepared by the authors with information taken and adapted from the Bloomberg platform (2018).

Results

Price series results

Figures 4a and 4b (see Appendix) offer the descriptive statistics and a correlogram (Box, Jenkins, Reinsel, & Ljung, 2015) for the selected commodity price series; as it can be seen, there seems to be the first-lag autocorrelation, meaning that the price on one day is correlated with that on the next. As a complement, Table 2 presents the finding formalized with the Breush–Godfrey test (Breusch, 1978; Godfrey, 1978), In this case, we detect autocorrelation of order 1 in the price residuals.

Table 2
Breusch–Godfrey serial correlation LM test

Variable	Coefficient	Std error	t-statistic	Prob.
C	-7.76And-05	0.034304	-0.002262	0.9982
RESID(-1)	0.963292	0.026130	36.86560	0.0000
RESID(-2)	0.035201	0.026130	1.347177	0.1781

Note: Two lags are included
Source: Prepared by the authors using EViews.

The result shows that the alternative hypothesis is not rejected and that there is a serial correlation of some order, in this case, for the first lag. Based on this, we can define, analyze, and model the variable using a system of stochastic differential equations of the GARCH-diffusion type. Once the equation for the price has been defined, the yields are estimated as an alternative to detect conditional volatility, as stated in the following Equation, which is

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derived from (1.3): $R_{t_i} = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}\sqrt{\Delta t_i}} = \mu\sqrt{\Delta t_i} + \sqrt{V_{t_{i-1}}}\eta_{t_{i-1}}$, where S_{t_i} is the price in period t_i, $S_{t_{i-1}}$, is the price in period t_{i-1}; r; is the risk-free rate; Δt_i is the time in period t_i; $V_{t_{i-1}}$; is the stochastic volatility process in period t_{i-1}; and $\eta_{t_{i-1}}$ is a random number with N (0,1) in period t_{i-1}; in addition,

$$\sqrt{V_{t_{i-1}}}\eta_{t_{i-1}} = h_{t_{i-1}}\eta_{t_{i-1}} = \varepsilon_{t_{i-1}} \tag{1.8}$$

Thus,

$$R_{t_i} = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}\sqrt{\Delta t_i}} = \mu\sqrt{\Delta t_i} + \varepsilon_{t_{i-1}} \tag{1.9}$$

Yield series results

In this section, we present the analysis of the WTI oil yield series estimated from Equation (1.9). We present the descriptive statistics (Figure 5a), and graphically check stationarity (Figure 5b), autocorrelation (Figure 6a), and heteroskedasticity (Figure 6b); suggesting conditional volatility, it should be estimated through ARCH models. As a complement, some tests are presented below, to verify the previous characteristics. The following statistical tests were performed: the Kwiatkowski–Phillips–Schmidt–Shin test (Kwiatkowski, Phillips, Schmidt, & Shin, 1992), to detect if the series was stationary on average; the Breush–Godfrey test, to detect presence of serial autocorrelation; and the Ljung–Box Q test (Ljung & Box, 1978) and Engle ARCH test (Engle, 1982), to check for heteroskedasticity in the series. The results are as follows:

Table 3
Kwiatkowski–Phillips–Schmidt–Shin test

		LM stat.
Kwiatkowski–Phillips–Schmidt–Shin test statistic		0.088610
Asymptotic critical values:	1% level	0.216000
	5% level	0.463000
	10% level	0.347000

Note: The trend and intercept are included.
Source: Prepared by the authors using EViews

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Table 4
Breusch–Godfrey serial correlation LM test

Variable	Coefficient	Std error	t-statistic	Prob.
C	6.69And-06	0.008658	0.000773	0.9994
RESID(-1)	-0.067052	0.026155	-2.563647	0.0105
RESID(-2)	-0.003275	0.026156	-0.125199	0.9004

Note: Two lags are included.
Source: Prepared by the authors using EViews

Table 5
Ljung–Box test

Stat.	p-value	c-value
127.8451	0	3.8415

Note: One lag is included.
Source: Prepared by the authors using MATLAB.

