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Fixed point theorem in a partial b-metric space applied to quantum operations

Теорема о фиксированной точке в частичном b-метрическом пространстве с применением в квантовых операциях

Теорема фиксирне тачке у делимичном b-метричком простору примењена на квантне операције

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ABSTRACT:

Introduction/purpose: A fixed point theorem of an order-preserving mapping on a complete partial b-metric space satisfying a contractive condition is constructed.

Methods: Extension of the results of Batsari et al.

Results: The fidelity of quantum states is used to construct the existence of a fixed quantum state.

Conclusions: The fixed quantum state is associated to an orderpreserving quantum operation.

KEYWORDS: partial b-metric space, order-preserving mapping, quantum operation, fidelity of quantum state, Bloch Vector.

AUTHOR NOTES

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Р е з ю м е :

Введение/цель: Сконструирована теорема о неподвижной точке с сохранением порядка в полном и частичном бметрическом пространстве при выполнении условий сжатия.

Методы: В данной статье применен метод расширения результатов Батсари и др.

Результаты: Точность квантового состояния используется для построения неподвижного квантового состояния.

Выводы: Неподвижное квантовое состояние связано с квантовой операцией, сохраняющей порядок.

К л ю ч е в ы е с л о в а : частичное б-метрическое пространство, отображение с сохранением порядка, квантовая операция, точность квантового состояния, вектор Блоха.

ABSTRACT:

Увод/циљ: Конструисана је теорема фиксне тачке мапирања с очувањем редоследа на комплетном парцијалном б-метричком простору уз задовољавање контрактивног услова.

Методе: Примењен метод проширен је резултатима Батсарија и других.

Резултати: Верност квантног стања користи се за конструисање фиксног квантног стања.

Закључак: Фиксно квантно стање повезано је са квантном операцијом која чува редослед.

KEYWORDS: парцијални б-метрички простор, мапирање с очувањем редоследа, квантна операција, верност квантног стања, Блохов вектор.

INTRODUCTION AND PRELIMINARIES

A partial metric space is a generalized metric space in which each object does not necessarily have a zero distance from itself (Aamri & El Moutawakil, 2002). Another angle of fixed point research emerged with the approach of the Knaster-Tarski fixed point theorem (Knaster, 1928; Tarski, 1955). The idea was first initiated from Knaster and Tarski in 1927 (Knaster, 1928), and later Tarski found some improvement of the work in 1939, which he discussed in some public lectures between 1939 and 1942 (Tarski, 1955, 1949). Finally, in 1955, Tarski (Tarski, 1955) published the comprehensive results together with some applications. A different property of this theorem is that it involves an order relation defined on the space of consideration. Indeed, the order relation serves as an alternative to the continuity and contraction of the mappings as found in the Brouwer (Brouwer, 1911) and Banach (Banach, 1922) fixed point theorems, respectively, see (Tarski, 1955).

After the approach of the Brouwer (Brouwer, 1911), Banach (Banach, 1922) and Knaster-Tarski (Tarski, 1955) fixed point theorems, many researchers become involved in extension (Browder, 1959; Leray & Schauder, 1934; Schauder, 1930), generalization (Batsari et al, 2018; Browder, 1959; Du et al, 2018) and improvements (Batsari et al, 2018; Batsari & Kumam, 2018; Kannan, 1972; Khan et al, 1984) of the theorems using different spaces and functions. In the way of generalizing spaces was Bourbaki-Bakhtin-Czerwik's b-metric space (Bakhtin, 1989; Bourbaki, 1974; Czerwik, 1993), Matthews's partial metric space (Matthews, 1994) and Shukla's Partial b-metric space (Shukla, 2014).

In the area of the quantum information theory, a qubit is seen as a quantum system, whereas a quantum operation can be inspected as the measurement of a quantum system; it describes the development of the system through the quantum states. Measurements have some errors which can be corrected through quantum error correction codes. The quantum error correction codes are easily developed through the information-preserving structures with the help of the fixed points set of the associated quantum operation. Therefore, the study of quantum operations is necessary in the field of the quantum information theory, at least in developing the error correction codes, knowing the state of the system (qubit) and the description of energy dissipation effects due to loss of energy from a quantum system (Nielsen & Chuang, 2000).

In 1951, Luders (Lüders, 1950) discussed the compatibility of quantum states in measurements (quantum operations). He also proved that the compatibility of quantum states in measurements is equivalent to the commutativity of the states with each quantum effects in the measurement.

In 1998, Busch et al. (Busch & Singh, 1998) generalized the Luders theorem. He also showed that a state is unchanged under a quantum operation if the state commutes with every quantum effect that relates the quantum operation. In 2002, Arias et al. (Arias et al, 2002) studied the fixed point sets of a quantum operation and gave some conditions for which the set is equal to a commutate set of the quantum effects that described the quantum operation. In 2011, Long and Zhang (Zhang & Ji, 2012) deliberated the fixed point set for quantum operations, they presented some necessary and sufficient conditions for the existence of a non-trivial fixed point set. Similarly, in 2012, Zhang and Ji (Long & Zhang, 2011) deliberated the existence of a non-trivial fixed point set of a generalized quantum operation. In 2016, Zhang and Si (Zhang & Si, 2016) explored the conditions for which the fixed point set of a quantum operation (ϕ_A) with respect to a row contraction A equals to the fixed point set of the power of the quantum operation (ϕ_A^j) for some $1 \leq j < +\infty$. Other useful references are (Agarwal et al, 2015; Debnath et al, 2021; Kirk & Shahzad, 2014).

