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Original scientific papers

New combinatorial proof of the multiple binomial coefficient identity

НОВОЕ КОМБИНАТОРНОЕ ДОКАЗАТЕЛЬСТВО ТОЖДЕСТВА С УЧАСТИЕМ КРАТНОГО БИНОМИАЛЬНОГО КОЭФФИЦИЕНТА

НОВ КОМБИНАТОРНИ ДОКАЗ ИДЕНТИТЕТА СА ВИШЕСТРУКИМ БИНОМНИМ КОЕФИЦИЈЕНТИМА

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Abstract:

Introduction/purpose: In this paper a new combinatorial proof of an al- ready existing multiple sum with multiple binomial coefficients is given. The derived identity is related to the Fibonacci numbers.
Methods: Combinatorial reasoning is used to obtain the results.
Results: The already known identity was obtained by using a new com- binatorial reasoning.
Conclusions: The new combinatorial reasoning led to the solution of the already existing identity.
Keywords: Fibonacci numbers, combinatorics.

Резюме:

Введение/цель: В данной статье представлено новое ком- бинаторное доказательство уже существующей кратной суммы тождества биномиальных коэффициентов, связан- ной с числами Фибоначчи.
Методы: В статье использованы комбинаторные рассуж- дения.
Результаты: Уже известное тождество было получено с помощью нового комбинаторного рассуждения.
Выводы: Новое комбинаторное рассуждение привело к ре- шению уже существующего тождества.
Ключевые слова: Числа Фибоначчи, комбинаторика.

Abstract:

Увод/циљ: У овом раду представљен је нов комбинатор- ни доказ већ постојећег вишеструког збира, вишеструких биномних коефицијената идентитета који је у вези са Фи- боначијевим бројевима
Методе: Како би се дошло до резултата користи се ком- бинаторно резоновање.
Резултати: Добија се већ познати идентитет коришће- њем новог комбинаторног резоновања.
Закључак: Ново комбинаторно резоновање довело је до ре- шења већ постојећег идентитета.
Keywords: Фибоначијеви бројеви, комбинаторика.

Introduction

The Fibonacci numbers date as early as 200 BC in Indian math- ematics, but they were named after the Italian mathematician Leonardo of Pisa, later known as Fibonacci. He published his masterpiece The Book of the Abacus in 1202 where he introduced a well-known problem of rabbits, the result of which was the Fibonacci numbers, known to many today. The Fibonacci numbers have been investigated by many and they occur in many areas. For more information about the Fibonacci numbers consult the following books (Flajolet & Sedgewick, 2009; Gessel, 1972; Grimaldi, 2012; Singh, 1985). The topic discussed in this paper is how to prove an already known multiple binomial coefficient sum, using different combinatorial reasoning than to the one already known. For the original proof see p.69 (Benjamin & Quinn, 2003).

Let us denote the notation which will be used throughout the paper. Writing

$$\sum_{\substack{\prod_{i=1}^{l} k_i=0 \\ l\geqslant 2}}^{n}$$

we mean that the number of sums depends on the parameter l, for example setting $l = 3$ we get

$$\sum_{\prod_{i-1}^{3} k_i=0}^{n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=0}^{n}$$

MAIN RESULTS

We give our new combinatorial proof of the multi binomial identity.

THEOREM 1. *The following equality holds*

$$\sum_{\substack{\prod_{i=1}^{l+1} k_i=0 \\ l\geqslant 2}}^{n} \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_l}{k_{l+1}} = (F_{l+1})^n \tag{1}$$

The F_l *sequence is defined as follows*

$$F_{l+1} = F_l + F_{l-1}, l \geq 2, F_1 = 2, F_2 = 3.$$

Where F_l *is a Fibonacci Sequence but with different initial conditions and shifted by two index places.*

