



Selecciones Matemáticas

ISSN: 2411-1783

selecmat@unitru.edu.pe

Universidad Nacional de Trujillo

Perú

Calderón, Calixto P.; Torchinsky, Alberto
The Hausdorff-Young Inequality for n-dimensional Hermite Expansions
Selecciones Matemáticas, vol. 9, núm. 02, 2022, Agosto-Diciembre, pp. 227-233
Universidad Nacional de Trujillo
Perú

DOI: <https://doi.org/10.17268/sel.mat.2022.02.01>

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The Hausdorff–Young Inequality for n –dimensional Hermite Expansions

La desigualdad de Hausdorff–Young para las expansiones de Hermite n -dimensionales

Calixto P. Calderón  and Alberto Torchinsky 

En reconocimiento a Alejandro Ortiz Fernández, por su amistad y sus contribuciones al análisis armónico.

Received, May. 08, 2022

Accepted, Set. 26, 2022



How to cite this article:

Calderón C, Torchinsky A. *The Hausdorff–Young Inequality for n –dimensional Hermite Expansions*. *Selecciones Matemáticas*. 2022;9(2):227–233. <http://dx.doi.org/10.17268/SEL.MAT.2022.02.01>

Abstract

We discuss a sharpened Hausdorff–Young inequality for n -dimensional Hermite expansions.

Keywords. Hausdorff–Young inequality, n -dimensional Hermite expansions.

Resumen

Consideramos una desigualdad de Hausdorff–Young refinada para expansiones de Hermite n -dimensionales.

Palabras clave. Desigualdad de Hausdorff–Young, expansiones de Hermite n -dimensionales.

1. Introduction. This note concerns the sharpened Hausdorff–Young inequality in the context of n -dimensional Hermite expansions. The corresponding 1–dimensional result was considered in [4].

The Hermite functions constitute an ONS in \mathbb{R} with respect to the Lebesgue measure there, and are defined as follows [5, 12, 14]. Szegő introduced the Hermite polynomials, $H_m(x)$, in Chapter V of [12]. Earlier, Hille had also considered the Hermite polynomials, and proved some remarkable formulas and estimates [5, 14]. In particular, Hille considered the generating formula

$$\sum_{m=0}^{\infty} H_m(x) \frac{u^m}{m!} = e^{2xu - u^2},$$

and defined the Hermite functions $\mathcal{H}_m(x)$, $m \geq 0$, by

$$\mathcal{H}_m(x) = \frac{1}{(m!)^{1/2}} \frac{1}{2^{m/2}} H_m(x) e^{-x^2/2}, \quad x \in \mathbb{R}.$$

The n –dimensional Hermite functions are obtained as products of the 1–dimensional Hermite functions [10, 14], and constitute an ONS in \mathbb{R}^n with respect to the Lebesgue measure there. To the point, given $x = (x_1, \dots, x_n)$ in \mathbb{R}^n and an n –tuple of nonnegative integers $m = (m_1, \dots, m_n)$, let the Hermite functions $\mathcal{H}_m(x)$ be given by

$$\mathcal{H}_m(x) = \mathcal{H}_{m_1}(x_1) \cdots \mathcal{H}_{m_n}(x_n).$$

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Now, for a function $f(x)$ defined on \mathbb{R}^n , the Hermite expansion of f is given by

$$f(x) \sim \sum_m C_m \mathcal{H}_m(x), \quad x \in \mathbb{R}^n,$$

where the Hermite coefficients of $f(x)$, C_m , are defined by

$$C_m = \int_{\mathbb{R}^n} f(x) \mathcal{H}_m(x) dx. \tag{1.1}$$

In order to verify that the n -dimensional Hermite functions are an ONS, it suffices to show that if all the Hermite coefficients of an $L^2(\mathbb{R}^n)$ function $f(x) \sim \sum_m C_m \mathcal{H}_m(x)$ vanish, then $f(x) = 0$ a.e. with respect to the Lebesgue measure on \mathbb{R}^n . For simplicity we assume that $n = 2$, and let $f(x_1, x_2)$ be such that $C_m = 0$ for all $m = (m_1, m_2)$ with $m_1, m_2 = 0, 1, 2, \dots$