DEFINITION 1. (Shukla, 2014) A **partial b-metric** on the set X is a function $p_s : X \times X \rightarrow \mathbb{R}_+$ such that,

- (1) For all $x, y \in X$, $x = y$ iff $p_s(x, x) = p_s(x, y) = p_s(y, y)$
- (2) For all $x, y \in X$, $p_s(x, x) \leq p_s(x, y)$
- (3) For all $x, y \in X$, $p_s(x, y) = p_s(y, x)$
- (4) There exists a real number $s \geq 1$ such that, for all $x, y, z \in X$, $p_s(x, z) \leq s[p_s(x, y) + p_s(y, z)] - p_s(y, y)$.

(X, p_s) denotes the partial b-metric space. Note that every partial metric is a partial b-metric with $s = 1$. Also, every b-metric is a partial b-metric with $p_s(x, x) = 0$, for all $x, y \in X$.

A sequence $\{x_n\}$ in the space (X, p_s) converges with respect to the topology τ_b to a point $x \in X$, if and only if

$$\lim_{n \rightarrow +\infty} p_s(x_n, x) = p_s(x, x). \quad (1)$$

The sequence $\{x_n\}$ is Cauchy in (X, p_s) if the below limit exists and is finite

$$\lim_{n, m \rightarrow +\infty} p_s(x_n, x_m) < +\infty. \quad (2)$$

A partial b-metric space (X, p_s) is complete, if every Cauchy sequence $\{x_n\}$ in (X, p_s) converges to a point $x \in X$ such that,

$$\lim_{n, m \rightarrow +\infty} p_s(x_n, x_m) = p_s(x, x). \quad (3)$$

DEFINITION 2. A mapping T is said to be order-preserving on X , whenever $x \# y$ implies $T(x) \# T(y)$ for all $x, y \in X$.

MAIN RESULT

The objective of this work is to establish a fixed point theorem in a complete partial b-metric space.

THEOREM 1. Let (X, p_s) be a complete partial b-metric space with $s \geq 1$ and associated with a partial order $\#$. Suppose an order preserving mapping $T : X \rightarrow X$ satisfies

$$p_s(T(x), T(y)) \leq \alpha \max\{p_s(x, y), p_s(x, T(y)), p_s(y, T(x))\} + \frac{\beta}{2} \min\{p_s(x, T(y)) + p_s(y, T(x)), p_s(x, T(x)) + p_s(y, T(y))\} \quad (4)$$

for all comparable $x, y \in X$, where $\alpha, \beta \in [0, \theta]$ and $\theta = \min\{\frac{1}{s}, \frac{2}{s+1}\}$. If there exists $x_0 \in X$ such that $x_0 \# T(x_0)$, then T has a unique fixed point $\hat{x} \in X$ such that $p_s(\hat{x}, \hat{x}) = 0$.

Proof. Suppose $x_0 \neq T(x_0)$, define a sequence $\{x_n\} \# X$ by $x_n = T^n(x_0)$ and let $q_n = p_s(x_n, x_{n+1})$. It is clear that if $x_n = x_{n+1}$ for some natural number n , then x_n is a fixed point of T , i.e., $x_{n+1} = T(x_n) = x_n$.

Let $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, we proceed as follows:

$$\begin{aligned} q_n &= p_s(x_n, x_{n+1}) = p_s(T(x_{n-1}), T(x_n)) \\ &\leq \alpha \max\{p_s(x_{n-1}, x_n), p_s(x_{n-1}, T(x_n)), p_s(x_n, T(x_{n-1}))\} \\ &\quad + \frac{\beta}{2} \min\{p_s(x_{n-1}, T(x_n)) + p_s(x_n, T(x_{n-1})), p_s(x_{n-1}, T(x_{n-1})) + p_s(x_n, T(x_n))\} \\ &= \alpha \max\{p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_{n+1}), p_s(x_n, x_n)\} \\ &\quad + \frac{\beta}{2} \min\{p_s(x_{n-1}, x_{n+1}) + p_s(x_n, x_n), p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})\} \\ &= \alpha \max\{p_s(x_{n-1}, x_n), s[p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})]\} \\ &\quad + \frac{\beta}{2} \left[\frac{s[p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})] + p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})}{2} \right] \\ &= \alpha(s[p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})]) + \frac{\beta}{2}(s+1) \left[\frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})}{2} \right] \\ &= \alpha s + \frac{\beta(s+1)}{4} (p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})) \\ &= \frac{4\alpha s + \beta s + \beta}{4} (p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})) \\ &= \frac{4\alpha s + \beta s + \beta}{4} (q_{n-1} + q_n). \end{aligned}$$

Thus, we have

$$q_n \leq \frac{4\alpha s + \beta s + \beta}{4} (q_{n-1} + q_n)$$

which implies

$$\left(\frac{4 - 4\alpha s - \beta s - \beta}{4}\right)q_n \leq \left(\frac{4\alpha s + \beta s + \beta}{4}\right)q_{n-1} \quad (5)$$

By simplifying (5), we have

$$q_n \leq \left(\frac{4\alpha s + \beta s + \beta}{4 - 4\alpha s - \beta s - \beta}\right)q_{n-1} \quad (6)$$

For $\theta \in \min\{\frac{1}{s}, \frac{2}{s+1}\}$, we deduce that

$$0 \leq \frac{4\alpha s + \beta s + \beta}{4 - 4\alpha s - \beta s - \beta} \leq 1.$$

Therefore, from (6), we conclude that $p_s(x_n, x_{n+1}) = q_n \leq q_{n-1} = p_s(x_{n-1}, x_n)$. Thus, $\{q_n\}_{n=1}^{+\infty}$ is a monotone non-increasing sequence of real numbers and bounded below by 0. Therefore, $\lim_{n \rightarrow +\infty} q_n = 0$, see Chidume et al. (Chidume & Chidume, 2014).