Table 6
Engle ARCH test

Variable	Coefficient	Std error	t-statistic	Prob.
C	0.077605	0.007062	10.98838	0.0000
RESID^2(-1)	0.295272	0.024988	11.81659	0.0000

Note: One lag is included.
Source: Prepared by the authors using EViews.

The results were evaluated with an Alpha value of 10%. Thus, the series is apparently stationary on average, since the null hypothesis should not be rejected for any of the critical values (Table 3); the series also apparently has autocorrelation of order 1, because the one residual lag is significant (Table 4); and, finally, there is heteroskedasticity as indicated by the two tests (Ljung–Box and Engle ARCH). We have satisfactory evidence to reject the null hypothesis that the series does not have significant ARCH effects (Tables 5 and 6).

Findings

Parameter estimation corresponding to the GARCH (1,1) process

Once the statistical tests had been applied to the yield series, we move on to estimate the value of each of the parameters using a GARCH (1,1) model as presented in Equation (1.2) to find their equivalents in the GARCH-diffusion model, as presented later.

Table 7
GARCH (1,1) estimation process for the WTI oil yield series

Variable	Coefficient	Std error	z-statistic	Prob.
Variance Equation				
C	0.000496	0.000224	2.219792	0.0264
RESID(-1)^2	0.061478	0.008451	7.274757	0.0000
GARCH(-1)	0.935966	0.008894	105.2315	0.0000

Note: With unconditional prevariance.
Source: Prepared by the authors using EViews.

Table 7 shows the results and indicates that the coefficients are significant with an Alpha value of 10%; thus, the estimation has the following structural form:

$$h_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}^2;\ \ h_t^2 = 0.000496 + 0.061478\varepsilon_{t-1}^2 + 0.935966 h_{t-1}^2$$

Figure 7a exhibits the WTI oil yield conditional volatility for the established dates; in addition, the yield residual squared correlogram was estimated to verify that there was white noise (Figure 7b).

It can be seen in Figure 7a, that there are three volatility clusters defined: The first has low volatility from 2013 to mid-2014, the second has high volatility until the end of 2016, and the third shows some lower volatility until the end. On the right side, Figure 7b presents the residuals squared correlogram, in which the series contains white noise, which is completely random. To conclude this section, the estimation of the WTI oil price versus the real prices for the given range is presented below. The forecast was constructed from the information offered by the GARCH (1,1) conditional volatility process and then a random number series was generated to estimate the respective residuals, as shown in Equations (1.8) and (1.9), the forecast is presented in Figure 8.

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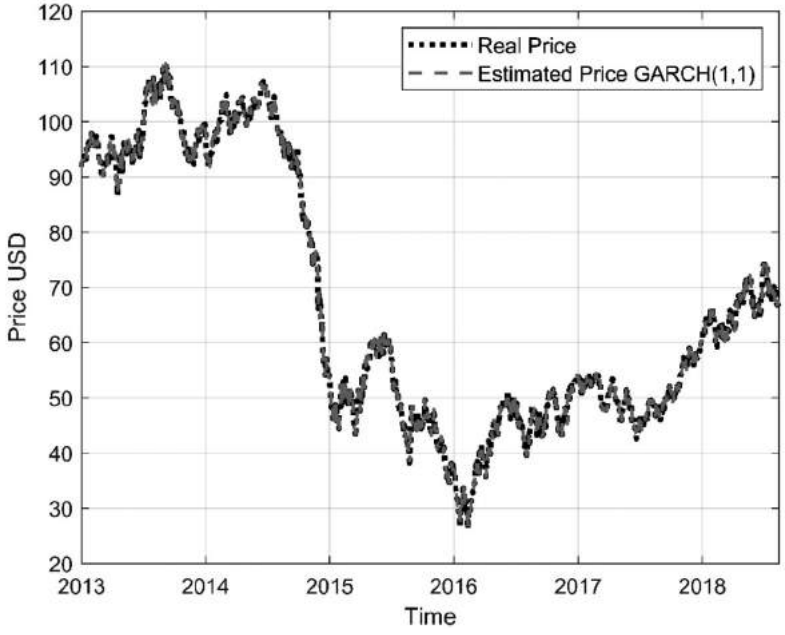

