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A different approach to $b_{(\alpha_n, \beta_n)}$-hypermetric spaces

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Original scientific papers

A different approach to $b_{(\alpha n,\beta n)}$-hypermetric spaces

Иной подход к $b_{(\alpha n,\beta n)}$-гиперметрическим пространствам

Другачији приступ према $b_{(\alpha n,\beta n)}$-хиперметричким просторима

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Abstract:

Introduction/purpose: The aim of this paper is to present the concept of b(αn,βn) -hypermetric spaces.
Methods: Conventional theoretical methods of functional analysis.
Results: This study presents the initial results on the topic of $b_{(\alpha n,\beta n)}$-hypermetric spaces. In the first part, we generalize an n-dimensional ($n \geq 2$) hypermetric distance over an arbitrary non-empty set X. The $b_{(\alpha n,\beta n)}$-hyperdistance function is defined in any way we like, the only constraint being the simultaneous satisfaction of the three properties, viz, non-negativity and positive-definiteness, symmetry and (αn, βn)-triangle inequality. In the second part, we discuss the concept of (αn, βn)-completeness, with respect to this $b_{(\alpha n,\beta n)}$-hypermetric, and the fixed point theorem which plays an important role in applied mathematics in a variety of fields.
Conclusion: With proper generalisations, it is possible to formulate well-known results of classical metric spaces to the case of $b_{(\alpha n,\beta n)}$-hypermetric spaces.

Keywords: $b_{(\alpha n,\beta n)}$-hypermetric spaces, G-metric, fixed point.

Резюме:

Введение/цель: Целью данной статьи является представление концепции $b_{(\alpha n,\beta n)}$-гиперметрических пространств.
Методы: В статье применены конвенциональные теоретические методы функционального анализа.

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Результаты: В статье представлены инициальные результаты в области $b_{(\alpha n,\beta n)}$-гиперметрических пространств. В первой части обобщается n-мерное (n ≥ 2) гиперметрическое расстояние на произвольном непустом множестве X. Функцию $b_{(\alpha n,\beta n)}$-гиперрастояния можно определить произвольно при наличии трех свойств: не отрицательность, положительная определенность, симметрия и (αn,βn)-неравенство треугольника. Во второй части статьи рассматривается концепция (αn,βn)-полноты по отношению к $b_{(\alpha n,\beta n)}$-гиперметрике и теореме о неподвижной точке, которая играет важную роль в прикладной математике в нескольких областях.
Выводы: С помощью соответствующих обобщений можно сформулировать известные результаты классических метрических пространств в случае $b_{(\alpha n,\beta n)}$-гиперметрических пространств.
Ключевые слова: $b_{(\alpha n,\beta n)}$-гиперметрические пространства, G-метрика, неподвижные точки.

Abstract:

Увод/циљ: Циљ овог рада јесте да се представи концепт $b_{(\alpha n,\beta n)}$-хиперметричких простора.
Методе: Примењене су конвенционалне теоретске методе функционалне анализе.
Резултати: У раду су представљени иницијални резултати који се односе на $b_{(\alpha n,\beta n)}$-хиперметричке просторе. У првом делу генерализује се n-димензионално (n ≥ 2) хиперметричко растојање на произвољном непразном скупу X. Функција $b_{(\alpha n,\beta n)}$-хиперрастојања може се дефинисати на произвољан начин докле год су задовољене три особине: ненегативност, позитивна дефинитност, симетрија и (αn, βn)-неједнакост троугла. У другом делу рада разматрани су концепт (αn, βn)-комплетности у односу на $b_{(\alpha n,\beta n)}$-хиперметрику и теорема фиксне тачке, која има значајну улогу у примењеној математици на више поља.
Закључак: Одговарајућим генерализацијама могуће је формулисати познате резултате класичних метричких простора на случај $b_{(\alpha n,\beta n)}$-хиперметричких простора.
Keywords: $b_{(\alpha n,\beta n)}$-хиперметрички простори, G-метрика, фиксне тачке.

Introduction

In human effort to describe the surrounding world, the concept of distance has long been fundamental. Our intuitive understanding of distance as an exact value may however differ from its mathematical definition and its properties. If one is to include the measurement error, encountered in real life attempt to measure the distance between two objects, the distance will be defined as an interval. This is, for example, where we may come across a set-valued distance function. This approach will in fact be our main motivation for presenting a generalized concept of the distance as a set-valued function in this paper.

The notion of 2-metric spaces, as a possible generalization of metric spaces, was introduced by Gähler (Gähler, 1963). The 2-metric $d(x, y, z)$ is a function of three variables, and Gähler geometrically interpreted it as an area of triangle with vertices at x , y and z, respectively.

B. C. Dhage, in his PhD thesis (1992), introduced the notion of D-metric (Dhage et al, 2000) spaces that generalize metric spaces. However, most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect, as shown in 2003 by Mustafa and Sims (Mustafa & Sims, 2003). This led them to introduce the notion of G-metric spaces (Mustafa & Sims, 2006), as a generalization of the metric spaces. In this type of spaces, a non-negative real number is assigned to every triplet of elements.

The G-metric spaces were generalized to universal metrics by Dehghan Nezhad et al, in a series of papers (Dehghan Nezhad & Aral, 2011; Dehghan Nezhad & Khajuee, 2013; Dehghan Nezhad et al, 2017; Dehghan Nezhad et al, 2021; Dehghan Nezhad & Mazaheri, 2010). The interpretation of the perimeter of a triangle is applied, but this time on G-metric spaces. Since then, many authors have obtained fixed point results for Gmetric spaces.