Next, we show $\{x_n\}_{n=1}^{+\infty}$ is Cauchy. Let $x_n, x_m \in X$, for all $n, m \in \mathbb{N}$.

$$\begin{aligned} p_s(x_n, x_m) &= p_s(T^n x_0, T^m x_0) \\ &= p_s(T(x_{n-1}), T(x_{m-1})) \\ &= \alpha \max\{p_s(x_{n-1}, x_{m-1}), p_s(x_{n-1}, x_m), p_s(x_n, x_{m-1})\} \\ &\quad + \frac{\beta}{2} \min\{p_s(x_{n-1}, x_m) + p_s(x_{m-1}, x_n), p_s(x_{n-1}, x_n) + p_s(x_{m-1}, x_m)\} \\ &= \alpha \max\{s(p_s(x_{n-1}, x_n) + p_s(x_n, x_{m-1})), p_s(x_{n-1}, x_m), p_s(x_n, x_{m-1})\} \\ &\quad + \frac{\beta}{2} \left\{ \frac{p_s(x_{n-1}, x_m) + p_s(x_{m-1}, x_n) + p_s(x_{n-1}, x_n) + p_s(x_{m-1}, x_m)}{2} \right\} \\ &= \alpha \max\{s(p_s(x_{n-1}, x_n) + s(p_s(x_n, x_m) + p_s(x_m, x_{m-1}))), p_s(x_{n-1}, x_n) \\ &\quad + p_s(x_{m-1}, x_m)\} + \frac{\beta}{4} (s(p_s(x_{n-1}, x_n) + p_s(x_n, x_m)) + s(p_s(x_{m-1}, x_m) \\ &\quad + p_s(x_m, x_n)) + p_s(x_{n-1}, x_n) + p_s(x_{m-1}, x_m)) \\ &= \alpha s(p_s(x_{n-1}, x_n) + s(p_s(x_n, x_m) + p_s(x_m, x_{m-1}))) \\ &\quad + \frac{\beta}{4} (s p_s(x_{n-1}, x_n) + 2s p_s(x_n, x_m) + s p_s(x_{m-1}, x_m) \\ &\quad + p_s(x_{n-1}, x_n) + p_s(x_{m-1}, x_m)) \\ &\leq (\alpha s + \frac{s\beta}{4} + \frac{\beta}{4}) p_s(x_{n-1}, x_n) + (\alpha s^2 + \frac{s\beta}{2}) p_s(x_n, x_m) \\ &\quad + (\alpha s^2 + \frac{s\beta}{4} + \frac{\beta}{4}) p_s(x_{m-1}, x_m), \end{aligned} \quad (7)$$

implies that

$$\begin{aligned}
 (1 - (\alpha s^2 + \frac{s\beta}{2}))p_s(x_n, x_m) &\leq (\alpha s + \frac{s\beta}{4} + \frac{\beta}{4})p_s(x_{n-1}, x_n) + \\
 &\quad (\alpha s^2 + \frac{s\beta}{4} + \frac{\beta}{4})p_s(x_{m-1}, x_m) \\
 &\leq \frac{2}{2 - 2\alpha s^2 - s\beta}((\alpha s + \frac{s\beta}{4} + \frac{\beta}{4})p_s(x_{n-1}, x_n) \\
 &\quad + (\alpha s^2 + \frac{s\beta}{4} + \frac{\beta}{4})p_s(x_{m-1}, x_m)).
 \end{aligned} \tag{8}$$

Now, taking the limit as $n, m \rightarrow +\infty$ in (7), we have

$$\lim_{n, m \rightarrow +\infty} p_s(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X . For X being complete, there exists $\hat{x} \in X$ such that

$$\lim_{n \rightarrow +\infty} p_s(x_n, \hat{x}) = \lim_{n, m \rightarrow +\infty} p_s(x_n, x_m) = p_s(\hat{x}, \hat{x}) = 0.$$

Now, we proceed to prove the existence of the fixed point of T satisfying (1). Let $x_0 \in X$ be such that $x_0 \# T(x_0)$. If $T(x_0) = x_0$ then, x_0 is a fixed point of T . Recall that, T is order-preserving and $x_0 \# T(x_0)$ then, we have $x_0 \# T(x_0) = x_1$, $x_1 \# T(x_1) = x_2$, $x_2 \# T(x_2) = x_3, \dots, x_n \# T(x_n) = x_{n+1}$. By transitivity of $\#$, we have $x_0 \# x_1 \# x_2 \# x_3 \# \dots \# x_n \# x_{n+1} \# \dots$.

For showing $\hat{x} \in X$ is a fixed point of T , we proceed as follows:

$$\begin{aligned}
 p_s(\hat{x}, T(\hat{x})) &\leq s[p_s(\hat{x}, x_{n+1}) + p_s(x_{n+1}, T(\hat{x}))] - p_s(x_{n+1}, x_{n+1}) \\
 &\leq s[p_s(\hat{x}, x_{n+1}) + p_s(T(x_n), T(\hat{x}))] \\
 &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha \max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\}] \\
 &\quad + \frac{\beta}{2} \min\{p_s(x_n, T(\hat{x})) + p_s(\hat{x}, T(x_n)), p_s(x_n, T(x_n)) + p_s(\hat{x}, T(\hat{x}))\} \\
 &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha \max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\}] \\
 &\quad + \frac{\beta}{4} (p_s(x_n, T(\hat{x})) + p_s(\hat{x}, T(x_n)) + p_s(x_n, T(x_n)) + p_s(\hat{x}, T(\hat{x}))).
 \end{aligned} \tag{9}$$

