



Vojnotehnicki glasnik/Military Technical Courier
ISSN: 0042-8469
ISSN: 2217-4753
vojnotehnicki.glasnik@mod.gov.rs
University of Defence
Serbia

Certain integrals involving generalized Mittag-Leffler type functions

Haq, Sirazul; Aphane, Maggie; Khan, Mohammad Saeed; Fabiano, Nicola

Certain integrals involving generalized Mittag-Leffler type functions

Vojnotehnicki glasnik/Military Technical Courier, vol. 70, no. 4, 2022

University of Defence, Serbia

Available in: <https://www.redalyc.org/articulo.oa?id=661773214002>

DOI: <https://doi.org/10.5937/vojtehg70-40296>

<http://www.vtg.mod.gov.rs/copyright-notice-and-self-archiving-policy.html>



This work is licensed under Creative Commons Attribution 4.0 International.

Certain integrals involving generalized Mittag-Leffler type functions

Некоторые интегралы с обобщенными функциями типа Миттаг-Леффлера

Неки интегралы који укључују генерализоване Миттаг-Лефлерове функције

Sirazul Haq ^a

J.S. University, India

sirajulhaq007@gmail.com

 <https://orcid.org/0000-0001-9297-2445>

DOI: <https://doi.org/10.5937/vojtehg70-40296>

Redalyc: <https://www.redalyc.org/articulo.oa?id=661773214002>

id=661773214002

Maggie Aphane ^b

Sefako Makgatho Health Sciences University, South Africa

maggie.aphane@smu.ac.za

 <https://orcid.org/0000-0002-9640-7846>

Mohammad Saeed Khan ^c

Sefako Makgatho Health Sciences University, South Africa

drsaeed9@gmail.com

 <https://orcid.org/0000-0003-0216-241X>

Nicola Fabiano ^d

University of Belgrade, Serbia

nicola.fabiano@gmail.com

 <https://orcid.org/0000-0003-1645-2071>

Received: 22 August 2022

Revised document received: 10 October 2022

Accepted: 12 October 2022

ABSTRACT:

Introduction/purpose: Certain integrals involving the generalized Mittag-Leffler function with different types of polynomials are established.

Methods: The properties of the generalized Mittag-Leffler function are used in conjunction with different kinds of polynomials such as Jacobi, Legendre, and Hermite in order to evaluate their integrals.

Results: Some integral formulae involving the Legendre function, the Bessel Maitland function and the generalized hypergeometric functions are derived.

Conclusions: The results obtained here are general in nature and could be useful to establish further integral formulae involving other kinds of polynomials.

KEYWORDS: Mittag-Leffler function, Generalized hypergeometric function, Bessel Maitland function, Jacobi polynomials, Hermite polynomials..

AUTHOR NOTES

^a J.S.University, Department of Applied Science, Shikohabad, Firozabad, U.P., Republic of India

^b Sefako Makgatho Health Sciences University, Department of Mathematics and Applied Mathematics, Ga-Rankuwa, Republic of South Africa

^c Sefako Makgatho Health Sciences University, Department of Mathematics and Applied Mathematics, Ga-Rankuwa, Republic of South Africa

^d University of Belgrade, "Vinča" Institute of Nuclear Sciences - Institute of National Importance for the Republic of Serbia, Belgrade, Republic of Serbia

nicola.fabiano@gmail.com

Р е з ю м е :

Введение/цель: В данной статье установлены определенные интегралы, включающие обобщенную функцию Миттага-Леффлера с различными типами многочленов.

Методы: Свойства обобщенной функции Миттага-Леффлера используются в сочетании с различными видами многочленов, такими как Якоби, Лежандр и Эрмит для оценки их интегралов.

Результаты: Получены некоторые интегральные формулы, включающие функцию Лежандра, функцию Бесселя Мейтланда и обобщенные гипергеометрические функции.

Выводы: Полученные результаты носят общий характер и могут быть полезны для установления дальнейших интегральных формул, включающих другие виды многочленов.

К л ю ч е в ы е с л о в а : функция Миттага-Леффлера, обобщенная гипергеометрическая функция, функция Бесселя-Мейтланда, многочлены Якоби, многочлены Эрмита.

ABSTRACT:

Увод/циљ: Дефинисани су неки интеграли који укључују генерализовану Миттаг-Лефлерову функцију са различитим врстама полинома.

Методе: Својства генерализоване Миттаг-Лефлерове функције користе се на различитим врстама полинома, као што су Јакобијеви, Лежандрови, Ермитови, како би одредили њихове интеграле.

Резултати: Изведене су неке интегралне формуле које укључују Лежандрову функцију, Бесел-Мејтландову функцију и генерализоване хипергеометријске функције.

Закључак: Добијени резултати су опште природе и могли би бити корисни за утврђивање других интегралних формула које укључују друге врсте полинома.

KEYWORDS: Миттаг-Лефлерова функција, генерализована хипергеометријска функција, Бесел-Мејтландова функција, Џекобијеви полиноми, Хермитови полиноми.