Let then

$$\varphi_{m_2}(x_1) = \int_{\mathbb{R}} f(x_1, x_2) \mathcal{H}_{m_2}(x_2) dx_2, \quad x_1 \in \mathbb{R}, m_2 = 0, 1, 2, \dots,$$

and observe that all the Hermite coefficients C'_{m_1} of φ_{m_2} vanish. Indeed, with $m = (m_1, m_2)$,

$$\begin{aligned} C'_{m_1} &= \int_{\mathbb{R}} \varphi_{m_2}(x_1) \mathcal{H}_{m_1}(x_1) dx_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2) \mathcal{H}_{m_2}(x_2) dx_2 \mathcal{H}_{m_1}(x_1) dx_1 = C_m = 0. \end{aligned}$$

Hence, by the completeness of the Hermite functions in \mathbb{R} , $\varphi_{m_2}(x_1)$ vanishes a.e. with respect to the Lebesgue measure on the line for each $m_2 = 0, 1, 2, \dots$. Let E_{m_2} be the set of Lebesgue measure 0 in the line outside of which φ_{m_2} vanishes, and let $E = \bigcup_{m_2=0}^{\infty} E_{m_2}$; E is a set of Lebesgue measure 0 in \mathbb{R} .

Now, for each $x_1 \in \mathbb{R} \setminus E$, we have

$$\int_{\mathbb{R}} f(x_1, x_2) \mathcal{H}_{m_2}(x_2) dx_2 = 0, \quad m_2 = 0, 1, 2, \dots,$$

and by the completeness of the Hermite expansion in \mathbb{R} , $f(x_1, x_2) = 0$ a.e. x_2 in \mathbb{R} whenever $x_1 \in \mathbb{R} \setminus E$.

Then, by Tonelli's theorem, on account of the above observations it follows that

$$\int_{\mathbb{R}^2} |f(x)|^2 dx = \int_{\mathbb{R}} \int_E |f(x_1, x_2)|^2 dx_1 dx_2 + \int_{\mathbb{R} \setminus E} \int_{\mathbb{R}} |f(x_1, x_2)|^2 dx_2 dx_1 = 0,$$

and so, $f(x_1, x_2) = 0$ a.e. on \mathbb{R}^2 .

Thus, in particular, the n -dimensional Hermite expansions satisfy the Parseval–Plancherel formula in \mathbb{R}^n ,

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_m |C_m|^2,$$

and, in order to discuss the Hausdorff–Young inequality, we introduce some preliminary material concerning Lebesgue, Lorentz and Orlicz spaces.

2. Preliminaries. Given a function f defined on \mathbb{R}^n , with ν the Lebesgue measure on R^n , let $m(f, \lambda)$ denote the *distribution function* of f ,

$$m(f, \lambda) = \nu(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}), \quad \lambda > 0.$$

$m(f, \lambda)$ is nonincreasing and right continuous, and the *nonincreasing rearrangement* f^* of f defined for $t > 0$ by

$$f^*(t) = \inf\{\lambda : m(f, \lambda) \leq t\}, \quad \inf \emptyset = 0,$$

is informally its inverse (this statement is made precise in [9, p. 43]). f^* is nonincreasing and right continuous and, at its points of continuity t , $f^*(t) = \lambda$ is equivalent to $m(f, \lambda) = t$.

The *Lorentz space* $L^{p,q}(\mathbb{R}^n) = L(p, q)$, $0 < p < \infty$, $0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left(\frac{q}{p} \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q}, \quad 0 < q < \infty,$$

and,

$$\|f\|_{p,\infty} = \sup_{t>0} (t^{1/p} f^*(t)) = \sup_{\lambda>0} \lambda m(f, \lambda)^{1/p}, \quad q = \infty.$$

The Lorentz spaces are monotone with respect to the second index, that is, if $0 < q < q_1 \leq \infty$, then $L(p, q) \subset L(p, q_1)$, and

$$\|f\|_{p,q_1} \lesssim \|f\|_{p,q}, \tag{2.1}$$

with $L(p, p)$ the Lebesgue space $L^p(\mathbb{R}^n)$, and $L(p, \infty)$ the space weak- $L^p(\mathbb{R}^n)$.