Case I: Suppose $\max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\} = p_s(x_n, \hat{x})$. Then, from inequality (9), we have

$$\begin{aligned}
 p_s(\hat{x}, T(\hat{x})) &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha p_s(x_n, \hat{x}) \\
 &+ \frac{\beta}{2}(\frac{p_s(x_n, T(\hat{x})) + p_s(\hat{x}, T(x_n)) + p_s(x_n, T(x_n)) + p_s(\hat{x}, T(\hat{x}))}{2})] \\
 &\leq s p_s(\hat{x}, x_{n+1}) + s \alpha p_s(x_n, \hat{x}) + \frac{s\beta}{4}(s(p_s(x_n, \hat{x}) + p_s(\hat{x}, T(\hat{x}))) \\
 &\quad + p_s(\hat{x}, x_{n+1}) + p_s(x_n, x_{n+1}) + p_s(\hat{x}, T(\hat{x}))) \\
 &= (s + \frac{s\beta}{4})p_s(\hat{x}, x_{n+1}) + (s\alpha + \frac{s^2\beta}{4})p_s(x_n, \hat{x}) \\
 &\quad + (\frac{s^2\beta}{4} + \frac{s\beta}{4})p_s(\hat{x}, T(\hat{x})) + \frac{s\beta}{4}p_s(x_n, x_{n+1}).
 \end{aligned}$$

From the above inequality, we have

$$\begin{aligned}
 (1 - \frac{s^2\beta}{4} - \frac{s\beta}{4})p_s(\hat{x}, T(\hat{x})) &\leq (s + \frac{s\beta}{4})p_s(\hat{x}, x_{n+1}) + \\
 &\quad (s\alpha + \frac{s^2\beta}{4})p_s(x_n, \hat{x}) + \frac{s\beta}{4}p_s(x_n, x_{n+1}),
 \end{aligned}$$

which implies

$$\begin{aligned}
 p_s(\hat{x}, T(\hat{x})) &\leq \frac{4}{4 - s^2\beta - s\beta}[(s + \frac{s\beta}{4})p_s(\hat{x}, x_{n+1}) + \\
 &\quad (s\alpha + \frac{s^2\beta}{4})p_s(x_n, \hat{x}) + \frac{s\beta}{4}p_s(x_n, x_{n+1})].
 \end{aligned} \tag{10}$$

We can observe that for $\beta \in \min\{\frac{1}{s^2}, \frac{2}{s+1}\}$,

$$\begin{aligned}
 4 - s^2\beta - s\beta &= 4 - s^2\beta - s\beta \\
 &= 4 - s\beta(s + 1).
 \end{aligned} \tag{11}$$

If $\beta = \frac{1}{s^2}$, then, from equality (11) we have

$$\begin{aligned}
4 - s^2\beta - s\beta &= 4 - s\beta(s+1) \\
&= 4 - s(s+1)\frac{1}{s^3} \\
&= 4 - \frac{s+1}{s^2} \\
&> 0 \text{ for all } s \geq 1.
\end{aligned} \tag{12}$$

Similarly, if $\beta = \frac{2}{s+1}$, then, from equality (11),

$$\begin{aligned}
4 - s^2\beta - s\beta &= 4 - s\beta(s+1) \\
&= 4 - (s+1)s\frac{2}{s+1} \\
&\leq 4 - s(s+1)\frac{1}{s^3} \\
&> 0 \text{ for all } s \geq 1.
\end{aligned} \tag{13}$$

From equalities (12) and (13), we conclude that the right-hand side of (10) is non-negative.

Case II: Suppose $\max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\} = p_s(x_n, T(\hat{x}))$. Then, from inequality (9), we have

$$\begin{aligned}
p_s(\hat{x}, T(\hat{x})) &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha p_s(x_n, T(\hat{x})) \\
&\quad + \frac{\beta}{2}(p_s(x_n, T(\hat{x})) + p_s(\hat{x}, T(x_n)) + p_s(x_n, T(x_n)) + p_s(\hat{x}, T(\hat{x})))] \\
&\leq s[p_s(\hat{x}, x_{n+1}) + \alpha s(p_s(x_n, \hat{x}) + p_s(\hat{x}, T(\hat{x}))) \\
&\quad + \frac{\beta}{4}(s(p_s(x_n, \hat{x}) + p_s(\hat{x}, T(\hat{x}))) + p_s(\hat{x}, x_{n+1}) + p_s(x_n, x_{n+1}) + p_s(\hat{x}, T(\hat{x})))] \\
&\leq (s + \frac{\beta}{4}s)p_s(\hat{x}, x_{n+1}) + (s^2\alpha + \frac{\beta}{4}s^2)p_s(x_n, \hat{x}) \\
&\quad + (s^2\alpha + \frac{\beta}{4}s^2 + \frac{\beta}{4}s)p_s(\hat{x}, T(\hat{x})) + \frac{\beta}{4}sp_s(x_n, x_{n+1}),
\end{aligned}$$

from the above inequality, we have

$$\begin{aligned} (1 - s^2\alpha - \frac{\beta}{4}s^2 - \frac{\beta}{4}s)p_s(\hat{x}, T(\hat{x})) &\leq (s + \frac{\beta}{4}s)p_s(\hat{x}, x_{n+1}) \\ &+ (s^2\alpha + \frac{\beta}{4}s^2)p_s(x_n, \hat{x}) + \frac{\beta}{4}sp_s(x_n, x_{n+1}), \end{aligned}$$

so that

$$\begin{aligned} p_s(\hat{x}, T(\hat{x})) &\leq \frac{4}{4 - 4s^2\alpha - \beta s^2 - \beta s} \left[(s + \frac{\beta}{4}s)p_s(\hat{x}, x_{n+1}) \right. \\ &\quad \left. + (s^2\alpha + \frac{\beta}{4}s^2)p_s(x_n, \hat{x}) + \frac{\beta}{4}sp_s(x_n, x_{n+1}) \right] \end{aligned} \quad (14)$$

from the fact that $\theta \in \min\{\frac{1}{s^3}, \frac{2}{s+1}\}$, we have if $\alpha > \beta$ then by (14), we have

$$\begin{aligned} 4 - 4s^2\alpha - \beta s^2 - \beta s &= 4 - 4s^2\beta + \beta s^2 + \beta s \\ &= 4 - (5s + 1)s\beta. \end{aligned} \quad (15)$$