In an attempt to generalize the notion of a G-metric space to more than three variables, Khan first introduced the notion of a K-metric, and later the notion of a generalized n-metric space(for any n ≥ 2) (Khan, 2012, 2014), in 1975. He also proved the common fixed point theorem for such spaces.

Bakhtin (Bakhtin, 1989) and Czerwik (Czerwik, 1993) generalized the structure of metric space by weakening the triangle inequality and called it the b-metric space. In 2017, Kamran et al. (Kamran et al,

2017) introduced the concept of extended b-metric space by further weakening the triangle inequality. For more details also see (Agarwal et al, 2015; Debnath et al, 2021; Kirk & Shahzad, 2014; Todorčević, 2019). Also, for a broader perspective on extended b-metric spaces, dislocated b-metric spaces, rectangular b-metric spaces, b-metric like spaces, and applications see (Younis et al, 2021a,b,c; Younis & Singh, 2021).

The main purpose of this paper is a generalization of universal metric spaces into $b_{(\alpha n,\beta n)}$ -hypermetric spaces of the n-dimension.

REMARK 1. An ordered ring is a (usually commutative) ring R with a total order # such that for all a, b, and c in R:

i) if a # b, then a + c # b + c

ii) if 0 # a and 0 # b, then 0 # a · b.

We denote R^+ a set of non-negative elements of R, namely $R^+ := \{g \in R : 0 \# g\}$.

The concept of a b-metric space is initiated by Bakhtin (Bakhtin, 1989) and later used by Czerwick (Czerwik, 1993).

DEFINITION 1. (*Czerwik, 1993) Let X be a non-empty set and* $d_b : X\times X \rightarrow [0, +\infty)$ *be a function satisfying the following conditions:*

(b1) $d_b(x, y) = 0$ *if and only if* $x = y$

(b2) $d_b(x, y) = d_b(y, x)$, *for all* $x, y, z \# X$,

(b3) $d_b(x, y) \leq s(d_b(x, z) + d_b(z, y))$ *for all* $x, y, z \# X$, *where* $s \# 1$.

The function d_b *is called a b-metric and the pair* (X, d_b) *is called a bmetric space.*

EXAMPLE 1. (Berinde, 1993) Let $X = l_p[0, 1]$ be the space of all real functions $\varphi(t)$ with $t \in [0, 1]$ such that $\int_0^1 |\phi(t)|^p < +\infty$ with $0 < p < 1$. Define $d_b : X \times X \rightarrow [0, +\infty)$ as:

$$d_b(\phi, \psi) = \left(\int_0^1 |\phi(t) - \psi(t)|^p dt \right)^{\frac{1}{p}}$$

Therefore, (X, d_b) is a b-metric space with $s = 2^{\frac{1}{p}}$.

REMARK 2. (Czerwik, 1993) The class of the b-metric space is larger than the class of the metric space. When s = 1, the concept of the b-metric space coincides with the concept of the metric space.

In the following we recall the definition of the extended b-metric space.

DEFINITION 2. *(Kamran et al, 2017) Let X be a non-empty set and* $r : X \times X \rightarrow [1, +\infty)$. *A function* $d_r : X \times X \rightarrow [0, +\infty)$ *is called an extended b-metric if for all* $x, y, z \# X$ *it satisfies the following conditions:*

(b1) $d_r(x, y) = 0$ *if and only if* $x = y$,

(b2) $d_r(x, y) = d_r(y, x)$,

(b3) $d_r(x, y) \leq r(x, y)(d_r(x, z) + d_r(z, y))$.

The pair (X, d_r) *is called the extended b-metric space.*

Main results

The goal of this section is to describe a few properties and the results of the $b_{(\alpha n,\beta n)}$-hypermetric spaces of the dimension n.

B(αn,βn) -hypermetric spaces of the dimension n

Now we first recall and introduce some notation. For n ≥ 2, let X^n denote the n-times Cartesian product $\underbrace{X \times \ldots \times X}_{n-times}$ and R be an ordered ring. Let $P^{\#}(R)$ denote the family of all non-empty subsets of R. We begin with the following definition.

DEFINITION 3. *Let X be a non-empty set and α_n, $\beta_n : X^n \dashrightarrow [1, +\infty)$. Let* $\mathbb{U}_{(\alpha_n,\beta_n)} : X^n \longrightarrow P^*(R^+)$ *be a function that satisfies the following conditions:*

(U1) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n) = \{0\}$, *if* $x_1 = \ldots = x_n$,

(U2) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n) \supseteq \{0\}$, *for all* $x_1, \ldots, x_n$ *with* $x_i \neq x_j$, *for some* $i, j \in \{1, \ldots, n\}$,

(U3) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n) = \mathbb{U}_{(\alpha_n,\beta_n)}(x_{\pi_1}, \ldots, x_{\pi_n})$, *for every permutation* $(\pi_{(1)}, \ldots, \pi_{(n)})$ *of* $(1, 2, \ldots, n)$,

(U4) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, x_2, \ldots, x_{n-1}, x_{n-1}) \subseteq \mathbb{U}_{(\alpha_n,\beta_n)}(x_1, x_2, \ldots, x_{n-1}, x_n)$, *for all* $x_1, \ldots, x_n \in X$,

(U5) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, x_2, \ldots, x_n) \subseteq \alpha_n(x_1, x_2, \ldots, x_n) \cdot \mathbb{U}_{(\alpha_n,\beta_n)}(x_1, a, \ldots, a) + \beta_n(x_1, x_2, \ldots, x_n) \cdot \mathbb{U}_{(\alpha_n,\beta_n)}(a, x_2, \ldots, x_n)$, *for all* $x_1, \ldots, x_n, a \in X$.