INTRODUCTION

This paper follows the lines of the companion paper (Haq et al, 2019) involving the generalized Galuè-type Struve function in which the same topics are dealt here with the generalized Mittag-Leffler functions. As it is well known, a special function:

$$\mathcal{E}_v(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(vk + 1)}, \quad z, v \in \mathbb{C}, \Re(v) > 0, \quad (1)$$

and its general form

$$\mathcal{E}_{v,\omega}(z) = \sum_{k=0}^{\infty} \frac{(z)^k}{\Gamma(vk + \omega)}, \quad z, v \in \mathbb{C}, \Re(v) > 0, \Re(\omega) > 0, \quad (2)$$

are called Mittag-Leffler functions (Erdelyi et al, 1953a), $\mathbb{C}$ being the set of complex numbers. The former was established by Mittag-Leffler (Mittag-Leffler, 1903) in connection with his method of summation of some divergent series. Certain properties of this function were studied and investigated. The function defined by (2) appeared for the first time in the work of Wiman (Wiman, 1905). The functions given by equations (1) and (2) are entire functions of order $\mu = 1/v$ and of type $\sigma = 1$ (see for example (Erdelyi et al, 1953b)).

By means of the series representations, a generalization of the functions defined by equations (1) and (2) is introduced by Prabhakar (Prabhakar, 1971) as:

$$\mathcal{E}_{v,\omega}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(vk + \omega)k!}, \quad v, \omega, \rho \in \mathbb{C}, \quad \Re(v) > 0, \Re(\omega) > 0, \quad (3)$$

where

$$(\rho)_k = \rho(\rho + 1) \dots (\rho + k - 1) = \frac{\Gamma(\rho + k)}{\Gamma(\rho)},$$

whenever $\Gamma(\rho)$ is defined, $(\rho)_0 = 1$, $\rho = 0$. It is an entire function of order $\mu = (1/v)[\Re(v)]^{-1/v}$. For various properties of this function with applications, see Prabhakar (Prabhakar, 1971). Further generalization of the Mittag-Leffler function $\mathcal{E}_{v,\omega}^{\rho}(z)$ was considered earlier by Shukla and Prajapati (Shukla & Prajapati, 2007) which is given as:

$$\mathcal{E}_{v,\omega}^{\rho,q}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_{kq} z^k}{\Gamma(vk + \omega)k!},$$

$$\text{with } z, \omega, \rho \in \mathbb{C}, \quad \Re(v) > \max(0, \Re(q) - 1), \Re(q) > 0, \quad (4)$$

which is the special case when $q \in (0, 1)$ and $\min\{\Re(\omega), \Re(\rho)\} > 0$.

In continuation of this study, Salim and Faraj (Salim & Faraj, 2012; Nadir et al, 2014) introduced a new generalization of the Mittag-Leffler function which was given as:

$$\mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_{kq} z^k}{\Gamma(vk + \omega)(\delta)_{pk}},$$

$$(\min\{\Re(v), \Re(\omega), \Re(\rho), \Re(\delta)\} > 0; \quad p, q > 0; \quad z, v, \omega, \rho, \delta \in \mathbb{C}). \quad (5)$$

Numerous generalizations and cases of the Mittag-Leffler function have been studied and investigated, see for details (Singh & Rawat, 2013; Wright, 1935b; Faraj et al, 2013; Dorrego & Cerutti, 2012; Srivastava & Tomovski, 2009; Saxena et al, 2011; Khan & Ahmed, 2012).

Integral formulae involving the Mittag-Leffler functions have been developed by many authors, see for example, (Prajapati & Shukla, 2012; Prajapati et al, 2013; Gehlot, 2021; Purohit et al, 2011). In this sequel, here, we aim to establish certain new generalized integral formulae involving the new generalization of the Mittag-Leffler function. The main result presented here is general enough to be specialized to give many interesting integral formulae which are derived as special cases.

INTEGRALS WITH THE JACOBI POLYNOMIALS

The Jacobi polynomials $P_n^{(\varrho, \sigma)}(y)$ (Rainville, 1960; Srivastava & Manocha, 1984) may be defined by

$$P_n^{(\varrho, \sigma)}(y) = \frac{(1 + \varrho)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 + \varrho + \sigma + n; \\ \varrho + 1; \end{matrix} \frac{1 - y}{2} \right]. \quad (6)$$

When $\# = \sigma = 0$, the polynomial in (6) becomes the Legendre polynomial (Rainville, 1960). From (6), it follows that $P_n^{(\varrho, \sigma)}(y)$ is a polynomial of the degree n and that

$$P_n^{(\varrho, \sigma)}(y) = \frac{(1 + \varrho)_n}{n!}. \quad (7)$$

Here, we obtain the following integrals.

THEOREM 1. *If $p, q > 0$, $z, v, \omega, \rho, \delta, \# C$, $\#(v) > 0$, $\#(\omega) > 0$, $\#(\rho) > 0$, $\#(\delta) > 0$ and $\#(\xi) > -1$, $\# > -1$, $\sigma > -1$ then the following integral formula holds true*

$$\begin{aligned} & \int_{-1}^1 y^\xi (1 - y)^\varrho (1 + y)^{\eta + hk} P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1 + y)^h] dy \\ &= \frac{(-1)^n 2^{\varrho + \eta + 1} \Gamma(\eta + hk + 1) \Gamma(n + \varrho + 1) \Gamma(\eta + hk + \sigma + 1)}{n! \Gamma(\eta + hk + \sigma + n + 1) \Gamma(\eta + hk + \varrho + n + 2)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^h z) \times {}_3F_2 \left[\begin{matrix} -\xi, \eta + hk + \sigma + 1, \eta + hk + 1; \\ \eta + hk + \sigma + n + 1, \eta + hk + \varrho + n + 2; \end{matrix} 1 \right]. \end{aligned} \quad (8)$$