As for the Lorentz sequence spaces, given n -tuples of non-negative integers m , and a sequence $c = \{c_m\}$, let $\{c_k^*\}$ denote the sequence obtained by ordering $\{|c_m|\}$ in a nonincreasing fashion. The Lorentz sequence space $\ell(p, q)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, consists of those sequences $c = \{c_m\}$ with finite quasinorm $\|c\|_{\ell^{p,q}}$ given by

$$\|c\|_{\ell^{p,q}} = \left(\sum_{k=1}^{\infty} (k^{1/p} c_k^*)^q \frac{1}{k} \right)^{1/q}, \quad 1 \leq q < \infty,$$

and, with μ the atomic measure concentrated on the lattice of n -tuples of nonnegative integer atoms m taking the value $\mu(m) = 1$ on each such atom,

$$\|c\|_{\ell^{p,\infty}} = \sup_{k \geq 1} k^{1/p} c_k^* = \sup_{\lambda > 0} \lambda \mu(\{m : |c_m| > \lambda\})^{1/p}, \quad q = \infty. \tag{2.2}$$

As for the Orlicz spaces, the letters A, B are reserved for Young's functions, i.e., for functions $A(t)$ defined for $t \geq 0$ that are zero at zero, increasing, and convex, or, more generally, $A(t)/t$ increasing to ∞ as $t \rightarrow \infty$. The Orlicz space $L^A(\mathbb{R}^n)$ consists of those measurable functions f (modulo equality a.e.) such that $\int_{\mathbb{R}^n} A(|f(x)|/M) dx < \infty$ for some M , normed by

$$\|f\|_A = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} A\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

The Orlicz sequence space ℓ^A consists of those sequence $c = \{c_m\}$ such that for some M ,

$$\sum_m A(|c_m|/M) < \infty,$$

normed by

$$\|c\|_{\ell^A} = \inf \left\{ \lambda > 0 : \sum_m A\left(\frac{|c_m|}{\lambda}\right) \leq 1 \right\}.$$

Finally, an operator T of a class of functions f on \mathbb{R}^n into a linear class of functions is said to be linear provided that, if T is defined for f_0, f_1 , and $\lambda \in \mathbb{R}$, then T is defined for $f_0 + \lambda f_1$, and $T(f_0 + \lambda f_1)(x) = T(f_0)(x) + \lambda T(f_1)(x)$.

A linear operator T defined for $f \in L^A(\mathbb{R}^n)$ and taking values $T(f) = \{c_m\}$ in ℓ^B is said to be bounded if there is a constant $K > 0$ such that

$$\sum_m B\left(\frac{|c_m|}{K}\right) \leq 1$$

, whenever

$$\int_{\mathbb{R}^n} A(|f(x)|) dx \leq 1.$$

A bounded operator T from ℓ^A to $L^B(\mathbb{R}^n)$ is defined similarly. In either case, the smallest K above is called the norm of T , is denoted by $\|T\|$, and the operator is said to be of type (A, B) . These operators satisfy $\|T(f)\|_{\ell^B} \lesssim \|f\|_A$, and $\|T(\{c_m\})\|_B \lesssim \|\{c_m\}\|_{\ell^A}$, respectively. When $A(t) = t^p$ and $B(t) = t^q$, we say that T is of type (p, q) . If the mapping T is from an $L^p(\mathbb{R}^n)$ space into an $L^q(\mathbb{R}^n)$, or a sequence space $\ell(q, \infty)$, the mapping is said to be of weak-type (p, q) . Similarly for mappings from ℓ^p into weak- $L^q(\mathbb{R}^n)$ spaces.

For further consideration of the Lorentz and Orlicz spaces the reader may consult [1, 6, 8, 9].