Figure 8. WTI oil real prices versus estimated prices using the conditional volatility of GARCH (1,1) model.
Source: Prepared by the authors using Matlab.

As it can be observed in Figure 8, the estimated prices and the real ones were close to each other, and to verify this result, the mean square error was used. The result of 1.61 indicates that the estimation was robust and followed the dynamics of the process. The estimated prices were built using Equations (1.8) and (1.9), as follows: $\hat{S}_{t_i} = S_{t_{i-1}} + \hat{\mu} S_{t_{i-1}} \Delta t_i + S_{t_{i-1}} \sqrt{\Delta t_i} \hat{h}_{t_{i-1}} \eta_{t_{i-1}}$.

Equivalence between the values of the parameters of the GARCH (1,1) model and those of the GARCH-diffusion model

From the equivalence between the variance of processes detailed in section 3, Table 8 offers the values of the parameters between the GARCH (1,1) model and the GARCH-diffusion model using Equations: $\alpha = \frac{(1-\alpha_1-\beta_1)}{\Delta t}$, $\theta = \frac{\alpha_0}{1-\alpha_1-\beta_1}$, and $\sigma = \frac{\alpha_1\sqrt{2}}{\sqrt{\Delta t}}$. The values in last column (Table 8), will be utilized to model non-constant volatility in the ROA case presented in the following section.

http://dx.doi.org/10.22201/fca.24488410e.2021.2331

Table 8
Equivalent values of the parameters for the GARCH (1,1) and GARCH-diffusion processes

Method				
	GARCH (1,1)		GARCH-diffusion	
Parameters	α_0	0.000496	α	0.002556
	α_1	0.061478	θ	0.194127
	β_1	0.935966	σ	0.086942

Source: Prepared by the authors using EViews and MATLAB.

Real option valuation: a counterfactual case

As a counterfactual case, in this section we discuss a real option valuation using the algebraic expression of an option of abandonment, as described in section 5.1. To find the value of the strategic NPV, the numerical method was employed by means of quadrinomial trees with the stochastic volatility model presented in section 4 and considering the values estimated for the GARCH-diffusion model exhibited in Table 8, which are the parameters of Equation (1.4). Finally, the solution was compared with the estimated results of the traditional multiplicative binomial method.

Suppose the oil company "X" based its efforts exclusively to upstream process. This firm signed an exploration and production contract (E&P), where a determinate area was assigned to them, which one is used to exploration, evaluation and oil extraction. The development of contract E&P counts with a geological and geophysical analysis and wanted to know the strategic NPV value of a project according to the ROA methodology and that its cash flows can be modeled using Equations (1.3) and (1.4). Assume the WTI oil prices used to estimate the cash flows are perfectly correlated with the project and volatility is the same that project without administrative flexibility has (Brandão, Dyer, & Hahn, 2012). The estimated static NPV of the project is affected by the technical probability and corresponds to $S_0 = T_0 = 91.82$, which was calculated according to the traditional DCF methodology with an appropriate risk-adjusted rate. Besides that, is estimated the technical probability or the geological success depends on the occurrence probability of these factors, as basement, caprock, reservoir rock, and its own dynamics. The previous information allowed to find the technical probability as the probability product sequence mentioned before.