Proof. Naming the left-hand side (LHS) of (8) as I_1 and using the definition (5), we have

$$\begin{aligned} I_1 &= \int_{-1}^1 y^\xi (1 - y)^\varrho (1 + y)^\eta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1 + y)^h] dy \\ I_1 &= \int_{-1}^1 y^\xi (1 - y)^\varrho (1 + y)^\eta P_n^{(\varrho, \sigma)}(y) \sum_{k=0}^{\infty} \frac{(\rho)_{kq} [z(1 + y)^h]^k}{\Gamma(vk + \omega) (\delta)_{pk}} dy, \end{aligned}$$

interchanging the order of integration and summation which is permissible under the conditions of the theorem, the above expression becomes

$$\sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk + \omega) (\delta)_{pk}} \int_{-1}^1 y^\xi (1 - y)^\varrho (1 + y)^{\eta + hk} P_n^{(\varrho, \sigma)}(y) dy. \quad (9)$$

Apply the following formula (Saxena, 2008) on (9)

$$\int_{-1}^1 y^{\xi}(1-y)^{\varrho}(1+y)^{\eta} P_n^{(\varrho, \sigma)}(y) dy = \frac{(-1)^n 2^{\varrho+\eta+1} \Gamma(\eta+1) \Gamma(n+\varrho+1) \Gamma(\eta+\sigma+1)}{n! \Gamma(\eta+\sigma+n+1) \Gamma(\eta+\varrho+n+2)} \quad (10a)$$

$$\times {}_3F_2 \left[\begin{matrix} -\xi, \eta+\sigma+1, \eta+1; \\ \eta+\sigma+n+1, \eta+\varrho+n+2; \end{matrix} 1 \right], \quad (10b)$$

provided that $\# > -1$ and $\sigma > -1$, and we get the desired result.

THEOREM 2. If $p, q > 0$, $z, v, \omega, \rho, \delta, \# C, \#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $\#(\xi) > -1, \# > -1, \sigma > -1$ then the following integral formula holds true

$$\begin{aligned} & \int_{-1}^1 (1-y)^{\eta}(1+y)^{\sigma} P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h] dy \\ &= \frac{2^{\eta+\sigma+1} \Gamma(1+\mu+m) \Gamma(1+\varrho+n)}{m! n!} \\ & \times \sum_{k=0}^{\infty} \frac{(-m)_k (1+\mu+\theta+m)_k}{\Gamma(1+\mu+k)(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l (1+\mu+\theta+m)_l}{\Gamma(1+\mu+k)(l!)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^h z) \mathbb{B}(1+\eta+hk+k+l, 1+\sigma). \end{aligned} \quad (11)$$

Proof. Denoting the LHS of (11) by I_2 and using definition (5), we get

$$\begin{aligned} I_2 &= \int_{-1}^1 (1-y)^{\eta}(1+y)^{\sigma} P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h] dy \\ I_2 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk+\omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^{\eta+hk} (1+y)^{\sigma} P_n^{(\varrho, \sigma)}(y) P_n^{(\mu, \theta)}(y) dy. \end{aligned} \quad (12)$$

Now, using (6) in (12), we get

$$I_2 = \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{(1 + \mu)_m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k(1 + \mu + \theta + m)_k}{(1 + \mu)_k 2^k k!} \times \int_{-1}^1 (1 - y)^{\eta + hk + k} (1 + y)^{\sigma} P_n^{(\varrho, \sigma)}(y) dy. \quad (13)$$

Again using (6) in (13), we attain

$$I_2 = \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{\Gamma(1 + \mu + m)\Gamma(1 + \varrho + n)}{m!n!} \sum_{k=0}^{\infty} \frac{(-m)_k(1 + \mu + \theta + m)_k}{\Gamma(1 + \mu + k)2^k(k!)} \sum_{l=0}^{\infty} \frac{(-m)_l(1 + \mu + \theta + m)_l}{\Gamma(1 + \mu + l)2^l(l!)} \quad (14a)$$

$$\times \int_{-1}^1 (1 - y)^{\eta + hk + k + l} (1 + y)^{\sigma} P_n^{(\varrho, \sigma)}(y) dy, \quad (14b)$$

but by the formula found in (Rainville, 1960; Srivastava & Manocha, 1984)

$$\int_{-1}^1 (1 - y)^{\eta + n} (1 + y)^{\sigma + n} dy = 2^{2n + \sigma + \eta + 1} \mathbb{B}(1 + \eta + n, 1 + \sigma + n), \quad (15)$$

using it in (14a, 14b), we get the required result.

THEOREM 3. *If $p, q > 0$, $v, \omega, \rho, \delta, \# C, \#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $\# > -1, \sigma > -1$, then*

$$\begin{aligned} & \int_{-1}^1 (1 - y)^{\mu} (1 + y)^{\theta} P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1 - y)^h(1 + y)^t] dy \\ &= \frac{2^{\mu + \theta + 1}}{n!} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k(k!)} \\ & \quad \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^{h+t}z) \mathbb{B}(1 + \mu + hk + k, 1 + \theta + tk). \end{aligned} \quad (16)$$

Proof. Denoting the LHS of (16) by I_3 ,

$$\begin{aligned}
I_3 &= \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1-y)^h (1+y)^t] dy \\
&= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^{\mu+hk} (1+y)^{\theta+tk} P_n^{(\varrho, \sigma)}(y) dy.
\end{aligned} \tag{17}$$

Now, using (6) in (17) we have

$$\begin{aligned}
I_3 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k 2^k k!} \\
&\quad \times \int_{-1}^1 (1-y)^{n+\mu+hk+k-n} (1+y)^{n+\theta+tk-n} dy,
\end{aligned} \tag{18}$$

further, using (15) in (18) we attain the desired result.