3. The Hausdorff–Young Inequality. The sharpened Hausdorff–Young inequality for $n = 1$ proved in [4, Theorem 4.1] rests on a remarkable estimate for the Hermite functions established by Hille [5, p. 436], [12, p. 240], to wit,

$$|\mathcal{H}_m(x)| \lesssim m^{-1/12}. \tag{3.1}$$

Hille notes that (3.1) is the best possible estimate, but that in applications he will only use the weaker formula $|\mathcal{H}_m(x)| \lesssim 1$. On the other hand, as in [4], we will use (3.1) to obtain a sharpened Hausdorff–Young inequality for Hermite expansions on \mathbb{R}^n . We refer to these estimates as sharpened because they are of type (p, q) with $q < p'$.

We then have,

Theorem 3.1. *Let $f(x) \sim \sum_m C_m \mathcal{H}_m(x)$ denote the expansion of a function f defined on \mathbb{R}^n in a Hermite series, and let T be the mapping that assigns to f its sequence of Hermite coefficients $\{C_m\}$. Then, T maps the Lorentz space $L(p, s)$ continuously into the Lorentz sequence space $\ell(q, s)$, $1 \leq s \leq \infty$, provided that p, q verify*

$$1 < p < 2, \quad \text{and,} \quad \left(1 - \frac{1}{6n}\right) \frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{12n}. \tag{3.2}$$

In particular,

$$\|T(f)\|_{\ell^q} = \|\{C_m\}\|_{\ell^q} \lesssim_p \|f\|_p, \tag{3.3}$$

and T is of type (p, q) whenever (3.2) holds.

Moreover, if A, B are Young’s functions such that $\int_0^t B(s)/s^{12n} ds/s \lesssim B(t)/t^{12n}$ and $B(t)/t^2$ increases, T is of type (A, B) , provided that A, B verify

$$B^{-1}(t) = t^{((12n-1)/12n)} A^{-1}\left(t^{((1-6n)/6n)}\right), \quad t > 0. \tag{3.4}$$

Proof: For simplicity, since no new ideas are required for general n , we will carry out the proof for $n = 2$. Let $f(x) \sim \sum_m C_m \mathcal{H}_m(x)$ denote the Hermite expansion of f in a Hermite series, where the $\{C_m\}$ are defined as in (1.1) above.

Note that since $\mathcal{H}_0(t) = 1$, it follows that if $m_0 = (0, 0)$, then $|C_{m_0}| \leq \|f\|_1$, and also that

$$|C_m| \lesssim \|f\|_1 m_1^{-1/12}, \quad m = (m_1, 0), m_1 \geq 1, \tag{3.5}$$

and,

$$|C_m| \lesssim \|f\|_1 m_2^{-1/12}, \quad m = (0, m_2), m_2 \geq 1. \tag{3.6}$$

Also, for $m = (m_1, m_2)$ with $m_1 \cdot m_2 \neq 0$, we have

$$|C_m| \lesssim \|f\|_1 (m_1 m_2)^{-1/12}, \quad m = (m_1, m_2), m_1 \cdot m_2 \neq 0. \tag{3.7}$$

Let μ denote the atomic measure concentrated on the lattice of 2-tuples of integer atoms $m = (m_1, m_2)$ with $m_1, m_2 = 0, 1, 2, \dots$, taking the value $\mu(m) = 1$ on each such atom.

Given $\lambda > 0$, let $\mathcal{I}_\lambda = \{m : |C_m| > \lambda\}$. Now, if $m = (m_1, m_2)$ is in \mathcal{I}_λ and $m_1 \cdot m_2 \neq 0$, by (3.7) we have

$$\lambda < |C_m| \lesssim \|f\|_1 (m_1 m_2)^{-\frac{1}{12}},$$

and, consequently,

$$m_1 m_2 \leq \left(\|f\|_1 / \lambda\right)^{12},$$

which, since $m_1, m_2 \geq 1$ implies that

$$m_1 \lesssim \left(\|f\|_1 / \lambda\right)^{12}, \quad m_2 \lesssim \left(\|f\|_1 / \lambda\right)^{12}.$$