Let A_i be the subsets of X,(i = 1, . . . , n), for any D, D′ ∈ $P^*(R^+)$ and α ∈ R^+. We define

$$\mathbb{U}_{(\alpha_n,\beta_n)}(A_1, \ldots, A_n) = \bigcup \Big\{ \mathbb{U}_n(x_1, \ldots, x_n) \mid x_i \in A_i, \quad i = 1, \ldots, n \Big\},$$

$$D + D' = \{d + d' \mid d \in D, d' \in D'\} \text{ and } \alpha \cdot D = \{\alpha \cdot d \mid d \in D, \alpha \in R^+\}.$$

We shall use the following abbreviated notation: The function $\mathbb{U}_n$ is called an ordered $b_{(\alpha n,\beta n)}$-hypermetric ring of the dimension n, or more specifically a $b_{(\alpha n,\beta n)}$-hypermetric on X. The pair $(X, \mathbb{U}_n)$ is called an $b_{(\alpha n,\beta n)}$-hypermetric space.

For example, we can place $R^+ = \mathbb{Z}_+^0$ or $\mathbb{R}_+^0$, where $\mathbb{Z}_+^0 := \mathbb{N} \cup \{0\}$ = {0, 1, 2, . . . } and $\mathbb{R}_+^0 := [0, +\infty)$. In the sequel, for simplicity we assume that $R^+ = \mathbb{R}_+^0$. The following useful properties of a b_n-hypermetric are easily derived from the axioms.

REMARK 3. If $\alpha_n(x_1, x_2, \ldots, x_n) = \beta_n(x_1, x_2, \ldots, x_n)$ = c for c ≥ 1 and n = 1, then we obtain the definition of a b-metric space (Czerwik, 1993). It is clear that for c = 1, this b-metric becomes a usual metric.

PROPOSITION 1. *(Example) Let X = [0, 1] and α_2, $\beta_2 : X \times X \dashrightarrow [1, +\infty)$, with* $\alpha_2(x,y) = 1 + \frac{1}{x+y}, \beta_2(x,y) = 1 + \frac{2}{x+y}$. *Define*

$$\mathbb{F}_{\alpha_2,\beta_2} : X \times X \rightarrow P^*(\mathbb{R}_+^0)$$

with,

$$\mathbb{F}_{(\alpha_2,\beta_2)}(x,y) = \begin{cases} [1, \frac{1}{xy}) & ; \quad x, y \in (0,1], \ x \neq y \\ \{0\} & ; \quad x, y \in [0,1], \ x = y \\ \mathbb{F}_{(\alpha_2,\beta_2)}(y,x) = [1, \frac{1}{x}) & ; \quad y = 0, x \in (0,1] \end{cases}$$

and also assume A + B = A # B, for all $A, B \subseteq P(\mathbb{R}_+^0)$. *Then* $(X, \mathbb{F}_{(\alpha_2,\beta_2)})$ *is a* $b_{(\alpha 2,\beta 2)}$ *-hypermetric space.*

Proof. It is sufficient to show that $\mathbb{F}_{(\alpha_2,\beta_2)}$ is satisfied in all properties (U1),(U2), . . . ,(U5) . The proofs of (U1), . . . ,(U4), follow immediately from the definition of $\mathbb{F}_{(\alpha_2,\beta_2)}$. We only need to show that $\mathbb{F}_{(\alpha_2,\beta_2)}$ is satisfied in

$$\mathbb{F}_{(\alpha_2,\beta_2)}(x,y) \subseteq \alpha_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(x,z) + \beta_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(z,y), \forall x,y,z \in X.$$

We distinguish the following cases:

(i) Let $x, y \in (0,1]$ For $z \in (0,1]$, we have $\mathbb{F}_{(\alpha_2,\beta_2)}(x,y) \subseteq \alpha_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(x,z) + \beta_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(z,y)$ if and only if $[1,\frac{1}{xy}) \subseteq (1+\frac{1}{x+y})[0,\frac{1}{xz}) + (1+\frac{2}{x+y})[0,\frac{1}{zy})$ if and only if $[1,\frac{1}{xy}) \subseteq (1+\frac{2}{x+y})([0,\frac{1}{xz}) + [0,\frac{1}{zy}))$ if and only if $[1,\frac{1}{xy}) \subseteq (\frac{x+y+2}{x+y})[0,\frac{x+y}{xyz})$ if and only if $z \leq 2+x+y$.
If $z = 0$, then $\mathbb{F}_{(\alpha_2,\beta_2)}(x,y) \subseteq \alpha_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(x,0) + \beta_2(x,y).\mathbb{F}_{(\alpha_2,\beta_2)}(0,y)$ if and only if $[1,\frac{1}{xy}) \subseteq (1+\frac{1}{x+y})[0,\frac{1}{x}) + (1+\frac{2}{x+y})[0,\frac{1}{y})$ if and only if $[1,\frac{1}{xy}) \subseteq (1+\frac{2}{x+y})([0,\frac{1}{x})+[0,\frac{1}{y}))$ if and only if $[1,\frac{1}{xy}) \subseteq (\frac{x+y+2}{x+y})[0,\frac{x+y}{xy})$ if and only if $2 \leq 2+x+y$.

