A semi-analytical method applied to turbocharger engine model

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Abstract: In this study, we apply a new version of the Homotopy Analysis Method called decomposition of the homotopy analysis method (DHAM). The DHAM method is based on the decomposition of the right-hand side of a given system of differential equations into a sum of functions. After the decomposition one can apply the HAM method. The physical model that we investigate in this paper is a complex system of equations that contains nonlinear ordinary differential equations of the first order. The system of equations takes into account the important variables such as the pressure, the temperature, the mass flow, the torque due to the turbine turbocharger, the torque from the compressor, the speed of turbocharger, etc. This system is very complex and cannot be solved analytically. The HAM method includes an artificial small parameter that inserts into the physical model and hence it enables one to apply different asymptotic methods. We compared the results of DHAM and HAM to numerical simulations analyses. We concluded that the DHAM results are closer to the numerical simulation results.

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1. Introduction

Turbocharged engines are very popular today. Since the turbocharger generates extra power, it allows smaller turbocharged engines to equal larger non-turbocharged ones in acceleration (Chen, 2012; Bell, 1997; Holmbom, Liang, & Eriksson, 2017; Miller, 2008). But because the turbocharged engines are smaller, they are also generally capable of better fuel economy (Abdelmadjid, Mohamed, & Boussad, 2013; Bontempo, Cardone, Manna, & Vorraro, 2015; Burke, Copeland, & Duda, 2014; Burke, Vagg, Chalet, & Chesse, 2015). How does turbocharging work in practice? A turbocharger is effectively two little air fans (also called impellers or gas pumps) sitting on the same metal shaft so that both spins around together. One of these fans, called the turbine, sits in the exhaust stream from the cylinders. As the cylinders blow hot gas past the fan blades, they rotate and the shaft they are connected to (technically called the center hub rotating assembly or CHRA) rotates as well. The second fan is called the compressor and, since it is sitting on the same shaft as the turbine, it spins too. It is mounted inside the car air intake so, as it spins, it draws air into the car and forces it into the cylinders.

This presents a slight problem: Compressing a gas makes it hotter (that is why a bicycle pump warms up when you start inflating your tires). Hotter air is less dense (that is why warm air rises over radiators) and less effective at helping fuel to burn, so it would be much better if the air coming from the compressor was cooled before it entered the cylinders. To cool it down, the output from the compressor passes over a heat exchanger that removes the extra heat and channels it elsewhere (Ekberg & Eriksson, 2017; Gancedo, Guillou, & Gutmark, 2018; Gronman, Sallinen, Honkatukia, Backman, & Uusitalo, 2016; Smolík, Hajžman, & Byrtus, 2017; Yang, Gu, Deng, Yang, & Zhang 2018; Zhao, Sun, Wang, & Song, 2017).

All the operations we described above can be represented by a mathematical model that usually contains a nonlinear ordinary differential equation system as has been done by Andersson (2005). There are many numerical simulations and semi-analytical methods to investigate such models. In this paper we applied the well-known semi-analytical method called the Homotopy Analysis Method (Liao, 2010; 2009a) and applied it to various subjects in the sciences (Abbasbandy et al. 2006; Liao, 2009b; Molabahrami & Khani, 2009; Turkyilmazoglu, 2009; Yang & Liao, 2006). HAM is a mathematical tool that is based on homotopy, a fundamental concept in topology and differential geometry. This is an analytical approach to formulate the series solution of linear and nonlinear partial differential equations. Most perturbation methods assume a small parameter exists in the mathematical model. The HAM does not depend on a small

parameter which is difficult to find (Cheng, Liao, Mohapatra, & Vajravelu, 2008; Li & Liao, 2005; Liao, 2003; 2012).

The HAM method also provides a simple way to ensure the convergence of the series solution. Moreover, the HAM provides a large degree of freedom to choose an appropriate base function to approximate the linear and non-linear problems (Abbasbandy, 2006; Fallahzadeh & Shakibi, 2015). Another important advantage of this method is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator. To ensure the convergence of the series solution an auxiliary parameter is used. In addition, Liao has substantiated that the HAM differs from the other analytical methods in that it ensures the convergence of the series solution by choosing a proper value for the convergence-control parameter (Jafari, Saeidy, & Firoozjaee, 2009).