Hence,

$$\begin{aligned} \mu(\{m = (m_1, m_2) \in \mathcal{I}_\lambda : m_1 \cdot m_2 \neq 0\}) \\ \lesssim \left(\|f\|_1 / \lambda\right)^{12} \left(\|f\|_1 / \lambda\right)^{12} = \left(\|f\|_1 / \lambda\right)^{24}. \end{aligned} \tag{3.8}$$

Also, since from (3.5) and (3.6) above

$$|C_m| \lesssim \|f\|_1 m_1^{-1/12} \lesssim \|f\|_1 m_1^{-1/24}, \quad m = (m_1, 0),$$

and

$$|C_m| \lesssim \|f\|_1 m_2^{-1/12} \lesssim \|f\|_1 m_2^{-1/24}, \quad m = 0, (m_2),$$

it follows that

$$\mu(\{m = (m_1, m_2) \in \mathcal{I}_\lambda : m_1 = 0 \text{ or } m_2 = 0\}) \lesssim \left(\|f\|_1/\lambda\right)^{24},$$

which combined with (3.8) above yields

$$\lambda^{24} \mu(\{m = (m_1, m_2), (m_1, m_2) \neq (0, 0) : |C_m| > \lambda\}) \lesssim \|f\|_1^{24}. \tag{3.9}$$

Now, if $m_0 = (0, 0) \in \mathcal{I}_\lambda$, since as observed above $|C_{m_0}| \leq \|f\|_1$, it follows that $\lambda < |C_{m_0}| \leq \|f\|_1$, and so

$$\lambda^{24} \mu(m_0) = \lambda^{24} \leq \|f\|_1^{24},$$

which combined with (3.9) above gives that

$$\lambda^{24} \mu(\mathcal{I}_\lambda) \lesssim \|f\|_1^{24}.$$

Therefore, by (2.2), it follows that

$$\|\{C_m\}\|_{\ell^{24,\infty}} = \sup_{\lambda>0} \lambda \mu(\{m : |C_m| > \lambda\})^{1/24} \lesssim \|f\|_1, \tag{3.10}$$

and T is continuous from $L^1(\mathbb{R}^2)$ into the sequence space $\ell(24, \infty)$.

Also, T is of type (2.2) as established by the Parseval–Plancherel formula, and, in particular,

$$\|\{C_m\}\|_{\ell^2} = \left(\sum_m |C_m|^2\right)^{1/2} \lesssim \|f\|_2. \tag{3.11}$$

We are, therefore, in the appropriate framework to interpolate for the Orlicz spaces. We remind the reader the underlying principle to obtain these interpolation results [13]. If a linear mapping T is of type, or weak-type, or mixed types, (p_0, q_0) and (p_1, q_1) , with $p_0 \neq p_1$, and the equation of the straight line passing through the points $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1)$ is given by $y = \varepsilon x + \gamma$, then, under appropriate growth conditions on the Young’s functions A, B , the mapping T is of type (A, B) provided that

$$B^{-1}(t) = t^\gamma A^{-1}(t^\varepsilon).$$

In our case T is of weak-type $(1, 24)$ and of type $(2, 2)$, and the equation of the line passing through the points $(1, 1/24)$ and $(1/2, 1/2)$ is given by

$$y = -\frac{11}{12}x + \frac{23}{24}.$$

Hence, by [12, Theorem 2.8, p. 184], T is of type (A, B) provided that

$$B^{-1}(t) = t^{23/24} A^{-1}(t^{-11/12}), \quad t > 0, \tag{3.12}$$

which is precisely (3.4) for $n = 2$.