THEOREM 4. *If $p, q > 0$, $z, v, \omega, \rho, \delta, \# C$, $\#(v) > 0$, $\#(\omega) > 0$, $\#(\rho) > 0$, $\#(\delta) > 0$ and $\# > -1$, $\sigma > -1$, then*

$$\int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1+y)^{-h}] dy \tag{19a}$$

$$\begin{aligned}
&= \frac{2^{\mu+\theta+1}}{n!} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k (k!)} \\
&\quad \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^{-h} z) \mathbb{B}(1 + \mu + k, 1 + \theta - hk).
\end{aligned} \tag{19b}$$

Proof. Denoting the LHS of (19a, 19b) by I_4 ,

$$\begin{aligned}
I_4 &= \int_{-1}^1 (1-y)^\mu (1+y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1+y)^{-h}] dy \\
&= \sum_{k=0}^{\infty} \frac{(\rho)_{kq} (z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \int_{-1}^1 (1-y)^\mu (1+y)^{\theta-hk} P_n^{(\varrho, \sigma)}(y) dy.
\end{aligned} \tag{20}$$

Now, using (6) in (20) we attain

$$I_4 = \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k 2^k k!} \times \int_{-1}^1 (1 - y)^{n+\mu+k-n} (1 + y)^{n+\theta-hk-n} dy \quad (21)$$

further, using (15) in (21) we attain the required result.

THEOREM 5. *If $p, q > 0$, $z, v, \omega, \rho, \delta, \# C, \#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $\# > -1, \sigma > -1$, then*

$$\begin{aligned} & \int_{-1}^1 (1 - y)^\mu (1 + y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1 - y)^h (1 + y)^{-t}] dy \\ &= \frac{2^{\mu+\theta+1}}{n!} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k (k!)} \\ & \times \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}(2^{h-t} z) \mathbb{B}(1 + \mu + hk + k, 1 + \theta - tk). \end{aligned} \quad (22)$$

Proof. Denoting the LHS of (22) by I_5 ,

$$\begin{aligned} I_5 &= \int_{-1}^1 (1 - y)^\mu (1 + y)^\theta P_n^{(\varrho, \sigma)}(y) \mathcal{E}_{v, \omega, p}^{\rho, \delta, q}[z(1 - y)^h (1 + y)^{-t}] dy \\ &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \int_{-1}^1 (1 - y)^{\mu+hk} (1 + y)^{\theta-tk} P_n^{(\varrho, \sigma)}(y) dy, \end{aligned} \quad (23)$$

now using (6) in (23) we attain

$$\begin{aligned} I_5 &= \sum_{k=0}^{\infty} \frac{(\rho)_{kq}(z)^k}{\Gamma(vk + \omega)(\delta)_{pk}} \frac{(1 + \varrho)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1 + \varrho + \sigma + n)_k}{(1 + \varrho)_k 2^k k!} \\ & \times \int_{-1}^1 (1 - y)^{n+\mu+hk+k-n} (1 + y)^{n+\theta-tk-n} dy \end{aligned} \quad (24)$$

further, using (15) in (24) we attain the required result.

SOME SPECIAL CASES

If we replace η by $\xi - 1$ and put $\# = \sigma = \mu = \theta = 0$ then the integral I_2 transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$\begin{aligned}
I_6 &= \int_{-1}^1 (1-y)^{\xi-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1-y)^h] dy \\
I_6 &= \sum_{k=0}^{\infty} \frac{2^{\xi} (-m)_k (1+m)_k}{(k!)^2} \times \sum_{l=0}^{\infty} \frac{(-n)_l (1+n)_l}{l!^2} \\
&\quad \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^h z) \mathbb{B}(1+\xi+hk+k+l, 1).
\end{aligned} \tag{25}$$

If $\sigma = \# = 0$, μ is replaced by $\mu - 1$ and θ by $\theta - 1$, then the integral I3 transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$\begin{aligned}
I_7 &= \int_{-1}^1 (1-y)^{\mu-1} (1+y)^{\theta-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1-y)^h (1+y)^t] dy \\
I_7 &= \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1} (-n)_k (1+n)_k}{(k!)^2} \\
&\quad \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{h+t} z) \mathbb{B}(1+\mu+hk+k, \theta+tk).
\end{aligned} \tag{26}$$

If $\# = \sigma = 0$, μ is replaced by $\mu - 1$ and θ by $\theta - 1$ then the integral I3 transforms into the following integral involving the Legendre polynomial (Rainville, 1960)

$$I_8 = \int_{-1}^1 (1-y)^{\mu-1} (1+y)^{\theta-1} P_n(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(1-y)^h (1+y)^{-t}] dy \tag{27a}$$

$$\begin{aligned}
I_8 &= \sum_{k=0}^{\infty} \frac{2^{\mu+\theta-1} (-n)_k (1+n)_k}{(k!)^2} \\
&\quad \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{h-t} z) \mathbb{B}(1+\mu+hk+k, \theta-tk).
\end{aligned} \tag{27b}$$

INTEGRAL WITH THE BESSEL MAITLAND FUNCTION

The special case of the Wright function (Erdelyi et al, 1953b), see also (Wright, 1935a,b) written in the form

$$\phi(-; A, a; z) = {}_0\psi_1 \left[\begin{matrix} -; \\ (A, a); \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Ak+a)} \frac{z^k}{k!}, \tag{28}$$

with complex z , $a \in \mathbb{C}$ and real $A \in \mathbb{R}$. When $A = \eta$, $a = \nu + 1$ and z is replaced by $-z$, then the function $\phi(\eta, \nu + 1; -z)$ is defined by $J_\nu^\eta(z)$

$$\phi(\eta, \nu + 1; -z) = J_\nu^\eta(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\eta k + \nu + 1)} \frac{(-z)^k}{k!}, \quad (29)$$

and such a function is known as the Bessel Maitland function, or the generalized Bessel function, or the Wright generalized Bessel function, see (Mcbride, 1995).