(ii) For $x \in (0,1]$ and $y = 0$, let $z \in (0,1]$, $\mathbb{F}_{(\alpha_2,\beta_2)}(x,0) \subseteq \alpha_2(x,0).\mathbb{F}_{(\alpha_2,\beta_2)}(x,z) + \beta_2(x,0).\mathbb{F}_{(\alpha_2,\beta_2)}(z,0)$ if and only if $[1,\frac{1}{x}) \subseteq (\frac{1+x}{x})[0,\frac{1}{xz}) + (\frac{2+x}{x})[0,\frac{1}{z})$ if and only if $[1,\frac{1}{x}) \subseteq (\frac{2+x}{x})([0,\frac{1}{xz}) + [0,\frac{1}{z}))$ if and only if $[1,\frac{1}{x}) \subseteq (\frac{x+2}{x})[0,\frac{x+1}{xz})$ if and only if $xz \leq (x+1)(x+2)$.

(iii) Let $x, y \in [0,1]$, $x = y$. Obviously, $\mathbb{F}_{(\alpha_2,\beta_2)}$ is satisfied in $(U5)$.

Hence $(X,\mathbb{F}_{(\alpha_2,\beta_2)})$ is a $b_{(\alpha2,\beta2)}$ -hypermetric space.

PROPOSITION 2. *Let* $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha_2,\beta_2)}$ *-hypermetric space, then for any x_1, ..., x_n, a # X it follows that:*

(1) If $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_n) = \{0\}$, *then* $x_1 = ... = x_n$,

(2) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_n) \subseteq \sum_{j=2}^{n} \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_1,x_j)$,

(3) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_n) \subseteq \sum_{j=1}^{n} \mathbb{U}_{(\alpha_n,\beta_n)}(x_j,a,...,a)$,

(4) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,x_2,...,x_2) \subseteq (n-1)\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_1,x_2)$.

PROPOSITION 3. *Let* $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha_2,\beta_2)}$ *-hypermetric space, then* $\{0\} \subseteq \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_n)$ *or all x_1, ..., x_n # X.*

Proof. By condition (U4) of the definition of a $b_{(\alpha n,\beta n)}$-hypermetric space, we have

$$\{0\} = \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_1) \subseteq \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,...,x_n).$$

PROPOSITION 4. *Every $b(_{\alpha n,\beta n})$-hypermetric space* $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ *defines a $b_{(\alpha2,\beta2)}$-hypermetric space* $(X,\mathbb{U}_{(\alpha_2,\beta_2)})$ *as follows:*

$$\mathbb{U}_{(\alpha_2,\beta_2)}(x,y) = \mathbb{U}_{(\alpha_n,\beta_n)}(x,y,\ldots,y) + \mathbb{U}_{(\alpha_n,\beta_n)}(y,x,\ldots,x); \forall x,y \in X.$$

Proof. Note that (U1), . . . ,(U4) trivially hold. We only need to show that $\mathbb{U}_{(\alpha_2,\beta_2)}$ is satisfied in

$$\mathbb{U}_{(\alpha_2,\beta_2)}(x,y) \subseteq \alpha_2(x,y).\mathbb{U}_{(\alpha_2,\beta_2)}(x,z) + \beta_2(x,y).\mathbb{U}_{(\alpha_2,\beta_2)}(z,y), \forall x,y,z \in X.$$

By setting

$$\alpha_2(x,y) = max\{\alpha_n(x,y,...,y), \alpha_n(y,x,...,x)\}$$

and

$$\beta_2(x,y) = max\{\beta_n(x,y,...,y), \beta_n(y,x,...,x)\}.$$

This completes the proof.

PROPOSITION 5. Let e be an arbitrary positive real number, and (X, d) be a metric space. We define an induced $b_{(\alpha 2,\beta 2)}$-hypermetric

$$\mathbb{U}^e_{(\alpha_2,\beta_2)} : X \times X \to P^*(\mathbb{R}^0_+)$$

$$\mathbb{U}^e_{(\alpha_2,\beta_2)}(x,y) = \begin{cases} (d(x,y)-e, d(x,y)+e) \cup \{0\}; & x \neq y, d(x,y) > e \\ (d(x,y)-e, d(x,y)+e) \cap \mathbb{R}^0_+; & x \neq y, d(x,y) < e \\ \{0\}; & x = y \text{ or } d(x,y) = e. \end{cases}$$

Then $(X, \mathbb{U}^e_{(\alpha_2,\beta_2)})$ is a $b_{(\alpha 2,\beta 2)}$-hypermetric spac

Quotient b(αn,βn)-hypermetric space

Let $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ be a $b_{(\alpha n,\beta n)}$-hypermetric space and $\tilde{X}$ be a partition of X. For each point p ∈ X, we denote $\tilde{p}$ a point in $\tilde{X}$ containing p, and we denote the equivalent relation induced by the relation by ∼ .

DEFINITION 4. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *e a* $b_{(\alpha n,\beta n)}$ *hypermetric space. Let* $p_1, \ldots, p_n$ *# X, and consider* $\tilde{p}_1, \ldots, \tilde{p}_n \in \tilde{X}$. *A quotient* $b_{(\alpha n,\beta n)}$*-hypermetric of points of* $\tilde{X}$ *induced by* $\mathbb{U}_{(\alpha_n,\beta_n)}$ *is the function*

$$\widetilde{\mathbb{U}}_{(\alpha_n,\beta_n)} : \widetilde{X}^n \longrightarrow P^*(\mathbb{R}^0_+)$$

given by

$$\widetilde{\mathbb{U}}_{(\alpha_n,\beta_n)}(\widetilde{p}_1,\ldots,\widetilde{p}_n) = \bigcap_{p_i \in \widetilde{p}_i} \mathbb{U}_{(\alpha_n,\beta_n)}(p_1,\ldots,p_n).$$