The firm also determined that it has an opportunity to assign the rights and property of the

project to a third party at the value of $K = 66$ when the market conditions are unfavorable. That is, this value corresponds to the salvage value and remains constant over the evaluation horizon of the firm's abandonment option (Pareja Vasseur & Cadavid Pérez, 2016). Assume, in addition, the following values that were obtained from the GARCH (1,1) estimation process of the WTI oil price yields between January 2013 and August 2018 and their equivalents for the GARCH-diffusion model (see Table 8): $V_0 = 0.194127$, $r = 0.0294$, $\alpha = 0.02556$, $\theta = 0.194127$, and $\sigma = 0.0869$, as well as $\sigma_{BS} = 0.3319$ was obtained with the standard deviation of the aforementioned yield series for the traditional multiplicative binomial methodology.

Table 9
Comparison of the value of the strategic NPV through the quadrinomial method and the binomial method

	Method/time	1	2	3	4	5
(1)	Quadrinomial tree	95.59	100.09	102.37	104.57	106.31
(2)	Binomial tree	91.88	96.93	96.96	99.99	100.01
(3)	Difference in percentage	4.04%	3.26%	5.58%	4.58%	6.30%

Source: Prepared by the authors using MATLAB and Excel.

The results of the strategic NPV for the periods from year one to year five, with annual steps, are summarized in Table 9. It can be seen in all the cases that value estimated by our proposed method is higher than the traditional one. This would indicate an undervaluation of the strategic NPV at estimating lower volatility than the one actually presented in the series. Specifically, the first row shows the values for our methodology; for example, for the fourth year, a strategic NPV is estimated at 104.57, with an abandonment option of 12.75, whereas using the traditional method with multiplicative binomial trees, that is, the second row, the results are 99.99 and 8.17, respectively. The last row of Table 9 shows the percentage differences between the NPV values of the two methods, concluding that there is an approximate average of 5% for all the years of the real option evaluation. It is important to state that, if constant volatility 44.06% had been used in the traditional methodology, which comes from $\sqrt{\theta}$ in the proposed method, the results between the two methodologies would be very close. This means that the real option value of the traditional method would be undervalued due to poorly estimation of volatility because such value is greater than estimated using the standard deviation of the yields.

Conclusions

The findings can be summarized as follows. First, the price analysis helps to detect first-lag autocorrelation, which suggested that there was evidence of conditional heteroskedasticity in the time series chosen as the sample. Second, with the price we detected three different volatility clusters for the WTI oil, two with moderate volatilities, the first between 2013 and 2014 and the second between 2017 and 2018, and one with high volatility for 2015 and 2016. This implies that volatility is not constant but dynamic and for that reason, it should be modeled in the latter form. Third, once price yields were estimated, it was possible to use a statistical test at the 10% significance Alpha level to look for stationarity, autocorrelation, and heteroskedasticity, suggesting that the price variations of WTI oil have conditional volatility depending on the time. Fourth, it was possible to estimate a GARCH process for the WTI oil commodity and model its variance using a GARCH (1,1) model. We found out that the residuals after the estimation were white noise, that is, completely random. Fifth, using mathematical and statistical development, it was possible to find an equivalence between the GARCH (1,1) conditional volatility and the stochastic variance of the GARCH-diffusion model. The latter provided elements to build and develop the numerical method by trees to assess options that are derived from the behavior of this commodity type. Sixth, it was possible to use an algebraic expression to depict the evolution of the price of an asset in presence of both dynamic and constant volatility in such a way that its effect is captured in an appropriate manner, to model the evolution or behavior of the underlying asset in the market. Seventh, it was concluded that the real option value is higher in our method than the binomial method, because "real" volatility is greater than that estimated by traditional simple standard yields deviation. It was also indicated that, when the traditional binomial method uses $\sqrt{\theta}$ as volatility, the values of the strategic NPV are similar.

Future research aims to analyze how it changes the quadrinomial method when there is a correlation between Brownian Motions. At the same time, using different commodities as real options applications with the proposed methodology is also necessary. The appropriate analysis is advocated to determine the initial volatility value for proposed method, since changes in this variable produces significant variations in real option assess.