If $\beta = \frac{1}{s^3}$ by (15), we have

$$\begin{aligned} 4 - 4s^2\alpha - \beta s^2 - \beta s &= 4 - (5s + 1)s\beta \\ &= 4 - (5s + 1)s\frac{1}{s^3} \\ &\geq 0 \quad \text{for all } s \geq 1. \end{aligned} \quad (16)$$

If $\beta = \frac{2}{s+1}$ by (15), we have

$$\begin{aligned} 4 - 4s^2\alpha - \beta s^2 - \beta s &= 4 - (5s + 1)s\beta \\ &= 4 - (5s + 1)s\frac{2}{s+1} \\ &\leq 4 - (5s + 1)s\frac{1}{s^3} \\ &\geq 0 \quad \text{for all } s \geq 1. \end{aligned} \quad (17)$$

From inequalities (16) and (17), we conclude that the right-hand side of (10) is non-negative.

If $\alpha < \beta$, then by (14), we have

$$\begin{aligned} 4 - 4s^2\alpha - \beta s^2 - \beta s &= 4 - 4s^2\alpha + \alpha s^2 + \beta s \\ &= 4 - 5s^2\alpha + s\alpha \\ &= 4 - (5s + 1)s\alpha. \end{aligned} \quad (18)$$

Similarly for (18), we conclude that the right-hand side of (10) is non-negative.

Case III: Suppose $\max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\} = p_s(\hat{x}, T(x_n))$. Then, from inequality (9), we have

$$\begin{aligned} p_s(\hat{x}, T(\hat{x})) &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha p_s(\hat{x}, T(x_n))] \\ &+ \frac{\beta}{4}(p_s(x_n, T(\hat{x})) + p_s(\hat{x}, T(x_n)) + p_s(x_n, T(x_n)) + p_s(\hat{x}, T(\hat{x}))) \\ &\leq s[p_s(\hat{x}, x_{n+1}) + \alpha p_s(\hat{x}, x_{n+1})] \\ &+ \frac{\beta}{4}(s(p_s(x_n, \hat{x}) + p_s(\hat{x}, T(\hat{x}))) + p_s(\hat{x}, x_{n+1}) + p_s(x_n, x_{n+1}) + p_s(\hat{x}, T(\hat{x}))) \\ &\leq (s + \alpha s + \frac{\beta}{4}s)p_s(\hat{x}, x_{n+1}) + \frac{\beta}{4}s^2p_s(x_n, \hat{x}) \\ &+ (\frac{\beta}{4}s^2 + \frac{\beta}{4}s)p_s(\hat{x}, T(\hat{x})) + \frac{\beta}{4}s p_s(x_n, x_{n+1}). \end{aligned}$$

By the simplification of the above equality, we have

$$\begin{aligned} p_s(\hat{x}, T(\hat{x})) &\leq \frac{4}{4 - s^2\beta - s\beta} \left[(s + \alpha s + \frac{\beta}{4}s)p_s(\hat{x}, x_{n+1}) \right. \\ &\left. + \frac{\beta}{4}s^2p_s(x_n, \hat{x}) + \frac{\beta}{4}s p_s(x_n, x_{n+1}) \right]. \end{aligned} \quad (19)$$

Note that, for any value of $\alpha, \beta \in [0, \theta)$ and $4 - s^2\beta - s\beta \geq 0$. Thus, the right-hand side of (10) is non-negative. Taking the limit as $n \rightarrow +\infty$ of both sides in the respective inequalities (10), (14) and (19), we conclude that

$$\begin{aligned} p_s(\hat{x}, T(\hat{x})) &= \lim_{n \rightarrow +\infty} p_s(\hat{x}, T(\hat{x})) \\ &= 0. \end{aligned}$$

Thus, $T(\hat{x}) = \hat{x}$.

Next, we prove that if $\hat{x} \in X$ is a fixed point of T , then $p_s(\hat{x}, \hat{x}) = 0$.

Suppose $p_s(\hat{x}, \hat{x}) \neq 0$. Then

$$\begin{aligned}
 p_s(\hat{x}, \hat{x}) &= p_s(T(\hat{x}, \hat{x})) \\
 &\leq \alpha \max\{p_s(\hat{x}, \hat{x}), p_s(\hat{x}, \hat{x}), p_s(\hat{x}, \hat{x})\} \\
 &+ \frac{\beta}{2} \min\{p_s(\hat{x}, \hat{x}) + p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, \hat{x}) + p_s(\hat{x}, T(\hat{x}))\} \\
 &= \alpha \max\{p_s(\hat{x}, \hat{x}), p_s(\hat{x}, \hat{x}), p_s(\hat{x}, \hat{x})\} \\
 &+ \frac{\beta}{2} \min\{p_s(\hat{x}, \hat{x}) + p_s(\hat{x}, \hat{x}), p_s(\hat{x}, \hat{x}) + p_s(\hat{x}, \hat{x})\} \\
 &= (\alpha + \beta)p_s(\hat{x}, \hat{x}) \\
 &= \theta p_s(\hat{x}, \hat{x}) \\
 &< p_s(\hat{x}, \hat{x}).
 \end{aligned}$$

This is contradicting the fact that $p_s(\hat{x}, \hat{x}) \neq 0$. Therefore, $p_s(\hat{x}, \hat{x}) = 0$.