THEOREM 6. *If $p, q > 0$, $z, \nu, \omega, \rho, \delta, \#C, \#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0, \# - \# \tau > -1, \# > 0, 0 < \tau < 1$ and $\#(\mu + 1) > 0$, then the following integral formula holds true.*

$$\int_{-1}^1 (y)^\mu J_\nu^\tau(y) \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}[z(y)^q] dy = \frac{\Gamma(\mu + \varrho k + 1)}{\Gamma(1 + \nu - \tau - \tau(\mu + \varrho k))} \times \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}(z). \quad (30)$$

Proof. Naming the LHS of (30) as I_9 , we obtain

$$\begin{aligned} I_9 &= \int_{-1}^1 (y)^\mu J_\nu^\tau(y) \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}[z(y)^q] dy \\ I_9 &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \int_0^\infty (y^{\mu + \varrho k}) J_\nu^\tau(y) dy. \end{aligned} \quad (31)$$

Now we know the formula, see (Saxena, 2008)

$$\int_0^\infty (y^\mu) J_\nu^\tau(y) dy = \frac{\Gamma(\mu + 1)}{\Gamma(1 + \nu - \tau - \tau\mu)}, \quad (32)$$

provided $\#(\mu) > -1, 0 < \tau < 1$.

Using (32) in (31), we attain

$$I_9 = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \frac{\Gamma(\mu + \varrho k + 1)}{\Gamma(1 + \nu - \tau - \tau(\mu + \varrho k))},$$

hence proved.

INTEGRALS WITH THE LEGENDRE FUNCTIONS

The Legendre functions are solution of Legendre's differential equation, see (Erdelyi et al, 1953a)

$$(1 - z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + [(\nu(\nu + 1)) - \omega^2(1 - z^2)^{-1}]w = 0, \quad (33)$$

where z, ν, ω are unrestricted.

Under the substitution $w = (z^2 - 1)^{\omega/2} \nu$ in (5.1) becomes

$$(1 - z^2) \frac{d^2 \nu}{dz^2} - 2(\omega + 1)z \frac{d\nu}{dz} + [(\nu + \omega)(\nu + \omega + 1)]\nu = 0, \quad (34)$$

and with $\lambda = 1/2 - z/2$ as the independent variable, this differential equation becomes

$$\lambda(1 - \lambda) \frac{d^2 \nu}{d\lambda^2} + (\omega + 1)(1 - 2\lambda) \frac{d\nu}{d\lambda} + [(\nu - \omega)(\nu + \omega + 1)]\nu = 0. \quad (35)$$

This is the Gauss hypergeometric type equation with $a = \omega - \nu, b = \nu + \omega + 1, c = \omega + 1$. Hence it follows that the function

$$W = P_\nu^\omega(z) = \frac{1}{\Gamma(1 - \omega)} \left(\frac{z + 1}{z - 1} \right)^{\omega/2} {}_2F_1 \left[\begin{matrix} -\nu, \nu + 1; \\ 1 - \omega; \end{matrix} \middle| 1/2 - z/2 \right], \quad (36)$$

for $|1 - z| < 2$

is a solution of (33).

The function $P_\nu^\omega(z)$ is known as the Legendre function of the first kind (Erdelyi et al, 1953a). It is one valued and regular on the z -plane, supposed cut along the real axis from 1 to $-\infty$.

THEOREM 7. *If $p, q > 0, z, \nu, \omega, \rho, \delta, \# C, \#(\nu) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $\vartheta > 0$ and η is a positive integer then*

$$\begin{aligned} & \int_0^1 (y)^{\theta-1} (1 - y^2)^{\eta/2} P_\nu^\eta(y) \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}[z(y)^q] dy \\ &= \frac{(-1)^\eta \sqrt{\pi} 2^{-\theta-\eta} \Gamma(\nu + \eta + 1)}{\Gamma(1 - \eta + \nu)} \times \\ & \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \rho k)}{\Gamma(1/2 + \frac{\theta + \rho k}{2} + \eta/2 - \nu/2) \Gamma(1 + \frac{\theta + \rho k}{2} + \eta/2 + \nu/2)} \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}(z/2^\theta). \end{aligned} \quad (37)$$

Proof. Denoting the LHS of (37) by I_{10} ,

$$I_{10} = \int_0^1 (y)^{\theta-1} (1-y^2)^{\eta/2} P_\nu^\eta(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^\varrho] dy$$

$$I_{10} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \int_0^1 y^{\theta-1+\varrho k} (1-y^2)^{\eta/2} P_\nu^\eta(y) dy. \quad (38)$$