Finally, this research connects the areas of econometrics and stochastic processes, verifying that there is existence of a relationship between variances of the GARCH (1,1) model and the GARCH-diffusion model; therefore, in this line of research, future research could demonstrate the equivalences that could exist between the different models of the ARCH family and

their equivalents in the stochastic differential equations system to obtain better estimates of derivatives' prices and accordingly to facilitate a deeper, developed, and efficient market.

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Annex

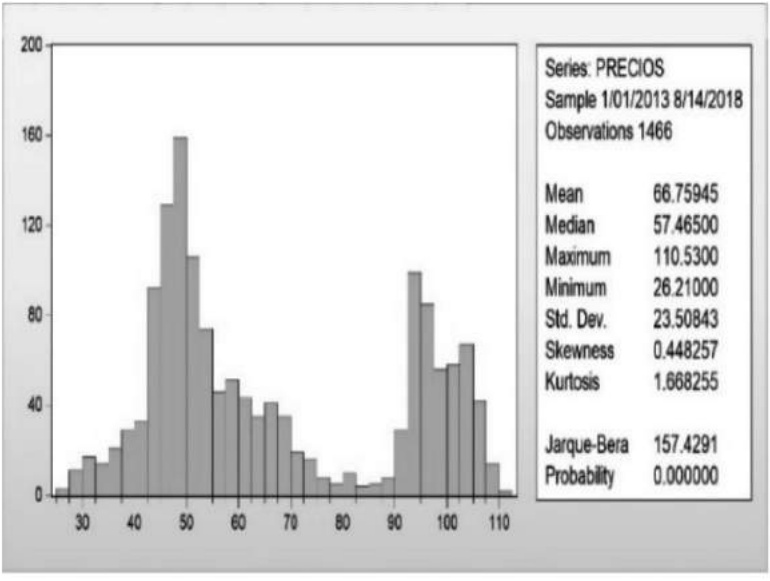
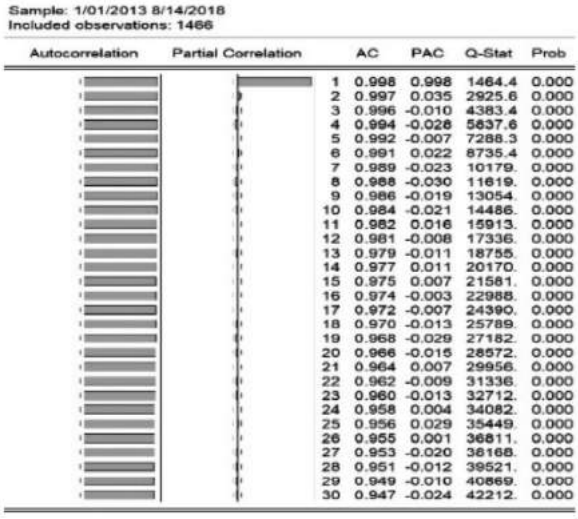


Figure 4a. WTI oil price descriptive statistics.
Source: Prepared by the authors using EViews.