Last, we will prove the uniqueness of the fixed point. Let $x_1, x_2 \in X$ be two distinct fixed points of T . Then

$$\begin{aligned}
 p_s(x_1, x_2) &= p_s(T(x_1), T(x_2)) \\
 &\leq \alpha \max\{p_s(x_1, x_2), p_s(x_1, T(x_2)), p_s(x_2, T(x_1))\} \\
 &+ \frac{\beta}{2} \min\{p_s(x_1, T(x_2)) + p_s(x_2, T(x_1)), p_s(x_1, T(x_1)) + p_s(x_2, T(x_2))\} \\
 &= \alpha \max\{p_s(x_1, x_2), p_s(x_1, x_2), p_s(x_1, x_2)\} \\
 &+ \frac{\beta}{2} \min\{p_s(x_1, x_2) + p_s(x_1, x_2), p_s(x_1, x_2) + p_s(x_1, x_2)\} \\
 &= (\alpha + \beta)p_s(x_1, x_2) \\
 &= \theta p_s(x_1, x_2) \\
 &< p_s(x_1, x_2).
 \end{aligned}$$

This is a contradiction. Therefore, the fixed point is unique.

REMARK 2. If we take $\alpha = \frac{\beta}{2}$ and $p_s(x, T(y)) + p_s(y, T(x)) \geq p_s(x, T(x)) + p_s(y, T(y))$ then we find Theorem 1 of Batsari et al. (Batsari & Kumam, 2020).

COROLLARY 3. Let (X, p) be a complete partial metric space associated with a partial order $\#$. Suppose an order-preserving mapping $T : X \rightarrow X$ satisfies

$$\begin{aligned}
 p_s(T(x), T(y)) &\leq \alpha \max\{p_s(x, y), p_s(x, T(y)), p_s(y, T(x))\} \\
 &+ \frac{\beta}{2} \min\{p_s(x, T(y)) + p_s(y, T(x)), p_s(x, T(x)) + p_s(y, T(y))\}
 \end{aligned} \tag{20}$$

for all comparable $x, y \in X$, where $\theta \in [0, 1]$. If there exists $x_0 \in X$ such that $x_0 \# T(x_0)$, then T has a unique fixed point $\hat{x} \in X$ and $p(\hat{x}, \hat{x}) = 0$.

Now we apply our main result similar to (Batsari & Kumam, 2020) as follows:

APPLICATION TO QUANTUM OPERATIONS

In quantum systems, measurements can be seen as quantum operations (Seevinck, 2003). Quantum operations are very important in narrating quantum systems that collaborate with the environment.

Let $\mathcal{B}(H)$ be the set of bounded linear operators on the separable complex Hilbert space H ; $\mathcal{B}(H)$ is the state space of consideration. Suppose $\mathcal{A} = \{A_i, A_i^* : i = 1, 2, 3, \dots\}$ is a collection of operators $A_i^* \in \mathcal{B}(H)$ satisfying $\sum A_i A_i^* \leq I$. A map $\phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ of the form $\phi_A(B) = \sum A_i B A_i^*$ is called a quantum operation (Arias et al, 2002), quantum operations can be used in quantum measurements of states. If the A_i 's are self adjoint then, ϕ_A is self-adjoint.

General quantum measurements that have more than two values are narrated by effect-valued measures (Arias et al, 2002). Denote the set of quantum effects by $\varepsilon(H) = \{A \in \mathcal{B}(H) : 0 \leq A \leq I\}$. Consider the discrete effect-valued measures narrated by a sequence of $E_i \in \varepsilon(H)$, $i = 1, 2, \dots$ satisfying $\sum E_i = I$ where the sum converges in the strong operator topology. Therefore, the probability that outcome i eventuates in the state ρ is $\rho(E_i)$ and the post-measurement state given that i eventuates is $\frac{E_i \rho E_i}{\text{tr}(\rho E_i)}$ (Arias et al, 2002). Furthermore, the resulting state after the implementation of measurement without making any consideration is given by

$$\phi(\rho) = \sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}} \quad (21)$$

If the measurement does not disturb the state ρ , then we have $\phi(\rho) = \rho$.

Furthermore, the probability that an effect A eventuates in the state ρ given that the measurement was conducted is

$$P_{\phi(\rho)}(A) = \text{tr} \left[A \sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}} \right] = \text{tr} \left(\sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}} \rho \right) \quad (22)$$

If A is not interrupted by the measurement in any state we have

$$\sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}} = A,$$

and by defining $\phi(A) = \sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}$, we end up with $\phi(A) = A$.

From now, we will be dealing with a bi-level ($|0\rangle$, $|1\rangle$) single qubit quantum system where a quantum state $|\Psi\rangle$ can be narrated as

$$|\Psi\rangle = a|0\rangle + b|1\rangle, \text{ with } a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1$$

see (Batsari & Kumam, 2020; Nielsen & Chuang, 2000). Considering the characterization of a bi-level quantum system by the Bloch sphere (Figure 1) above, a quantum state ($|\Psi\rangle$) can be represented with the density matrix below (ρ),

$$|\Psi\rangle = \rho = \frac{1}{2} \begin{pmatrix} 1 + \eta \cos \theta & \eta \cos \phi \sin \theta - i \eta \sin \phi \sin \theta \\ \eta \cos \phi \sin \theta + i \eta \sin \phi \sin \theta & 1 - \eta \cos \theta \end{pmatrix},$$

$\eta \in [0, 1]$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$.