Now the integral in (38) can be solved by using the formula (Erdelyi et al, 1953a)

$$\int_0^1 y^{\theta-1} (1-y^2)^{\eta/2} P_\nu^\eta(y) dy$$

$$= \frac{(-1)^\eta \sqrt{\pi} 2^{-\theta-\eta} \Gamma(\theta) \Gamma(\nu + \eta + 1)}{(\Gamma(1/2 + \theta/2 + \eta/2 - \nu/2) \Gamma(1 + \theta/2 + \eta/2 + \nu/2) (1 - \eta + \nu))}, \quad (39)$$

provided $\Re(\theta) > 0, \eta = 1, 2, 3, \dots$

Now (38) becomes

$$I_{10} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times$$

$$\frac{(-1)^\eta \sqrt{\pi} 2^{-(\theta+\varrho k)-\eta} \Gamma(\theta + \varrho k) \Gamma(\eta + \nu + 1)}{\Gamma(1/2 + \frac{\theta+\varrho k}{2} + \eta/2 - \nu/2) \Gamma(1 + \frac{\theta+\varrho k}{2} + \eta/2 + \nu/2) \Gamma(1 - \eta + \nu)},$$

which is the desired result.

THEOREM 8. If $p, q > 0, z, v, \omega, \rho, \delta, \# C, \Re(v) > 0, \Re(\omega) > 0, \Re(\rho) > 0, \Re(\delta) > 0$ and $\vartheta > 0$ and η is a positive integer then

$$\int_0^1 (y)^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^\varrho] dy \quad (40a)$$

$$= \sqrt{\pi} 2^{-\theta+\eta} \sum_{k=0}^{\infty} \frac{\Gamma(\theta + \varrho k)}{\Gamma(1/2 + \frac{\theta+\varrho k}{2} - \eta/2 - \nu/2) \Gamma(1 + \frac{\theta+\varrho k}{2} - \eta/2 - \nu/2)}$$

$$\times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(z/2^\theta). \quad (40b)$$

Proof. Denoting the LHS of (40a, 40b) by I_{11} ,

$$I_{11} = \int_0^1 (y)^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) \mathcal{E}_{\nu, \omega, p}^{\rho, \delta, q}[z(y)^q] dy$$

$$I_{11} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times \int_0^1 y^{\theta-1+qk} (1-y^2)^{-\eta/2} P_\nu^\eta(y) dy, \quad (41)$$

now the integral in (41) can be solved by using the formula (Erdelyi et al, 1953a)

$$\int_0^\infty y^{\theta-1} (1-y^2)^{-\eta/2} P_\nu^\eta(y) dy$$

$$= \frac{\sqrt{\pi} 2^{-\theta+\eta} \Gamma(\theta)}{\Gamma(1/2 + \theta/2 - \eta/2 - \nu/2) \Gamma(1 + \theta/2 - \eta/2 - \nu/2)}, \quad (42)$$

provided $\Re(\theta) > 0, \eta = 1, 2, 3, \dots$

Again (41) becomes

$$I_{11} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\nu k + \omega)(\delta)_{pk}} \times$$

$$\frac{\sqrt{\pi} 2^{-(\theta+qk)+\eta} \Gamma(\theta + qk)}{\Gamma(1/2 + \frac{\theta+qk}{2} - \eta/2 - \nu/2) \Gamma(1 + \frac{\theta+qk}{2} - \eta/2 - \nu/2)}.$$

INTEGRALS WITH THE HERMITE POLYNOMIALS

The Hermite polynomials $H_n(y)$, see (Rainville, 1960; Srivastava & Manocha, 1984) may be defined by means of the relation

$$\exp(2yt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(y) t^n}{n!}, \quad (43)$$

valid for all finite y and t . Since

$$\exp(2yt - t^2) = \exp(2yt) \exp(-t^2)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(2y)^n t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2y)^{n-2k} t^n}{(n-2k)! k!}.
\end{aligned}$$

It follows from (43) that

$$H_n(y) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2y)^{n-2k} t^n}{(n-2k)! k!}. \quad (44)$$

The examination of equation (44) shows that $H_n(y)$ is a polynomial of degree precisely n in y and that

$$H_n(y) = 2^n y^n + \pi_{n-2}(y) \quad (45)$$

in which $\pi_{n-2}(y)$ is a polynomial of the degree $(n-2)$ in y .

THEOREM 9. *If $p, q > 0$, $z, v, \omega, \rho, \delta, \#C, \#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $h > 0, \#(\mu) = 0, 1, 2, \dots$ then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{-2h}] dy \\
&= \sqrt{\pi} 2^{2(\nu-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu - hk + 1)}{\Gamma(\mu - hk - \nu + 1)} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{2h} z).
\end{aligned} \quad (46)$$

Proof. Denoting the LHS of (9) by I_{12} , we have

$$\begin{aligned}
I_{12} &= \int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{-2h}] dy \\
I_{12} &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk + \omega)(\delta)_{pk}} \times \int_{-\infty}^{\infty} y^{2\mu-2hk} (\exp)^{-y^2} H_{2\nu}(y) dy,
\end{aligned}$$

(47)

now the integral in (47) can be solved by using the formula (Saxena, 2008)

$$\int_{-\infty}^{\infty} y^{2\mu} (\exp)^{-y^2} H_{2\nu}(y) dy = \frac{\sqrt{\pi} 2^{2(\nu-\mu)} \Gamma(2\mu+1)}{\Gamma(\mu-\nu+1)}. \quad (48)$$

Again (47) becomes

$$I_{12} = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk+\omega)(\delta)_{pk}} \times \frac{\sqrt{\pi} 2^{2(\nu-(2\mu+2hk))} \Gamma(2\mu-2hk+1)}{\Gamma(\mu-hk-\nu+1)}.$$