http://dx.doi.org/10.22201/fca.24488410e.2021.2331


Sample: 1/01/2013 8/14/2018
Included observations: 1466

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.998	0.998	1464.4	0.000
		2	0.997	0.035	2925.6	0.000
		3	0.996	-0.010	4383.4	0.000
		4	0.994	-0.028	5837.6	0.000
		5	0.992	-0.007	7288.3	0.000
		6	0.991	0.022	8735.4	0.000
		7	0.989	-0.023	10179.	0.000
		8	0.988	-0.030	11619.	0.000
		9	0.986	-0.019	13054.	0.000
		10	0.984	-0.021	14486.	0.000
		11	0.982	0.016	15913.	0.000
		12	0.981	-0.008	17336.	0.000
		13	0.979	-0.011	18755.	0.000
		14	0.977	0.011	20170.	0.000
		15	0.975	0.007	21581.	0.000
		16	0.974	-0.003	22988.	0.000
		17	0.972	-0.007	24390.	0.000
		18	0.970	-0.013	25789.	0.000
		19	0.968	-0.029	27182.	0.000
		20	0.966	-0.015	28572.	0.000
		21	0.964	0.007	29956.	0.000
		22	0.962	-0.009	31336.	0.000
		23	0.960	-0.013	32712.	0.000
		24	0.958	0.004	34082.	0.000
		25	0.956	0.029	35449.	0.000
		26	0.955	0.001	36811.	0.000
		27	0.953	-0.020	38168.	0.000
		28	0.951	-0.012	39521.	0.000
		29	0.949	-0.010	40869.	0.000
		30	0.947	-0.024	42212.	0.000

Figure 4b. WTI oil price residuals autocorrelation and partial autocorrelation correlogram.
Source: Prepared by the authors using EViews.

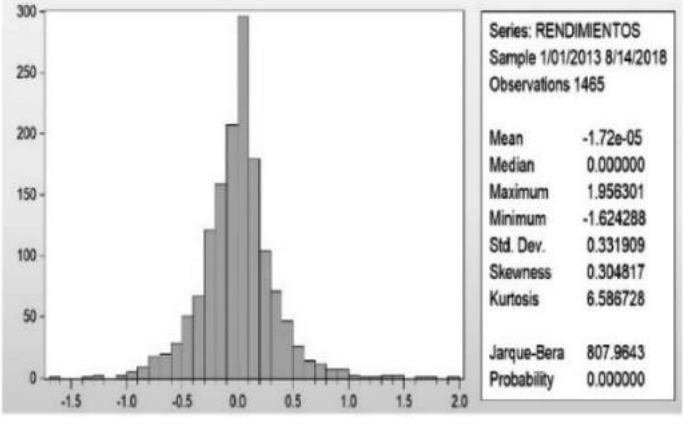


Figure 5a. Descriptive statistics.
Source: Prepared by the authors using EViews.

http://dx.doi.org/10.22201/fca.24488410e.2021.2331

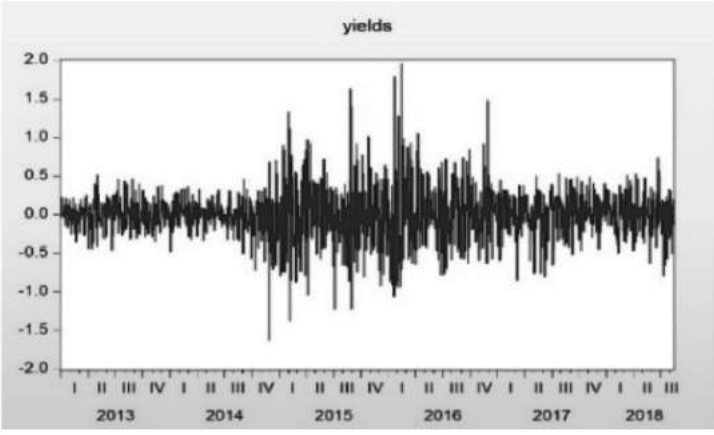


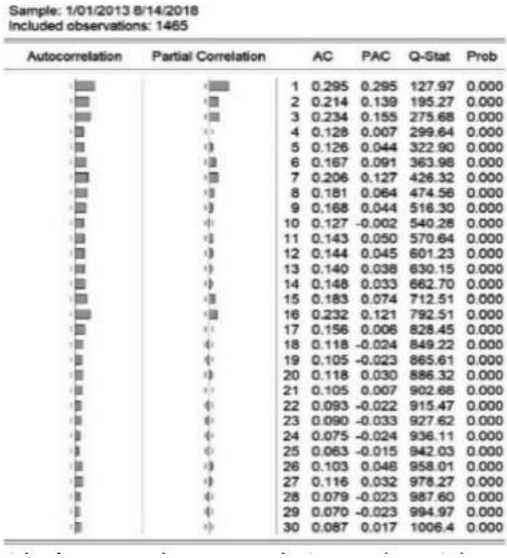
Figure 5b. Yield graph.
Source: Prepared by the authors using EViews.