(23)

Also, the density (ρ) matrix is,

$$\rho = \frac{1}{2}[I + \bar{r}_\rho \cdot \bar{\sigma}] = \frac{1}{2} \begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix} \quad (24)$$

where $\bar{r}_\rho = [r_x, r_y, r_z]$ is the Bloch vector with $\|\bar{r}_\rho\| \leq 1$, and $\sigma = [\sigma_x, \sigma_y, \sigma_z]$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

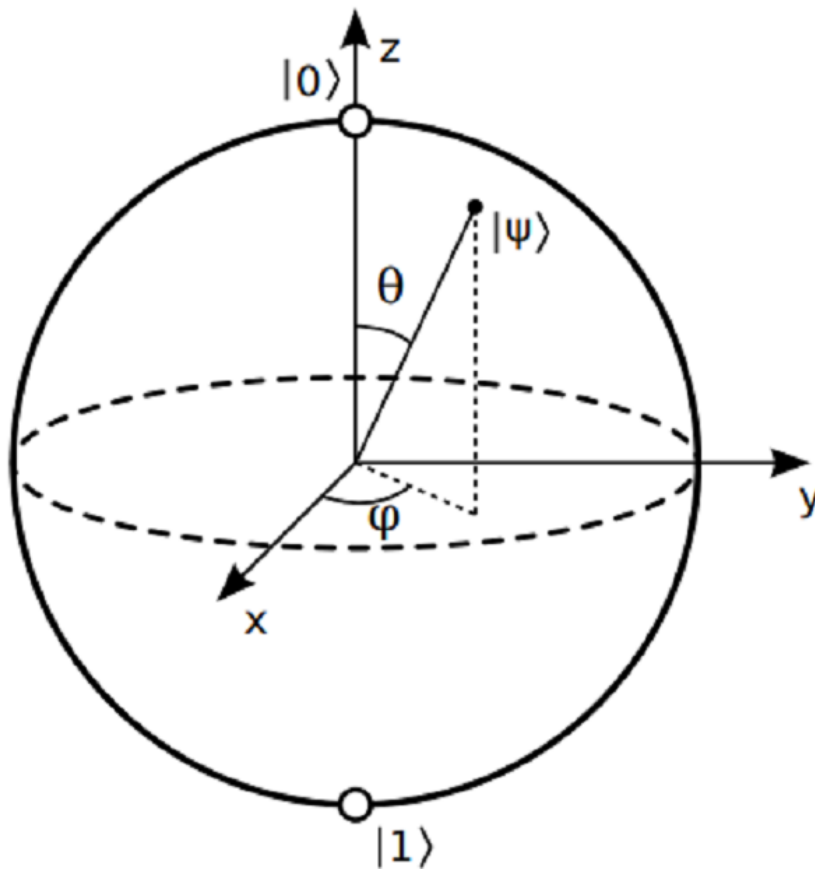


FIGURE 1
Bloch sphere

Let ρ, σ be two quantum states in a bi-level quantum system. Then, the Bures fidelity (Bures, 1969) between the quantum states ρ and σ is defined as

$$F(\rho, \sigma) = [\text{tr} \sqrt{\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}}]^2.$$

The Bures fidelity satisfies $0 \leq F(\rho, \sigma) \leq 1$, if $\rho = \sigma$ it takes the value 1 and 0 if ρ and σ have an orthogonal support (Nielsen & Chuang, 2000).

Now consider a two-level quantum system X represented with the collection of density matrices $\{\rho : \rho \text{ is as defined in Equation (24)}\}$. Define the function $p_s : X \times X \rightarrow \mathbb{R}_+$ as follows:

$$p_s(\rho, \delta) = \begin{cases} \max\{\|\overline{r_\rho}\|, \|\overline{r_\delta}\|\} e^{\frac{1}{6}(1-F(\rho, \delta))}, & \rho \neq \delta, \\ 0, & \rho = \delta. \end{cases}$$

It is easy to show that p_s is a b-metric on X with s taking the value 1 approximately. They also define an order relation $\#$ on X by

$$\rho \preceq \delta \text{ iff the line from the origin joining the point } \overline{r_\delta} \text{ passes through } \overline{r_\rho}. \quad (25)$$

It is easy to show that the order relation defined above is a partial order (Batsari & Kumam, 2020).

As in (Batsari & Kumam, 2020), we find the following corollary.

COROLLARY 4. Let (p_s, X) be a complete partial b-metric space associated with the above order $\#$. Suppose an order-preserving quantum operation $T : X \rightarrow X$ that satisfies conditions in Theorems 1. Then, T has a fixed point.

The following example validates our main result.

EXAMPLE 0.1. Consider the depolarizing quantum operation T on the Bloch sphere X ; $T(\rho) = \frac{1}{2}p + (1-p)\rho$ with the depolarizing parameter $p \in [0, 1]$. Let the comparable quantum states satisfy (25).

We examine that $T : X \rightarrow X$ satisfies all the conditions of our theorem. Now, let $\rho, \delta \in X$. We show that T is order preserving with definition (25). For this, we will prove that if $\rho \# \delta$ then $T(\rho) \# T(\delta)$.