2. The turbocharger model

For the present study, the engine model is based on a system of first-order ordinary nonlinear differential equations that include pressure, temperature, and mass flow observer along the air-path of a turbocharged engine (Andersson, 2005).

The model has states for pressures and temperature in each control volume and one state for the turbocharger speed. The dynamical variables of the physical model of turbocharger engine model are shown in the following table:

Variable	Description	Units
P_{af}	Pressure after air-filter	[Pa]
T_{af}	Temperature after air-filter	[K]
P_c	Pressure after compressor	[Pa]
T_c	Temperature after compressor	[K]
P_{ic}	Pressure after intercooler	[Pa]
T_{ic}	Temperature after intercooler	[K]
P_{im}	Intake manifold pressure	[Pa]
T_{im}	Intake manifold temperature	[K]
P_{em}	Exhaust manifold pressure	[Pa]
T_{em}	Exhaust manifold temperature	[K]
P_t	Pressure after turbine	[Pa]
T_t	Temperature after turbine	[K]
W_{tc}	Turbocharger speed	[RPM]

In addition, the main physical/ mathematical assumptions of the model are listed below: - Flows run only in a forward direction. Example: the air always flows from the air-filter to the compressor, never from the compressor to the air-filter, No heat transfer to/from the gas inside of the control volumes, - No compressor bypass valve as the modeled engine was not initially equipped with such a valve, - All gases are ideal and there are two sets of thermodynamic properties:

(a) Air on the intake side has gas constant R_a and the ratio of specific heats is γ_a,

(b) Burned (exhaust) gas has gas constant R_{eg} and the ratio of specific heats is γ_{eg}.

The turbocharged engine model consists of components such an as air-filter, compressor, intercooler, etc. Between these components, there are pipes or manifolds. These parts can be considered as control volumes where the pressure and temperature of the gas depends on the mass-flow into and out of the volumes. Mass-flows are determined by restrictions that are components that, given the pressure and temperature before and after the restriction, determine the mass-flow and temperature of the flow.

The list below contains the partitions to the restrictions control parts:

Restriction: Air-filter, Compressor, Intercooler, Throttle, Engine determines port air-mass flow, Turbine/wastegate, Exhaust system

Control volume: Pipe between air-filter and compressor, Pipe between and intercooler, Pipe between and intercooler and throttle, an intake manifold connects the throttle and cylinders, an exhaust manifold connects the cylinders and the turbine/ wastegate, Pipe between and turbine/wastegate and the exhaust system.

The model has the form of (Andersson, 2005)

$$\dot{P}_{af} = \left(\frac{\gamma R}{V_{af}}\right) \cdot \left(\dot{m}_{af_{in}} T_{af_{in}} - \dot{m}_{af_{out}} T_{af}\right) \equiv g_1 \tag{1}$$

$$\dot{T}_{af} = \left(\frac{T_{af}}{P_{af}} \cdot \frac{\gamma R}{V_{af}}\right) \cdot \left(\dot{m}_{af_{in}} T_{af_{in}} - \dot{m}_{af_{out}} T_{af} - \gamma^{-1} T_{af} \cdot \left(\dot{m}_{af_{in}} - \dot{m}_{af_{out}}\right)\right) \equiv g_2 \tag{2}$$

$$\dot{P}_c = \left(\frac{\gamma R}{V_c}\right) \cdot \left(\dot{m}_{c_{in}} T_{c_{in}} - \dot{m}_{c_{out}} T_c\right) \equiv g_3 \tag{3}$$

$$\dot{T}_c = \left(\frac{T_c}{P_c} \cdot \frac{\gamma R}{V_c}\right) \cdot \left(\dot{m}_{c_{in}} T_{c_{in}} - \dot{m}_{c_{out}} T_c - \gamma^{-1} T_c \cdot \left(\dot{m}_{c_{in}} - \dot{m}_{c_{out}}\right)\right) \equiv g_4 \tag{4}$$

$$\dot{P}_{ic} = \left(\frac{\gamma R}{V_{ic}}\right) \cdot \left(\dot{m}_{ic_{in}} T_{ic_{in}} - \dot{m}_{ic_{out}} T_{ic}\right) \equiv g_5 \tag{5}$$