Sample: 1/01/2013 8/14/2018
Included observations: 1465

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.067	-0.067	6.5555	0.010
		2	0.001	-0.003	6.5576	0.038
		3	0.039	0.039	8.7651	0.033
		4	0.001	0.006	8.7655	0.067
		5	-0.021	-0.021	9.4422	0.093
		6	0.023	0.019	10.227	0.115
		7	0.021	0.024	10.867	0.145
		8	-0.002	0.003	10.873	0.209
		9	0.009	0.007	10.991	0.276
		10	-0.002	-0.003	10.997	0.358
		11	-0.023	-0.022	11.751	0.383
		12	-0.030	-0.033	13.087	0.363
		13	-0.002	-0.007	13.092	0.441
		14	-0.002	-0.001	13.099	0.519
		15	0.023	0.025	13.899	0.533
		16	0.031	0.034	15.336	0.500
		17	-0.001	0.003	15.337	0.571
		18	0.024	0.024	16.169	0.581
		19	-0.009	-0.007	16.292	0.638
		20	-0.047	-0.047	19.577	0.485
		21	0.015	0.007	19.892	0.528
		22	0.044	0.043	22.754	0.416
		23	0.016	0.024	23.136	0.453
		24	-0.011	-0.012	23.306	0.502
		25	0.007	-0.001	23.378	0.555
		26	0.028	0.032	24.586	0.542
		27	-0.005	0.006	24.625	0.595
		28	-0.018	-0.017	25.085	0.623
		29	0.008	0.002	25.177	0.669
		30	-0.002	-0.001	25.181	0.716

Figure 6a. WTI oil yield residuals autocorrelation and partial autocorrelation correlogram.
Source: Prepared by the authors using EViews.

http://dx.doi.org/10.22201/fca.24488410e.2021.2331


Sample: 1/01/2013 8/14/2018
Included observations: 1465

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.295	0.295	127.97	0.000
		2	0.214	0.139	195.27	0.000
		3	0.234	0.155	275.68	0.000
		4	0.128	0.007	299.64	0.000
		5	0.126	0.044	322.90	0.000
		6	0.167	0.091	363.98	0.000
		7	0.206	0.127	426.32	0.000
		8	0.181	0.064	474.56	0.000
		9	0.168	0.044	516.30	0.000
		10	0.127	-0.002	540.28	0.000
		11	0.143	0.050	570.64	0.000
		12	0.144	0.045	601.23	0.000
		13	0.140	0.038	630.15	0.000
		14	0.148	0.033	662.70	0.000
		15	0.183	0.074	712.51	0.000
		16	0.232	0.121	792.51	0.000
		17	0.156	0.006	828.45	0.000
		18	0.118	-0.024	849.22	0.000
		19	0.105	-0.023	865.61	0.000
		20	0.118	0.030	886.32	0.000
		21	0.105	0.007	902.68	0.000
		22	0.093	-0.022	915.47	0.000
		23	0.090	-0.033	927.62	0.000
		24	0.075	-0.024	936.11	0.000
		25	0.063	-0.015	942.03	0.000
		26	0.103	0.046	958.01	0.000
		27	0.116	0.032	978.27	0.000
		28	0.079	-0.023	987.60	0.000
		29	0.070	-0.023	994.97	0.000
		30	0.087	0.017	1006.4	0.000

Figure 6b. WTI oil yield residuals squared autocorrelation and partial autocorrelation correlogram.
Source: Prepared by the authors using EViews.