Proof. We will prove our Theorem using combinatorial reasoning. Let n denote the number of white balls and *k.* the number of colors we paint the balls with. The base case l=1 is trivial, we have n balls and we choose i of them to paint yellow $\binom{n}{i}$. On the right side we choose between two colors, yellow and white for each ball. Therefore, the right side is2^n. For simplicity let us consider the case $l = 2.$. Therefore, we have n balls and two colors. We pick k_1 . white balls and paint them yellow which gives us $\binom{n}{k_1}$. Then we pick the balls that have not been painted yellow and paint them blue, which gives us $\binom{n-k_1}{k_2}$. The number of ways to paint the balls like this gives us the left-hand side. Alternatively, we can choose 3 colors for each ball to be painted with, white, yellow and blue, that is 3^n . Therefore, we get

$$\sum_{k_1=0}^{n}\binom{n}{k_1}\sum_{k_2=0}^{n}\binom{n-k_1}{k_2} = 3^n$$

Let us now consider the case when $l = 3$, where we paint balls with three colors. We pick k_1. white balls to be painted yellow, which gives us $\binom{n}{k_1}$, then we paint those that have not been painted yellow to be painted blue $\binom{n-k_1}{k_2}$, now we choose balls which have not been painted blue to be painted red $\binom{n-k_2}{k_3}$. A ball that has been painted both yellow and red becomes an orange ball. The number of ways to paint the balls like this forms the left-hand side. On the other hand, each ball can be painted white,yellow,blue,red and orange. Therefore, each ball has 5 ways to be painted, and we obtain the equality

$$\sum_{k_1=0}^{n}\binom{n}{k_1}\sum_{k_2=0}^{n}\binom{n-k_1}{k_2}\sum_{k_3=0}^{n}\binom{n-k_2}{k_3} = 5^n.$$

Consider the case $l = 4$ with 4 colors. We choose k_1 . white balls to be painted yellow $\binom{n}{k_1}$, and k_2 balls to be painted blue which have not been painted yellow $\binom{n-k_1}{k_2}$, k_3 balls to be painted red which have not been painted blue $\binom{n-k_2}{k_3}$, k_4 balls to be painted purple which have not been painted red $\binom{n-k_3}{k_4}$.By adding a new color , in this case purple, we paint all the balls which have not been painted with red, remember we painted a yellow one red to get an orange one and we have a red one itself, therefore we paint yellow blue and white, which is the number of colors we got in the $l = 2.$ case. Therefore, by adding a new color, we have the relation $F_4 = F_3 + F_2 \Rightarrow F_4 = 5+3 \Rightarrow F_4 = 8.$, which means that by adding a new color we have the old number of colors plus the painted ones which have not been painted with the previous color. This means we have 8 colors to choose from, which in combination with n balls gives us the following equality

$$\sum_{k_1=0}^{n}\binom{n}{k_1}\sum_{k_2=0}^{n}\binom{n-k_1}{k_2}\sum_{k_3=0}^{n}\binom{n-k_2}{k_3}\sum_{k_4=0}^{n}\binom{n-k_3}{k_4} = 8^n$$

Now observing the general case, let F_n denote the number of colors

$$\underline{F_1}\ \underline{F_2}\cdots\underline{F_{n-1}}\ \underline{F_n}\ \underline{F_{n+1}}$$

.

Adding a new color, we get F_{n+1} , which means we have F_n colors and we paint all the balls in F_n that have not been painted with the n-th color inF_n, which in turn gives us that the new color may paint all the balls which have not been painted with the n-th color. Therefore, we obtain a general formula

$$F_{n+1} = F_n + F_{n-1}.$$

The right-hand side is obtained by the fact that for each ball we can choose F_{n+1} colors, therefore we get $(F_{n+1})^n$. The proof is done.

In the following Corollary, we show the usage of the derived Theorem.

EXAMPLE 1. Setting $l = 6, n = 3$ in the previously derived Theorem, we get the following.

$$\sum_{\prod_{i=1}^{7} k_i=0}^{3} \binom{3}{k_1}\binom{3-k_1}{k_2}\binom{3-k_2}{k_3}\binom{3-k_3}{k_4}\binom{3-k_4}{k_5}\binom{3-k_5}{k_6}\binom{3-k_6}{k_7} = (34)^3$$

$$\sum_{\prod_{i=1}^{7} k_i=0}^{3} \binom{3}{k_1}\binom{3-k_1}{k_2}\binom{3-k_2}{k_3}\binom{3-k_3}{k_4}\binom{3-k_4}{k_5}\binom{3-k_5}{k_6}\binom{3-k_6}{k_7} = 3930$$

CONCLUSION

1. In this paper we have shown that the number of sums and the number of binomial coefficients is related to painting balls whose result is related to the Fibonacci numbers raised to the number of balls.
2. This paper motivates further research in a direction of painting various objects and the sums they can represent.

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