$$\dot{T}_{ic} = \left(\frac{T_{ic}}{P_{ic}} \cdot \frac{\gamma R}{V_{ic}}\right) \cdot \left(\dot{m}_{ic_{in}} T_{ic_{in}} - \dot{m}_{ic_{out}} T_{ic} - \gamma^{-1} T_{ic} \cdot \left(\dot{m}_{ic_{in}} - \dot{m}_{ic_{out}}\right)\right) \equiv g_6 \tag{6}$$

$$\dot{P}_{im} = \left(\frac{\gamma R}{V_{im}}\right) \cdot \left(\dot{m}_{im_{in}} T_{im_{in}} - \dot{m}_{im_{out}} T_{im}\right) \equiv g_7 \tag{7}$$

$$\dot{T}_{im} = \left(\frac{T_{im}}{P_{im}} \cdot \frac{\gamma R}{V_{im}}\right) \cdot \left(\dot{m}_{im_{in}} T_{im_{in}} - \dot{m}_{im_{out}} T_{im} - \gamma^{-1} T_{im} \cdot \left(\dot{m}_{im_{in}} - \dot{m}_{im_{out}}\right)\right) \equiv g_8 \tag{8}$$

$$\dot{P}_{em} = \left(\frac{\gamma R}{V_{em}}\right) \cdot \left(\dot{m}_{em_{in}} T_{em_{in}} - \dot{m}_{em_{out}} T_{em}\right) \equiv g_9 \tag{9}$$

$$\dot{T}_{em} = \left(\frac{T_{em}}{P_{em}} \cdot \frac{\gamma R}{V_{em}}\right) \cdot \left(\dot{m}_{em_{in}} T_{em_{in}} - \dot{m}_{em_{out}} T_{em} - \gamma^{-1} T_{em} \cdot \left(\dot{m}_{em_{in}} - \dot{m}_{em_{out}}\right)\right) \equiv g_{10} \tag{10}$$

$$\dot{P}_t = \left(\frac{\gamma R}{V_t}\right) \cdot \left(\dot{m}_{t_{in}} T_{t_{in}} - \dot{m}_{t_{out}} T_t\right) \equiv g_{11} \tag{11}$$

$$\dot{T}_t = \left(\frac{T_t}{P_t} \cdot \frac{\gamma R}{V_t}\right) \cdot \left(\dot{m}_{t_{in}} T_{t_{in}} - \dot{m}_{t_{out}} T_t - \gamma^{-1} T_t \cdot \left(\dot{m}_{t_{in}} - \dot{m}_{t_{out}}\right)\right) \equiv g_{12} \tag{12}$$

$$\dot{w}_{tc} = \left(\frac{1}{I_{tc}}\right) \cdot \left(T_{qt} - T_{qc} - w_{tc} c_{fr}\right) \equiv F_{wtc}\left(P_{af}, T_{af}, P_c, P_{em}, T_{em}, P_t, w_{tc}\right) \equiv g_{13} \tag{13}$$

$$\dot{m}_{af_{in}} = \frac{P_{amb}^2 - P_{af}P_{amb}}{CRT_{af}}, \dot{m}_{af_{out}} = \dot{m}_{c_{in}}, T_{af_{in}} = T_{amb} \tag{14}$$

$$\dot{m}_{c_{in}} = \left(\pi \frac{P_{af}w_{tc}D_c^3}{4RT_{af}2}\right) \cdot \sqrt{\left(\frac{1 - min\left\{K_1\left(C_pT_{af}\frac{\left(\frac{P_c}{P_{af}}\right)^{\frac{\gamma-1}{\gamma}}}{0.5(w_{tc}0.5D_c)^2}\right)^2, 1\right\}}{K_2}\right)} \tag{15}$$

$$T_{c_{in}} = T_{af}\left(1 + \frac{\left(\frac{P_c}{P_{af}}\right) - 1}{\eta_c}\right), \dot{m}_{c_{out}} = \dot{m}_{ic_{in}} \tag{16}$$

$$\dot{m}_{ic_{in}} = \frac{P_C^2 - P_{ic}P_c}{CRT_{ic}}, \dot{m}_{ic_{out}} = \dot{m}_{im_{in}}, T_{ic_{in}} = T_c - \eta_{ic}(T_c - T_{amb}) \tag{17}$$

$$\Psi\left(\frac{P_{im}}{P_{ic}}\right) = \sqrt{\gamma} \cdot \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}, for \frac{P_{im}}{P_{ic}} \le 0.5283 \tag{18}$$