PROPOSITION 6. *The quotient* $b_{(\alpha n,\beta n)}$*-hypermetric induced by* $\mathbb{U}_{(\alpha_n,\beta_n)}$ *is well defined and is a* $b_{(\alpha n,\beta n)}$*-hypermetric on X*

Proof. $\tilde{\mathbb{U}}_{(\alpha_n,\beta_n)}$ is satisfied in all properties (U1), till (U4),

$$\widetilde{\mathbb{U}}_{(\alpha_n,\beta_n)}(\widetilde{p}_1,\ldots,\widetilde{p}_n) \subseteq \widetilde{\mathbb{U}}_{(\alpha_n,\beta_n)}(\widetilde{p}_1,\widetilde{q},\ldots,\widetilde{q}) + \mathbb{U}_{(\alpha_n,\beta_n)}(\widetilde{q},\widetilde{p}_2,\ldots,\widetilde{p}_n)$$

$$\bigcap_{p_i\in\widetilde{P}_i} \mathbb{U}_{(\alpha_n,\beta_n)}(p_1,\ldots,p_n) \subseteq \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \left(\mathbb{U}_{(\alpha_n,\beta_n)}(p_1,q,\ldots,q) + \mathbb{U}_{(\alpha_n,\beta_n)}(q,p_2,\ldots,p_n)\right)$$

$$= \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \mathbb{U}_{(\alpha_n,\beta_n)}(p_1,q,\ldots,q) + \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \mathbb{U}_{(\alpha_n,\beta_n)}(q,p_2,\ldots,p_n)$$

$$= \bigcap_{\substack{p_i\in\widetilde{P}_i\\ q\in\widetilde{q}}} \left(\mathbb{U}_{(\alpha_n,\beta_n)}(p_1,q,\ldots,q) + \mathbb{U}_{(\alpha_n,\beta_n)}(q,p_2,\ldots,p_n)\right).$$

Let $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ be a $b_{(\alpha n,\beta n)}$-hypermetric space of a dimension n > 2. For any arbitrary a in X, define the function $\mathbb{U}_{(\alpha_{n-1},\beta_{n-1})}$ on X^{n-1} by $\mathbb{U}_{(\alpha_{n-1},\beta_{n-1})}(x_1,\ldots,x_{n-1}) := \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_{n-1},a)$. Then we have the following result.

PROPOSITION 7. *The function* $\mathbb{U}_{(\alpha_{n-1},\beta_{n-1})}$ *efine a* $b_{(\alpha_{n-1},\beta_{n-1})}$ *-hypermetric on X.*

Proof. We will verify that $\mathbb{U}_{(\alpha_{n-1},\beta_{n-1})}$ satisfies the five properties of a $b_{(\alpha_{n-1},\beta_{n-1})}$-hypermetric.

PROPOSITION 8. *Let Π : X → Y be an injection from a set X to a set Y. If* $\mathbb{U}_{(\alpha_n,\beta_n)} : Y^n \to P^*(\mathbb{R}^0_+)$ *is a* $b_{(\alpha n,\beta n)}$*-hypermetric on the set Y. Then* $\overline{\mathbb{U}}_{(\alpha_n,\beta_n)} : X^n \to P^*(\mathbb{R}^0_+)$, *given by the formula* $\overline{\mathbb{U}}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_n) = \mathbb{U}_{(\alpha_n,\beta_n)}(\Pi(x_1),\ldots,\Pi(x_n))$ *for all* $x_1, \ldots, x_n$ *# X, is a* $b_{(\alpha n,\beta n)}$*-hypermetric on the set X.*

PROPOSITION 9. *Let* $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ *be any* $b(_{\alpha n,\beta n})$*-hypermetric space and* $\lambda \in \mathbb{R}^0_+$. *Then* $(X,\mathbb{U}^{\lambda}_{(\alpha_n,\beta_n)})$ *is also a* $b_{(\alpha n,\beta n)}$*-hypermetric space where* $\mathbb{U}^{\lambda}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_n) := \{A\cap[0,\lambda) | A \in \mathbb{U}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_n)\}$.

So, on the same X many intances of the $b_{(\alpha n,\beta n)}$-hypermetric can be defined, as a result of which the same set X is endowed with different metric structures. Another structure in the next proposition is useful for scaling the $b_{(\alpha n,\beta n)}$-hypermetric, so we need the following explanation.

For any non-empty subset A of $\mathbb{R}^0_+$, and $\lambda \in \mathbb{R}^+$ we define a set $\lambda \cdot A$ to be $\lambda \cdot A := \{\lambda \cdot a \mid a \in A\}$.

PROPOSITION 10. *Let* $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ *e any* $b_{(\alpha n,\beta n)}$*-hypermetric space. Let Λ be any positive real number. We define* $\dot{\mathbb{U}}^{\Lambda}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_n) = \lambda.\mathbb{U}_{(\alpha_n,\beta_n)}(x_1,\ldots,x_n)$. *Then* $(X,\dot{\mathbb{U}}^{\lambda}_{(\alpha_n,\beta_n)})$ *is also a* $b_{(\alpha n,\beta n)}$*-hypermetric space.*

A sequence $\{x_m\}$ in a $b_{(\alpha n,\beta n)}$-hypermetric space $(X,\mathbb{U}_{(\alpha_n,\beta_n)})$ is said to converge to a point s in X, if for any # > 0 there exists a natural number N such that for every $m_1, \ldots, m_{n-1} \geq N$.

$$\mathbb{U}_{(\alpha_n,\beta_n)}(x_{m_1},\ldots,x_{m_{n-1}},s) \subseteq [0,\epsilon),$$

then we shall write

$$\lim_{m_1,\ldots,m_{n-1}\longrightarrow+\infty} \mathbb{U}_{(\alpha_n,\beta_n)}(x_{m_1},\ldots,x_{m_{n-1}},s) = \{0\}.$$

We shall say that a sequence $\{x_m\}$ has a cluster point x if there exists a subsequence $\{x_{mk}\}$ of $\{x_m\}$ that converges to x.