THEOREM 10. *If $p, q > 0$, $z, v, \omega, \rho, \delta, \# C$, $\#(v) > 0$, $\#(\omega) > 0$, $\#(\rho) > 0$, $\#(\delta) > 0$ and $h > 0$, $\#(\mu) = 0$, $1, 2, \dots$ then*

$$\begin{aligned} & \int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{2h}] dy \\ &= \sqrt{\pi} 2^{2(\nu-\mu)} \sum_{k=0}^{\infty} \frac{\Gamma(2\mu+2hk+1)}{\Gamma(\mu+hk-\nu+1)} \times \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(2^{-2h}z). \end{aligned} \quad (49)$$

Proof. Denoting the LHS of (10a, 10b) by I_{13} , we have

$$\begin{aligned} I_{13} &= \int_{-\infty}^{\infty} (y)^{2\mu} \exp(-y^2) H_{2\nu}(y) \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[z(y)^{2h}] dy \\ I_{13} &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(vk+\omega)(\delta)_{pk}} \times \int_{-\infty}^{\infty} y^{2\mu+2hk} (\exp)^{-y^2} H_{2\nu}(y) dy, \end{aligned} \quad (50)$$

using the formula mentioned in (48), then the above expression (50), we get the desired result.

INTEGRALS WITH THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

A generalized hypergeometric function (Rainville, 1960) may be defined by

$${}_pF_q \left[\begin{matrix} (\varrho)_1, (\varrho)_2, \dots, (\varrho)_p; \\ (\sigma)_1, (\sigma)_2, \dots, (\sigma)_q; \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (\varrho_i)_n}{\prod_{j=1}^q (\sigma_j)_n} \frac{z^n}{n!}, \quad (51)$$

in which no denominator parameter σ_j is allowed to be zero or a negative integer. If any numerator parameter $\#_i$ in (51) is zero or a negative integer, the series terminates.

THEOREM 11. *The following integral formula holds true,*

$$\begin{aligned} & \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^v] dy \\ &= \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{u+v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu + uk + \varrho k, \theta + vk + \sigma k), \end{aligned} \quad (52)$$

where

$$f(k) = \frac{(l_1)_k, \dots, (l_p)_k}{(m_1)_k, \dots, (m_q)_k} \quad (53)$$

provided

- (1) $\#(v) > 0$, $\#(\omega) > 0$, $\#(\rho) > 0$, $\#(\delta) > 0$ and $p, q > 0$,
- (2) $\#(\#) \geq 0$, $\#(v) \geq 0$ (both are not zero simultaneously),
- (3) $\#$ and σ are positive integers such that $\# + \sigma \geq 1$.

Proof. Representing the LHS of (11) by I_{15} , we have

$$\begin{aligned} I_{15} &= \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^v] dy \\ I_{15} &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k t^{\theta+vk-1}}{\Gamma(vk + \omega)(\delta)_{pk}} \times \\ & \int_0^t y^{\mu+uk-1} (1-x/t)^{\theta+vk-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] dy, \end{aligned}$$

putting $x = st$ and $dx = tds$, then we get

$$\begin{aligned} I_{15} &= \sum_{k=0}^{\infty} \frac{(\rho)_{qk} (zt^{u+v})^k t^{\mu+\theta-1}}{\Gamma(vk + \omega)(\delta)_{pk}} \times \\ & \int_0^1 (s)^{\mu+uk-1} (1-s)^{\theta+vk-1} {}_pF_q[(l_p); (m_q) : at^{\varrho+\sigma} s^{\varrho}(1-s)^{\sigma}] ds. \end{aligned}$$

The remaining theorems could be proved in a completely analogous fashion.

THEOREM 12. *The following integral formula holds true,*

$$I_{16} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^{-u}(t-y)^{-v}] dy$$

$$I_{16} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{-u-v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu - uk + \varrho k, \theta - vk + \sigma k), \quad (54)$$

where $f(k)$ is defined in (53)

provided

- (1) $\#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $p, q > 0$,
- (2) $\#(\#) \geq 0, \#(v) \geq 0$ (both are not zero simultaneously),
- (3) $\#$ and σ are positive integers such that $\# + \sigma \geq 1$.

THEOREM 13. The following integral formula holds true,

$$I_{17} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^u(t-y)^{-v}] dy$$

$$I_{17} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{u-v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu + uk + \varrho k, \theta - vk + \sigma k), \quad (55)$$

where $f(k)$ is defined in (53)

provided

- (1) $\#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $p, q > 0$,
- (2) $\#(\#) \geq 0, \#(v) \geq 0$ (both are not zero simultaneously),
- (3) $\#$ and σ are positive integers such that $\# + \sigma \geq 1$.

THEOREM 14. The following integral formula holds true,

$$I_{18} = \int_0^t (y)^{\mu-1} (t-y)^{\theta-1} {}_pF_q[(l_p); (m_q) : ay^{\varrho}(t-y)^{\sigma}] \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}[zy^{-u}(t-y)^v] dy$$

$$I_{18} = \mathcal{E}_{v,\omega,p}^{\rho,\delta,q}(zt^{-u+v}) t^{\mu+\theta-1} \sum_{k=0}^{\infty} f(k) t^{(\varrho+\sigma)k} \times \mathbb{B}(\mu - uk + \varrho k, \theta + vk + \sigma k), \quad (56)$$

provided

- (1) $\#(v) > 0, \#(\omega) > 0, \#(\rho) > 0, \#(\delta) > 0$ and $p, q > 0$,
- (2) $\#(\#) \geq 0, \#(v) \geq 0$ (both are not zero simultaneously),
- (3) $\#$ and σ are positive integers such that $\# + \sigma \geq 1$.