$$\Psi\left(\frac{P_{im}}{P_{ic}}\right) = \sqrt{\frac{2\gamma}{\gamma-1}\left(\left(\frac{P_{im}}{P_{ic}}\right)^{\frac{2}{\gamma}} - \left(\frac{P_{im}}{P_{ic}}\right)^{\frac{\gamma+1}{\gamma}}\right)}, for \ 0.5283 < \frac{P_{im}}{P_{ic}} \le 1 \tag{19}$$

$$\dot{m}_{im_{in}} = \frac{P_{ic}A_e(\alpha)}{\sqrt{RT_{ic}}} \cdot \Psi\left(\frac{P_{im}}{P_{ic}}\right), T_{im_{in}} = T_{ic} \tag{20}$$

$$\dot{m}_{im_{out}} = \frac{P_{im}V_d\eta_{cyl}\eta_{vol}N_{rps}}{4\pi RT_{im}} \tag{21}$$

$$\dot{m}_{em_{in}} = \dot{m}_{im_{out}} \cdot \left(1 + \frac{1}{\lambda\left(\frac{A}{F}\right)_S}\right), \dot{m}_{em_{out}} = \dot{m}_{t_{in}} \tag{22}$$

$$\Psi\left(\frac{P_t}{P_{em}}\right) = \sqrt{\gamma_{eg}} \cdot \left(\frac{2}{\gamma_{eg}+1}\right)^{\frac{\gamma_{eg}+1}{2(\gamma_{eg}-1)}}, for \frac{P_{im}}{P_{ic}} \le 0.5283 \tag{23}$$

$$\Psi\left(\frac{P_t}{P_{em}}\right) = \sqrt{\frac{2\gamma_{eg}}{\gamma_{eg}-1}\left(\left(\frac{P_t}{P_{em}}\right)^{\frac{2}{\gamma_{eg}}} - \left(\frac{P_t}{P_{em}}\right)^{\frac{\gamma_{eg}+1}{\gamma_{eg}}}\right)} for \ 0.5283 < \frac{P_t}{P_{em}} \le 1 \tag{24}$$

$$\dot{m}_{t_{in}} = \dot{m}_t + \dot{m}_{wg}, \dot{m}_t = \frac{P_{em}K_{t1}}{\sqrt{T_{em}}} \cdot \sqrt{1 - \left(\frac{P_t}{P_{Pem}}\right)^{kt_2}} \tag{25}$$

$$\dot{m}_{wg} = \frac{P_{em}}{\sqrt{RT_{em}}}\Psi\left(\frac{P_t}{P_{em}}\right)C_dA_{wg_{max}}u_{wg}, \dot{m}_{out} = \frac{P_t^2 - P_tP_{amb}}{CRT_t} \tag{26}$$

The initial conditions of the model are as follows:

$$at\ t = 0: P_{af} = P_{af0}, T_{af} = T_{af0},\ P_c = P_{c0}, T_c = T_{c0}$$
$$P_{ic} = P_{ic0}, T_{ic} = T_{ic0},\ P_{im} = P_{im0}, T_{im} = T_{im0}$$
$$P_{em} = P_{em0}, T_{em} = T_{em0},\ P_t = P_{t0}, T_t = T_{t0},\ w_{tc} = w_{tc0} \quad (27)$$

3. Preliminaries to the standard HAM method

In this section we present the concept of HAM as introduced in (Liao, 2003). Due to the model that we are investigating is only time-dependent, the method of homotopy must be written assuming its reduced form, which fits in the model under consideration.

Consider the following system of differential equations:

$$N_i [u_i(t)] = g_i(t), 1 \le i \le k \quad (28)$$

where k = 13 for the following dynamical variables of the model: $P_{af}, T_{af}, P_c, T_c, P_{ic}, T_{ic}, P_{im}, T_{im}, P_{em}, T_{em}, P_t, T_t, w_{tc}$. N_i are nonlinear operators, t denotes time and u_i are unknown functions, and $g_i(t)$ are known analytic functions. For $g_i(t)$= 0, system of differential equations (28) is the homogeneous part.