Furthermore, since the Lorentz norms are monotone with respect to the second index, from (2.1) it follows that

$$\|\{C_n\}\|_{\ell^{2,\infty}} \lesssim \|\{C_n\}\|_{\ell^2} \lesssim \|f\|_2 \lesssim \|f\|_{2,1},$$

and, thus, together with (3.10) we are in the right framework to interpolate for the Lorentz spaces, and so, by [3, Corollary to Theorem 10, p. 293] it follows that T maps the Lorentz space $L(p, s)$ continuously into the Lorentz sequence space $\ell(q, s)$, $1 \leq s \leq \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \theta + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{24} + \frac{1-\theta}{2}.$$

Now, replacing θ above by its value,

$$\theta = 2\left(1 - \frac{1}{p}\right),$$

gives

$$\frac{11}{12} \frac{1}{p} + \frac{1}{q} = \frac{23}{24},$$

which is (3.2) for $n = 2$. And so, for these values of p, q we have,

$$\|\{C_n\}\|_{\ell^{q,s}} \lesssim_{p,s} \|f\|_{p,s}, \quad 1 \leq s \leq \infty. \tag{3.13}$$

Moreover, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p, q verifying (3.2) we have $p < 2 < q$, setting $s = q$ in (3.13) (3.14), it follows that

$$\|\{C_n\}\|_{\ell^q} \lesssim \|\{C_n\}\|_{\ell^{q,q}} \lesssim_p \|f\|_{p,q} \lesssim_p \|f\|_{p,p} \lesssim_p \|f\|_p,$$

and T is of type (p, q) . This conclusion also follows letting $A(t) = t^p$ in (3.12) above. This proves (3.3), and we have finished.

A companion result to the Hausdorff–Young inequality addresses under what conditions $\{c_m\}$ is the sequence of Fourier coefficients of a function f in the Hausdorff–Young range [2], [15, Vol.2, Theorem 2.3, p, 101]. For the Hermite expansions in \mathbb{R} , this is done in [4, Theorem 4.2].

In our context, for the Hermite expansions in n dimensions we have,

Theorem 3.2. *Suppose that p, q verify,*

$$\frac{12n}{12n - 1} < p < 2, \quad \text{and}, \quad \frac{1}{p} + \left(1 - \frac{1}{6n}\right) \frac{1}{q} = 1 - \frac{1}{12n}. \tag{3.14}$$

Then, given $\{c_m\}$ in the Lorentz sequence space $\ell(p, s)$, there is f in the Lorentz space $L(q, s)$, $1 \leq s \leq \infty$, such that $f(x) \sim \sum_m c_m \mathcal{H}_m(x)$, and

$$\|f\|_{q,s} \lesssim_{p,s} \|\{c_m\}\|_{\ell^{p,s}}. \tag{3.15}$$

In particular, if τ denotes the mapping that assigns f to the sequence $\{c_m\}$, τ is of type (p, q) whenever (3.14) holds.

Moreover, if A, B are Young’s functions such that $B(t)/t^2$ increases, and for some $r > 2$, $B(t)/t^r$ decreases and $\int_t^\infty B(s)/s^r ds/s \lesssim B(t)/t^r$, then τ is of type (A, B) , provided that A, B verify

$$B^{-1}(t) = t^{1/2((1-12n)/(1-6n))} A^{-1}(t^{-6n/(1-6n)}), \quad t > 0. \tag{3.16}$$

Proof: For simplicity we argue the case $n = 2$ as no new ideas are required for general n . Let $b(x) = \{\mathcal{H}_m(x)\}$. Then, as it was shown in the argument leading to (3.10), $b(x)$ is in the Lorentz sequence space $\ell(24, \infty)$, uniformly in x . Therefore, for a sequence $\{c_m\}$ in its conjugate Lorentz sequence space, $\ell(24/23, 1)$, it follows that

$$\left| \sum_m c_m \mathcal{H}_m(x) \right| \lesssim \|\{c_m\}\|_{\ell^{24/23,1}}, \quad \text{uniformly in } x \in \mathbb{R}^2.$$