Therefore, as (Batsari & Kumam, 2020) using the Bloch sphere representation of states in a bi-level quantum system below

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + \mu \cos \theta & \mu \cos \phi \sin \theta - i\mu \sin \phi \sin \theta \\ \mu \cos \phi \sin \theta + i\mu \sin \phi \sin \theta & 1 - \mu \cos \theta \end{pmatrix},$$

$$\mu \in [0, 1], 0 \leq \theta \leq \pi, \text{ and } 0 \leq \phi \leq 2\pi,$$

So

$$\begin{aligned} T(\rho) &= \frac{1}{2} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \\ &\quad \frac{1-p}{2} \begin{pmatrix} 1 + \mu \cos \theta & \mu \cos \phi \sin \theta - i\mu \sin \phi \sin \theta \\ \mu \cos \phi \sin \theta + i\mu \sin \phi \sin \theta & 1 - \mu \cos \theta \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 + \mu \cos \theta & \mu \cos \phi \sin \theta - i\mu \sin \phi \sin \theta \\ \mu \cos \phi \sin \theta + i\mu \sin \phi \sin \theta & 1 - \mu \cos \theta \end{pmatrix} \\ &\quad - \frac{1}{2} \begin{pmatrix} p + p\mu \cos \theta & p\mu \cos \phi \sin \theta - ip\mu \sin \phi \sin \theta \\ p\mu \cos \phi \sin \theta + ip\mu \sin \phi \sin \theta & p - p\mu \cos \theta \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1+(1-p)\mu \cos \theta & (1-p)[\mu \cos \phi \sin \theta - i\mu \sin \phi \sin \theta] \\ (1-p)[\mu \cos \phi \sin \theta + i\mu \sin \phi \sin \theta] & 1-(1-p)\mu \cos \theta \end{pmatrix} \end{aligned}$$

Clearly, the angles θ and ϕ do not change by the depolarizing quantum operation T . Also, we can deduce that the distance of the quantum state ρ from the origin given by μ is greater than or equal to the distance

of the new quantum state $T(\rho)$ from the origin given by $(1-p)\mu$, $p \in [0, 1]$. Consequently, for any two quantum states which are comparable $\rho, \delta \in X(\rho \# \delta)$, with respective distances from the origin μ_ρ and μ_δ such that, $\mu_\rho \leq \mu_\delta$, the depolarizing quantum operation T constructs two quantum states $T(\rho), T(\delta) \in X$, have distances $(1-p)\mu_\rho$ and $(1-p)\mu_\delta$ from the origin for $p \in [0, 1]$ respectively. Since $\mu_\rho \leq \mu_\delta$, then $(1-p)\mu_\rho \leq (1-p)\mu_\delta$, for all $p \in [0, 1]$. Thus, $T(\rho) \# T(\delta)$, which proves T is order-preserving.

The fidelity of any two quantum states $\rho = \frac{1}{2}(I_2 + \vec{r}_\rho \cdot \vec{\sigma})$ and $\delta = \frac{1}{2}(I_2 + \vec{r}_\delta \cdot \vec{\sigma})$ is,

$$F(\rho, \delta) = \frac{1}{2} [1 + \vec{r}_\rho \cdot \vec{r}_\delta + \sqrt{1 - \|\vec{r}_\rho\|^2} \sqrt{1 - \|\vec{r}_\delta\|^2}] \quad (26)$$

see (Batsari & Kumam, 2020; Chen et al, 2002), where $\vec{r}_\rho \cdot \vec{r}_\delta$ is the inner dot product between the vectors $\vec{r}_\rho$ and $\vec{r}_\delta$. So, for any comparable quantum states $\rho = \frac{1}{2}(I_2 + \vec{r}_\rho \cdot \vec{\sigma})$ and $\delta = \frac{1}{2}(I_2 + \vec{r}_\delta \cdot \vec{\sigma})$, $\vec{r}_\rho \cdot \vec{r}_\delta = \|\vec{r}_\rho\| \|\vec{r}_\delta\| \cos \vartheta$ for ϑ being the angle between $\vec{r}_\rho$ and $\vec{r}_\delta$. Using Equation (26), we have,

- (i). $F(\rho, \rho) = 1$.
- (ii) $F(\rho, o) = \frac{1}{2}$; for ρ a pure state and o the completely mixed state.
- (iii) $F(\rho, \rho-) = 0$; for ρ a pure state that is 180° separated from ρ ,

see (Davies, 1976; Göhde, 1965). Thus, $1.000 \leq e^{\frac{1}{5}(1-F(\rho, \delta))} \leq 1.181$ for $\rho, \delta \in X$. Now, using $s = 1$ and $\theta \in [0, 1]$ on Theorems 1. We have

$$\begin{aligned} p_s(T(\rho), T(\delta)) &= \max\{\|\vec{r}_\rho\|, \|\vec{r}_\delta\|\} e^{\frac{1}{5}(1-F(T(\rho), T(\delta)))} \\ &= \frac{1}{4} \|\delta\| e^{\frac{1}{5}(1-F(T(\rho), T(\delta)))} \\ &\leq \frac{1}{4} (\|\delta\| e^{\frac{1}{5}(1-F(T(\rho), \delta))} + \|\rho\| e^{\frac{1}{5}(1-F(T(\rho), \rho))}) \\ &= \frac{1}{2} \left(\frac{1}{2} (p_s(T(\rho), \delta) + p_s(T(\rho), \rho)) \right) \\ &= \frac{1}{2} \left(\frac{1}{2} \max\{p_s(\rho, \delta), p_s(\rho, T(\delta)), p_s(T(\rho), \delta)\} \right. \\ &\quad \left. + \min\left\{ \frac{p_s(\rho, T(\delta)) + p_s(T(\rho), \delta)}{2}, \frac{p_s(\rho, T(\rho)) + p_s(\delta, T(\delta))}{2} \right\} \right). \end{aligned}$$

Taking $\alpha = \frac{1}{2}$ and $\beta = 1$, condition (1) in Theorem 1 is satisfied. So T has a unique fixed point in X .

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