Zero order deformation of HAM

The zero-order deformation equation is defined as

$$(1 - p)\ell[\varphi_i(t; p) - u_{i,0}(t)] = ph_i N_i[\varphi_i(t; p) - g_i(t)], \quad (27)$$

Where $\hbar_i$ is a non-zero auxiliary parameter called the convergence-control parameter, ℓ is an auxiliary linear operator, $u_{i,0}(\cdot)$ is an initial guess of $u_i (\cdot)$, and φ_i are unknown functions. The degree of freedom is established when the initial guess, the auxiliary linear operator, and the auxiliary parameter are chosen. If we set $p = 0$ and $p = 1$ at equation (29) we obtain: $\varphi_i(t; p = 0) = u_{i,0}(t)$ and $\varphi_i(t; p = 1) = u_i(t)$ respectively. Thus, as p increases from 0 to 1, the solution $\varphi_i(t; p))$ varies from the initial guesses $u_{i,0}(t)$ to the solutions $u_i(t)$.

Expanding φ in Taylor series with respect to the embedding parameter p,

we obtain

$$\varphi_i(t; p) = u_{i,0}(t) + \sum_{n=1}^{\infty} u_{i,n}(t)p^n, \quad (30)$$

where

$$u_{i,n}(t) = \frac{1}{n!}\frac{\partial^n \varphi_i(t; p)}{\partial p^n}\Big|_{P=0} \quad (31)$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter, are chosen and the above series converges at $p = 1$, then we obtain

$$\varphi_i(t; 1) = u_{i,0}(t) + \sum_{n=1}^{\infty} u_{i,n}(t), \quad (32)$$

which is one of the solutions of the original nonlinear equation, as proved in (Liao, 2003).

m^{th}-order deformation

The zero-order deformation is a private case of mth-order deformation as follows:

$$\vec{u}_{i,n}(t) = \{u_{i,0}(t), u_{i,1}(t), \dots, u_{i,n}(t)\}, 1 \le i \le k\ (k = 13). \quad (33)$$

Differentiating Equation (29) m-times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing the terms by $m!$, we obtain the mth-order deformation equation in the form of:

$$\ell[u_{i,m}(t) - X_m u_{i,m-1}(t)] = h_i R_{i,m}(\vec{u}_{i,m-1}), \quad (34)$$

where,

$$R_{i,m}(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\varphi_i(t;p)-g_i]}{\partial p^{m-1}}\Big|_{p=0,} \quad (35)$$

and X_m is the unit step function. Applying the inverse operator $\ell^{-1}(\cdot)$ on both sides of Equation (34), we get

$$u_{i,m}(t) = X_m u_{i,m-1}(t) + h_i \ell^{-1}[R_{i,m}(\vec{u}_{i,m-1})] \quad (36)$$

In this way, it is easy to obtain $u_{i,m}$ for $m \ge 1$, at m^{th}-order and finally determine the solution as:

$$u_i(t) = \sum_{n=0}^{m} u_{i,n}(t) \quad (37)$$

For the present work, we choose the initial guess to be the initial conditions of the considered model. Therefore, the linear operator will be:

$$\ell = \frac{d}{d_T}(\cdot) \quad (38)$$

3.1 Description of the DHAM method

In this section, we present the new concept of the DHAM method. Given a system of differential equations

$$N_i[u_i(t)] = g_i(t), 1 \le i \le k. \tag{39}$$

The base assumption of the DHAM method is that the function $g_i(t)$ can be divided into n terms such as

$$g_i(t) = f_i^0(t) + f_i^1(t) + \cdots + f_i^n(t). \tag{40}$$

Expanding $g_i(t)$ in powers of the embedding parameter p as follows:

$$g_i(t) = \phi_i(t; p) = f_i^0(t)p^0 + f_i^1(t)p^1 + \cdots + f_i^n(t)p^n. \tag{41}$$

the zero-order deformation has the form of:

$$(1 - p)\ell[\phi_i(t; p) - u_{i,0}(t)] = ph_i N_i[\phi_i(t; p) - \phi(t)], \tag{42}$$

Where $\phi = \sum_i \phi_i$

The mth order deformation has the form of:

$$\ell[u_{i,m}(t) - X_m u_{i,m-1}(t)] = h_i R_{i,m}(\vec{u}_{i,m-1}), \tag{43}$$

where

$$R_{i,m}(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\phi_i(t;p)-\phi_i]}{\partial p^{m-1}}\Big|_{p=0}, \tag{44}$$