Hence, if $f(x) \sim \sum_m c_m \mathcal{H}_m(x)$, then $f \in L^\infty(\mathbb{R}^2)$, and

$$\|f\|_{\infty,\infty} = \|f\|_\infty \lesssim \|\{c_m\}\|_{\ell^{14/23,1}}.$$

And, since by the Parseval–Plancherel formula τ is of type $(2,2)$ and we have $\|f\|_{2,\infty} \lesssim \|f\|_2 \lesssim \|\{c_m\}\|_2 \lesssim \|\{c_m\}\|_{\ell^{2,1}}$, interpolating, by [3, Corollary to Theorem 10, p. 293] it follows that τ maps the Lorentz sequence space $\ell(p, s)$ continuously into the Lorentz space $L(q, s)$, $1 \leq s \leq \infty$, where, $24/23 < p < 2$, and for $0 < \theta < 1$,

$$\frac{1}{p} = \frac{23}{24} \theta + \frac{1 - \theta}{2}, \quad \frac{1}{q} = \frac{1 - \theta}{2}.$$

Now, eliminating θ in the above relations gives (3.14) for $n = 2$, and, provided that (3.14) holds, we get that

$$\|f\|_{p,s} \lesssim_{p,s} \|\{c_n\}\|_{\ell^{q,s}}, \quad 1 \leq s \leq \infty.$$

And, since $p < q$, setting $s = q$ in (3.15) gives that τ is of type (p, q) , provided that (3.14) holds.

The result for the Orlicz spaces follows now by interpolation. In our case the equation of the line that passes through $(23/24, 0)$ and $(1/2, 1/2)$ is given by

$$y = -\frac{12}{11}x + \frac{23}{22},$$

and, consequently, by [12, Theorem 2.8, p. 184], T is of type (A, B) provided that

$$B^{-1}(t) = t^{23/22} A^{-1}(t^{-12/11}), \quad t > 0,$$

which is (3.16) for $n = 2$, and the proof is finished.

The reader will observe that as $n \rightarrow \infty$, the expressions (3.2) and (3.4) above relating p, q become $1/p + 1/q = 1$, which is precisely the Hausdorff–Young range in the case of Fourier expansions. And, naturally, the expressions (3.4) and (3.15) above approach the formula $B^{-1}(t) = t A^{-1}(1/t)$, which is the condition for the Hausdorff–Young inequality to hold in the case of the Fourier transform [7].

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References

- [1] Bennett C and Sharpley R. Interpolation of Operators, Academic Press, Orlando, Florida, 1988.
- [2] Butzer PL. The Hausdorff–Young theorems of Fourier analysis and their impact, J. Fourier Anal. Appl. 1994; (1):113–130.
- [3] Calderón AP. Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. XXVI. 1966; 273–299.
- [4] Calderón CP and Torchinsky A. Maximal integral inequalities and Hausdorff–Young, preprint.
- [5] Hille E. A class of reciprocal functions, Ann. of Math. 1926; (2): 27(4):427-464.
- [6] Hunt RA. On $L(p, q)$ spaces, Enseign. Math. 1966; 12(2):249–276.
- [7] Jodeit M Jr. and Torchinsky A. Inequalities for Fourier transforms, Studia Math. XXXVII. 1971; 245–276.
- [8] Krasnosels’kii MA and Rutickii Ya. B. Convex functions and Orlicz spaces, Nordhoff, Groningen, 1961.
- [9] Oklander ET. Interpolación, Espacios de Lorentz, y el Teorema de Marcinkiewicz, Cursos y Seminarios 20, U. de Buenos Aires, 1965.
- [10] Pinsky MA and Prather C. Pointwise convergence of n -dimensional Hermite expansions, Jour. Math. Anal. and Appl. 1996; 199:620-628.
- [11] Stein EM and Weiss G. Introduction to Fourier Analysis on Euclidean Spaces, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1971.
- [12] Szegő G. Orthogonal polynomials, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc, Providence, R. I., 1959.
- [13] Torchinsky A. Interpolation of operations and Orlicz classes, Studia Math. LIX. 1976/77; 2:177–207.
- [14] Urbina–Romero W. Gaussian Harmonic Analysis, Springer International Publishing, 2019.
- [15] Zygmund A. Trigonometric Series. Vol. I, II, 3rd edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.