PROPOSITION 11. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *and* $(X', \mathbb{U}'_{(\alpha_n,\beta_n)})$ *be two* $b_{(\alpha n,\beta n)}$*-hypermetric spaces. Then a function* $T : X \to X\#$ *is* $b_{(\alpha n,\beta n)}$ *-continuous at a point* $x \# X$*, if and only if it is* $b_{(\alpha n,\beta n)}$*-sequentially continuous at x; that is, whenever sequence* $\{x_m\}$ *is* $b_{(\alpha n,\beta n)}$ *-convergent to x one has* $\{T(x_m)\}$ *is* $U_{(\alpha n,\beta n)}$ *-convergent to* $T(x)$*.*

DEFINITION 5. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$ *-hypermetric space, and* $A \# X$*. The set A is* $b_{(\alpha n,\beta n)}$ *-compact if for every* $b_{(\alpha n,\beta n)}$ *-sequence* $\{x_m\}$ *in A, there exists a subsequence* $\{x_{mk}\}$ *of* $\{x_m\}$ *such that* $b_{(\alpha n,\beta n)}$ *-convergences to some* $x_0 \# A$*.*

PROPOSITION 12. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *and* $(X', \mathbb{U}'_{(\alpha_n,\beta_n)})$ *be two* $b_{(\alpha n,\beta n)}$*-hypermetric spaces and* $T : X \to X\#$ *a* $b_{(\alpha n,\beta n)}$*-continuous function on X. If X is* $b_{(\alpha n,\beta n)}$*-compact, then* $T(X)$ *is* $b_{(\alpha n,\beta n)}$*-compact.*

DEFINITION 6. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$ *-hypermetric space, then for* $x_0 \# X$*,* $r > 0$*, the* $b_{(\alpha n,\beta n)}$ *-hyperball with the centre* x_0 *and the radius r is*

$$B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x_0, r) = \{y \in X : \mathbb{U}_{(\alpha_n,\beta_n)}(x_0, y, ..., y) \subseteq [0, r)\}.$$

PROPOSITION 13. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$ *-hypermetric space, then for* $x_0 \# X$*,* $r > 0$*,*

(i) If $\mathbb{U}_{(\alpha_n,\beta_n)}(x_0, x_2, ..., x_n) \subseteq [0, r)$*, then* $x_2, ..., x_n \in B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x_0, r)$*,*
(ii) If $y \in B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x_0, r)$*, then there exists,* $\delta > 0$ *such that* $B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(y, \delta) \subseteq B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x_0, r)$*.*

PROPOSITION 14. *The set of all* $\mathbb{U}_{(\alpha_n,\beta_n)}$ *-balls,* $\mathcal{B}_n = \{B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x, r) : x \in X, r > 0\}$*, forms a basis for a topology* $\mathcal{T}(\mathbb{U}_{(\alpha_n,\beta_n)})$ *on X.*

DEFINITION 7. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$*-hypermetric space. The sequence* $\{x_n\} \# X$ *is* $b_{(\alpha n,\beta n)}$*-convergent to x if it* $b_{(\alpha n,\beta n)}$*-converges to x in the* $b_{(\alpha n,\beta n)}$*-hypermetric topology,* $\mathcal{T}(\mathbb{U}_{(\alpha_n,\beta_n)})$ *.*

PROPOSITION 15. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$*-hypermetric space. Then for a sequence* $\{x_m\} \# X$*, and a point* $x \# X$ *the following are equivalent:*

(1) $\{x_m\}$ *is* $\mathbb{U}_{(\alpha_n,\beta_n)}$*-convergent to* x*,*

(2) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, ..., x_m, x) \to 0$*, and*
(3) $\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x, ..., x) \to 0$*.*

DEFINITION 8. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *,* $(Y, \mathbb{V}_{(\alpha_m,\beta_m)})$ *be universal hypermetric spaces of the dimensions n and m respectively; a function* $T : X \longrightarrow Y$ *is* $b_{(\alpha n,\beta n),(\alpha m,\beta m)}$*-continuous at the point* $x_0 \# X$*, if* $\mathcal{T}(U_n)$ *, for all* $r > 0$*.*

We say f is $b_{(\alpha n,\beta n),(\alpha m,\beta m)}$-continuous if it is $b_{(\alpha n,\beta n),(\alpha m,\beta m)}$-continuous at all points of X; that is, continuous as a function from X with the $\mathcal{T}(\mathbb{U}_{(\alpha_n,\beta_n)})$-topology to Y with the $\mathcal{T}(\mathbb{V}_{(\alpha_m,\beta_m)})$-topology.