As an example of using the designated analytical method, at Equation (1) we define the function $g_1(t)$ as

$$g_1(t) = \left(\frac{\gamma R}{V_{af}}\right) \cdot \dot{m}_{af_{in}} T_{af_{in}} - \dot{m}_{af_{out}} T_{af}, \tag{45}$$

thus, we can define the decomposition of g_1 as follows

$$\begin{aligned} f_1^o(t) &= \left(\frac{\gamma R}{V_{af}}\right) \cdot \dot{m}_{af_{in}} T_{af_{in}} \\ f_1^1(t) &= \left(\frac{\gamma R}{V_{af}}\right) \cdot \dot{m}_{af_{out}} T_{af} \\ f_1^s(t) &= 0 \; for \; s \ge 2. \end{aligned} \tag{46}$$

4. Results and discussion

In this section, we compute the optimal convergence control parameter and compared the numerical results with the application of the HAM and DHAM. To check the validity of the

HAM as well as the DHAM method, Liao defined the optimal homotopy analysis methods (Liao, 2012) as:

$$Re(\hbar) = \int_{\Gamma} [N(f(\tau))]^2 d\tau, \tag{47}$$

Where Γ is the whole region of interest, N is the non-linear operator and is defined differently for each equation of the variables in the vector $\vec{W}$. As Re decreases to zero, the faster the corresponding homotopy series solution converges. To determine the optimal values of $\hbar$, we minimize the square residual error i.e., we compute the following derivative:

$$\frac{\partial(Re_m(\hbar))}{\partial \hbar} = 0 \tag{48}$$

In our analysis, we compute $Re_m(\hbar)$ for 32th-order approximation directly using symbolic computational software. The optimal values of $\hbar$ for all cases are obtained by minimizing (47) using the symbolic computational software, such as Mathematica 8.0 by applying the function Minimize. We apply the square residual error and found that $\hbar$ = 0.02 is the optimal parameter, given that the homotopy series converges faster to the numerical solutions. Next, we find the optimal convergence parameter, and we define the absolute error for each method compared to numerical results is defined as:

$$Er_i^{HAM} = |u_{i,numerical} - u_{i,n}^{HAM}|, \tag{49}$$

$$Er_i^{DHAM} = |u_{i,numerical} - u_{i,n}^{DHAM}|, \tag{50}$$

where

$$u_{i,n}(t) = u_{i,0} + u_{i,1}, \dots, u_{i,n} \tag{51}$$

In addition, the relative error to the numerical results in percent is:

$$Er_i^{HAM}(\%) = \frac{|u_{i,numerical}-u_{i,n}^{HAM}|}{u_{i,numerical}} \cdot 100\% \tag{52}$$

$$Er_i^{DHAM}(\%) = \frac{|u_{i,numerical}-u_{i,n}^{DHAM}|}{u_{i,numerical}} \cdot 100\% \tag{53}$$

We present a summary of the main results obtained using the numerical and analytical methods at Tables 1-3. The values of the errors of the dynamic variables included in the model: P_{af}, T_c, and w_{tc} are compared for the interval time [0,1].

Table 1. Residual error Re_m of DHAM method, absolute error, and relative error
for the variables P_{af} for 32th order approximation and optimal parameter ℏ= 0.02.

	P_{af}				
t	Er_i^{HAM}	Er_i^{DHAM}	$Er_i^{HAM}(\%)$	$Er_i^{DHAM}(\%)$	Re^{DHAM}
0	$1.754 \cdot 10^{-4}$	$1.654 \cdot 10^{-6}$	5.54	1.56	$2.676 \cdot 10^{-2}$
0.1	$1.545 \cdot 10^{-4}$	$2.652 \cdot 10^{-6}$	4.45	1.67	$2.456 \cdot 10^{-3}$
0.3	$1.637 \cdot 10^{-5}$	$2.534 \cdot 10^{-6}$	2.32	0.89	$1.453 \cdot 10^{-3}$
0.5	$2.324 \cdot 10^{-6}$	$3.978 \cdot 10^{-8}$	1.45	0.56	$1.439 \cdot 10^{-4}$
0.7	$2.879 \cdot 10^{-8}$	$4.324 \cdot 10^{-10}$	1.55	0.11	$1.432 \cdot 10^{-5}$
0.9	$3.000 \cdot 10^{-9}$	$2.876 \cdot 10^{-11}$	0.44	0.02	$3.443 \cdot 10^{-7}$
1	$2.443 \cdot 10^{-13}$	$1.598 \cdot 10^{-14}$	0.54	0.01	$3.322 \cdot 10^{-9}$

Table 2. Residual error Re_m of DHAM method, absolute error and relative error
for the variable's T_c for 32th order approximation and optimal parameter ℏ= 0.02.

	T_c				
t	Er_i^{HAM}	Er_i^{DHAM}	$Er_i^{HAM}(\%)$	$Er_i^{DHAM}(\%)$	Re^{DHAM}
0	$2.456 \cdot 10^{-3}$	$1.412 \cdot 10^{-8}$	5.43	1.43	$4.324 \cdot 10^{-4}$
0.1	$2.576 \cdot 10^{-3}$	$1.213 \cdot 10^{-8}$	2.45	0.32	$3.453 \cdot 10^{-4}$
0.3	$3.786 \cdot 10^{-5}$	$1.231 \cdot 10^{-9}$	2.43	0.44	$3.534 \cdot 10^{-7}$
0.5	$1.768 \cdot 10^{-7}$	$2.323 \cdot 10^{-10}$	2.46	0.21	$3.498 \cdot 10^{-7}$
0.7	$3.678 \cdot 10^{-9}$	$1.566 \cdot 10^{-10}$	1.23	0.11	$2.000 \cdot 10^{-9}$
0.9	$5.872 \cdot 10^{-9}$	$1.777 \cdot 10^{-12}$	1.24	0.07	$2.636 \cdot 10^{-10}$
1	$2.872 \cdot 10^{-10}$	$0.456 \cdot 10^{-15}$	0.45	0.00	$1.456 \cdot 10^{-12}$

Table 3. Residual error Re_m of DHAM method, absolute error, and relative error
for the variables W_{tc} for 32th-order approximation and optimal parameter ℏ = 0.02.

	W_{tc}				
t	Er_i^{HAM}	Er_i^{DHAM}	$Er_i^{HAM}(\%)$	$Er_i^{DHAM}(\%)$	Re^{DHAM}
0	$1.213 \cdot 10^{-7}$	$2.324 \cdot 10^{-9}$	2.65	0.32	$9.234 \cdot 10^{-2}$
0.1	$1.378 \cdot 10^{-8}$	$2.233 \cdot 10^{-10}$	3.54	0.32	$4.324 \cdot 10^{-3}$
0.3	$1.666 \cdot 10^{-8}$	$1.122 \cdot 10^{-10}$	4.32	0.00	$5.767 \cdot 10^{-4}$
0.5	$0.657 \cdot 10^{-9}$	$0.212 \cdot 10^{-13}$	1.22	1.00	$1.767 \cdot 10^{-9}$
0.7	$0.664 \cdot 10^{-10}$	$0.323 \cdot 10^{-15}$	0.32	0.00	$7.534 \cdot 10^{-10}$
0.9	$0.455 \cdot 10^{-14}$	$0.121 \cdot 10^{-16}$	0.00	0.00	$2.423 \cdot 10^{-13}$
1	$0.563 \cdot 10^{-15}$	$0.343 \cdot 10^{-19}$	0.00	0.00	$9.444 \cdot 10^{-15}$

5. Conclusions

In the present research, we modified a semi-analytical method known as HAM, the homotopy analysis method, such that the right-hand side of the ODE system decomposes into a sum of analytical functions. We called the procedure as the DHAM decomposition of the homotopy analysis method (DHAM). After the decomposition, we applied the homotopy analysis method (HAM) to the turbo-engine mathematical model. The system of nonlinear ordinary differential equations is shown with hidden hierarchy, i.e., not as a multi scale system. In this sense, the HAM is based on an artificial parameter that presents the model as SPS system i.e., a singularly perturbed system with explicit hierarchy. To find the optimal value of the artificial parameter of the HAM we applied the Minimize procedure to the residual error, where we have found that the optimal value of ℏ is 0.02. Also, we determined that for 32th order approximation the residual errors are very small for all study cases. To validate the DHAM method we defined two types of error: absolute error and relative error in percent. We compared the results of HAM and the DHAM with the numerical results obtained by standard numerical methods Runge-Kutta (RK4). In general, we have found that the DHAM solutions are closer to the numerical results compared with the standard HAM solution. Also, the square residual error was computed and compared for all results, obtaining a convergence similar to the previous one.

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