In the sequel, for simplicity we have assumed that n = m. Since $b_{(\alpha n,\beta n)}$-hypermetric topologies are metric topologies, we have:

DEFINITION 9. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *and* $(Y, \mathbb{V}_{(\alpha_m,\beta_m)})$ *be two* $b_{(\alpha n,\beta n)}$*-hypermetric spaces and* $T : (X, \mathbb{U}_{(\alpha_n,\beta_n)}) \to (Y, \mathbb{V}_{(\alpha_n,\beta_n)})$ *be a function. The function f is called* $b_{(\alpha n,\beta n)}$*-continuous at a point* $a \# X$ *if and only if, for given* $\# > 0$*, there exists* $\delta > 0$ *such that* $x_1, \ldots, x_{n-1} \# X$ *and the subset relation* $\mathbb{U}_{(\alpha_n,\beta_n)}(a, x_1, \ldots, x_{n-1}) \subseteq [0, \delta)$ *implies that* $\mathbb{V}_{(\alpha_n,\beta_n)}(T(a), T(x_1), \ldots, T(x_{n-1})) \subseteq [0, \epsilon)$ *.*

A function f is $b_{(\alpha n,\beta n)}$ -continuous on X if and only if it is $b_{(\alpha n,\beta n)}$-continuous at all $a \in X$.

PROPOSITION 16. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *,* $(Y, \mathbb{V}_{(\alpha_m,\beta_m)})$ *be* $b_{(\alpha n,\beta n)}$*-hypermetric spaces, a function* $T : X \longrightarrow Y$ *is* $b_{(\alpha n,\beta n)}$*-continuous at point* $x \# X$ *if and only if it is* $b_{(\alpha n,\beta n)}$*-sequentially continuous at x; that is, whenever the* $\{x_n\}$ *is* $b_{(\alpha n,\beta n)}$*-convergent to x we have* $(T(x_n))$ *is* $b_{(\alpha n,\beta n)}$*-convergent to* $T(x)$*.*

PROPOSITION 17. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$*-hypermetric space. Then the function*

$\mathbb{U}_{(\alpha_n,\beta_n)}(z_1, z_2, \ldots, z_n)$ *is jointly* $b_{(\alpha n,\beta n)}$ *-continuous in all n of its variables.*

DEFINITION 10. *A map* $T : X \dashrightarrow Y$ *between* $b_{(\alpha n,\beta n)}$*-hypermetric spaces* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *and* $(Y, \mathbb{U}'_{(\alpha_n,\beta_n)})$ *is an iso-hypermetric when* $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n) = \mathbb{U}'_{(\alpha_n,\beta_n)}(T(x_1), \ldots, T(x_n))$ *for all* $x_1, \ldots, x_n \# X$. *If the iso-*$b_{(\alpha n,\beta n)}$*-hypermetric is injective, we call it iso-*$b_{(\alpha n,\beta n)}$*-hypermetric embedding. A bijective iso-*$b_{(\alpha n,\beta n)}$*-hypermetric is called a* $b_{(\alpha n,\beta n)}$*-hypermetric isomorphism.*

Fixed Point Theorem in b(αn,βn)-hypermetric spaces

In a $b_{(\alpha n,\beta n}$-hypermetric space, the concepts of basic topological notions, such as: $b_{(\alpha n,\beta n)}$-Cauchy sequence, $b_{(\alpha n,\beta n)}$-convergent sequence and $b_{(\alpha n,\beta n)}$-complete $b_{(\alpha n,\beta n)}$-hypermetric space can be easily adopted as shown below. We discuss about the concept of $b_{(\alpha n,\beta n)}$-completeness of $b_{(\alpha n,\beta n)}$-hypermetric spaces.

DEFINITION 11. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$*-hypermetric space, then a sequence {xm} # X is called* $b_{(\alpha n,\beta n)}$*-Cauchy if for every ε > 0, there exists* $N \in \mathbb{N}$ such that $\mathbb{U}_{(\alpha_n,\beta_n)}(x_{m_1}, x_{m_2}, \ldots, x_{m_n}) < \varepsilon$ for all $m_1, m_2, \ldots, m_n \geq N$.

The next proposition follows directly from the definitions.

PROPOSITION 18. *In a* $b_{(\alpha n,\beta n)}$*-hypermetric space,* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *, the following are equivalent*

(i) The sequence $\{x_m\}$ is $b_{(\alpha_n,\beta_n)}$-Cauchy.
(ii) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mathbb{U}_{(\alpha_n,\beta_n)}(x_l, x_m, \ldots, x_m) < \varepsilon$, for all $l, m \geq N$.
(iii) $\{x_m\}$ is a Cauchy sequence in the metric space $(X, d_{\mathbb{U}_{(\alpha_n,\beta_n)}})$.

COROLLARY 1. *(i) Every* $b_{(\alpha n,\beta n)}$*-convergent sequence in a* $b_{(\alpha n,\beta n)}$*- hypermetric space is* $b_{(\alpha n,\beta n)}$*-Cauchy. (ii) If a* $b(_{\alpha n,\beta n)}$*-Cauchy sequence in a* $b_{(\alpha n,\beta n)}$*-hypermetric space* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *contains a* $b_{(\alpha n,\beta n)}$*-convergent subsequence, then the sequence itself is b(αn,βn)-convergent.*

DEFINITION 12. *A* $b_{(\alpha n,\beta n)}$*-hypermetric space* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *is called* $b_{(\alpha n,\beta n)}$*-complete if every* $b_{(\alpha n,\beta n)}$*-Cauchy sequence in* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *is* $b_{(\alpha n,\beta n)}$*-convergent in* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$.

PROPOSITION 19. *A* $b_{(\alpha n,\beta n)}$*-hypermetric space* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *is* $b_{(\alpha n,\beta n)}$*-complete if and only if* $(X, d_{\mathbb{U}_{(\alpha_n,\beta_n)}})$ *is a complete metric space.*

DEFINITION 13. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *and* $(Y, \mathbb{U}'_{(\alpha_n,\beta_n)})$ *be two* $b_{(\alpha n,\beta n)}$*-hypermetric spaces. A function f : X ⇢ Y is called a* $b_{(\alpha n,\beta n)}$*-contraction if there exists a constant k # [0, 1) such that* $\mathbb{U}'_{(\alpha_n,\beta_n)}(f(x_1), \ldots, f(x_n)) = k\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n)$ *for all* $x_1, \ldots, x_n \# X$.

It follows that f is $b_{(\alpha n,\beta n)}$-continuous because; $\mathbb{U}_{(\alpha_n,\beta_n)}(x_1, \ldots, x_n)$ # [0, δ) with $k \neq 0$ and $\delta := \epsilon/k$ implies $\mathbb{U}'_{(\alpha_n,\beta_n)}(f(x_1), \ldots, f(x_n)) \subseteq [0, \epsilon)$.

THEOREM 1. *Let* $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$ *be a* $b_{(\alpha n,\beta n)}$ *-complete space and let T : X → X be a* $b_{(\alpha n,\beta n)}$*-contraction map. Then T has a unique fixed point T(x) = x.*

Proof. We consider $x_{m+1} = T(x_m)$, with x_0 being any point in X. By repeated use of the (αn, βn)-rectangle inequality and the application of the contraction property, we obtain

$$\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{m+1}, \ldots, x_{m+1}) \subseteq k^m \mathbb{U}_{(\alpha_n,\beta_n)}(x_0, x_1, \ldots, x_1)$$

for all m, $s_1 \in \mathbb{N}$ which m < s_1 and k ∈ [0, 1). It follows from the above that

$$\begin{aligned}\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, \ldots, x_{s_1}) \subseteq \Gamma_1 \mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{m+1}, \ldots, x_{m+1}) \\ + \Gamma_2 \mathbb{U}_{(\alpha_n,\beta_n)}(x_{m+1}, x_{m+2}, \ldots, x_{m+2}) \\ + \Gamma_3 \mathbb{U}_{(\alpha_n,\beta_n)}(x_{m+2}, x_{m+3}, \ldots, x_{m+3}) \\ + \ldots + \Gamma_{s_1-m} \mathbb{U}_{(\alpha_n,\beta_n)}(x_{s_1-1}, x_{s_1}, \ldots, x_{s_1}) \\ \subseteq \Gamma(k^m + k^{m+1} + \ldots + k^{s_1-1}) \mathbb{U}_{(\alpha_n,\beta_n)}(x_0, x_1, \ldots, x_1) \\ = \Gamma \frac{k^m(1-k^{s_1-m})}{1-k} \mathbb{U}_{(\alpha_n,\beta_n)}(x_0, x_1, \ldots, x_1)\end{aligned}$$

where $\Gamma 1 = \alpha_n(x_m, x_{s1}, \ldots, x_{s1})$,
and $\Gamma = max\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{s_1-m}\}$ for all $x_m, \ldots, x_{s_1} \in B_{\mathbb{U}_{(\alpha_n,\beta_n)}}(x_0, r)$.
Then we have

$$\lim_{m\,,\,s_1 \to +\infty} \mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, \ldots, x_{s_1}) = \{0\}$$

since

$$\lim_{m\,,\,s_1 \to +\infty} \Gamma \frac{k^m(1-k^{s_1-m})}{1-k} \mathbb{U}_{(\alpha_n,\beta_n)}(x_0, x_1, \ldots, x_1) = \{0\}.$$

For $m \le s_1 \le s_2 \in \mathbb{N}$ and (U5) it implies that

$$\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) \subseteq$$

$$\alpha_n(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) \mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, \ldots, x_{s_1})$$
$$+ \beta_n(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) \mathbb{U}_{(\alpha_n,\beta_n)}(x_{s_1}, x_{s_2}, \ldots, x_{s_2}),$$

now taking a limit as $m, s_1, s_2 \to +\infty$, we get

$$\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, x_{s_2} \ldots, x_{s_2}) \to \{0\}.$$

Now for $m \le s_1 \le s_2 \le \ldots \le s_{n-1} \in$, we will have

$$\mathbb{U}_{(\alpha_n,\beta_n)}(x_m, x_{s_1}, \ldots, x_{s_n}) \to \{0\}; \quad whenever, \quad m, s_1, \ldots, s_{n-1} \to +\infty,$$

then $\{x_m\}$ is a Cauchy sequence. By completeness of $(X, \mathbb{U}_{(\alpha_n,\beta_n)})$, there exists a $\in$ X such that $\{x_n\}$ is $b_{(\alpha n,\beta n)}$-convergent to a. It follows that the limit x_m is a fixed point of T following the $b_{(\alpha n,\beta n)}$-continuity of T, and

$$Ta = T \lim_{m \to +\infty} x_m = \lim_{m \to +\infty} Tx_m = \lim_{m \to +\infty} x_{m+1} = a.$$

Finally, if a and b are two fixed points, then

$$\{0\} \subseteq \mathbb{U}_{(\alpha_n,\beta_n)}(a,b,\ldots,b) = \mathbb{U}_{(\alpha_n,\beta_n)}\Big(T(a),T(b),\ldots,T(b)\Big) \subseteq k\mathbb{U}_{(\alpha_n,\beta_n)}(a,b,\ldots,b).$$

We conclude from k < 1 that $\mathbb{U}_n(a,b,\ldots,b) = \{0\}$. Consequently, a = b and the fixed point is unique.

Conclusion

The objective of this paper is to bring about the study of $b_{(\alpha n,\beta n)}$-hypermetric spaces and to introduce certain fixed point results of mappings in the setting of $b_{(\alpha n,\beta n)}$-hypermetric spaces. This study presents the initial results in this topic and more refined results can be derived in the near future. Also in the future, we will consider engineering applications of the considered topic.

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