CONCLUSIONS

Certain new generalized integral formulae involving the Generalized Mittag-Leffler Type functions with many types of polynomials were established in this study. The results obtained here are general in nature and yield to many interesting formulae which are derived as particular cases.

REFERENCES

- Dorrego, G.A. & Cerutti R.A. 2012. The k-Mittag-Leffler function. *International Journal of Contemporary Mathematics Sciences*, 7(15), pp.705-716 [online]. Available at: <http://www.m-hikari.com/ijcms/ijcms-2012/13-16-2012/ceruttiIJCMS13-16-2012-2.pdf> [Accessed: 20 August 2022].
- Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F.G. 1953a. *Higher transcendental functions Volume 1 (Bateman Manuscript Project)* [online]. New York, Toronto and London: McGraw-Hill Book Company. Available at: <https://authors.library.caltech.edu/43491/19/Volume%201.pdf> [Accessed: 20 August 2022].
- Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F.G. 1953b. *Higher transcendental functions Volume 3 (Bateman Manuscript Project)* [online]. New York, Toronto and London: McGraw-Hill Book Company. Available at: <https://authors.library.caltech.edu/43491/10/Volume%203.pdf> [Accessed: 20 August 2022].
- Faraj, A.W., Salim, T.O., Sadek, S. & Ismail, J. 2013. Generalized Mittag-Leffler Function Associated with Weyl Fractional Calculus Operators. *Journal of Mathematics*, 2013 (art ID:821762). Available at: <https://doi.org/10.1155/2013/821762>.
- Gehlot, K.S. 2021. The generalized k-Mittag-Leffler function. *International Journal of Contemporary Mathematical Sciences*, 7, pp.2213-2219.
- Haq, S., Khan, A.H. & Nisar, K.S. 2019. A study of new class of integrals associated with generalized Struve function and polynomials. *Communications of the Korean Mathematical Society*, 34(1), pp.169-183. Available at: <https://doi.org/10.4134/CKMS.c170490>.
- Khan, M.A. & Ahmed, S. 2012. Fractional calculus operators involving generalized Mittag-Leffler function. *World Applied Programming*, 2(12), pp.492-499.
- McBride, A.C. 1995. V. Kiryakova Generalized fractional calculus and applications (Pitman Research Notes in Mathematics Vol. 301, Longman 1994), 388 pp., 0 582 21977 9, £39. *Proceedings of the Edinburgh Mathematical Society*, 38(1), pp.189-190. Available at: <https://doi.org/10.1017/S0013091500006325>.
- Mittag-Leffler, G.M. 1903. Sur la nouvelle fonction $Ea(x)$. *CR Acad. Sci. Paris*, 137(2), pp.554-558.
- Nadir, A., Khan, A. & Kalim, M. 2014. Integral transforms of the generalized Mittag-Leffler function. *Applied Mathematical Science*, 8(103), pp.5145-5154. Available at: <https://doi.org/10.12988/ams.2014.43218>.
- Prabhakar, T.R. 1971. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mathematical Journal*, 19(1), pp.7-15 [online]. Available at: https://ynu.repo.nii.ac.jp/?action=pages_view_main&active_action=repository_view_main_item_detail&item_id=6514&item_no=1&page_id=15&block_id=22 [Accessed: 20 August 2022].
- Prajapati, J.C., Jana, R.K., Saxena, R.K. & Shukla, A.K. 2013. Some results on the generalized Mittag-Leffler function operator. *Journal of Inequalities and Applications*, 2013 (art.number:33). Available at: <https://doi.org/10.1186/1029-242X-2013-33>.
- Prajapati, J.C. & Shukla, A.K., 2012. Decomposition of Generalized Mittag-Leffler Function and Its Properties. *Advances in Pure Mathematics*, 2(1), p.8-14. Available at: <https://doi.org/10.4236/apm.2012.21003>.
- Purohit, S.D., Kalla, S.L. & Suthar, D.L. 2011. Fractional integral operators and the multiindex Mittag-Leffler functions. *SCIENTIA Series A: Mathematical Sciences*, 21, pp.87-96 [online]. Available at: <http://scientia.mat.utfsm.cl/archivos/vol21/vol21art9.pdf> [Accessed: 20 August 2022].
- Rainville, E.D. 1960. *Special functions (Vol. 5)*. New York: The Macmillan Company.
- Salim, T.O. & Faraj, A.W. 2012. A generalization of Mittag-Leffler function and integral operator associated with fractional calculus. *Journal of Fractional Calculus and Applications*, 3(5), pp.1-13 [online]. Available at: <https://www.naturalspublishing.com/download.asp?ArtID=1893> [Accessed: 20 August 2022].
- Saxena, V.P. 2008. *The I-function*. New Delhi: Anamaya publisher.
- Saxena, R.K., Pogany, T.K., Ram, J. & Daiya, J. 2011. Dirichlet Averages of Generalized Multiindex Mittag-Leffler Functions. *Armenian Journal of Mathematics*, 3(4), pp.174-187 [online]. Available at: <http://armjmath.sci.am/index.php/ajm/article/view/79> [Accessed: 20 August 2022].

- Shukla A.K. & Prajapati J.C. 2007. On a generalization of Mittag-Leffler function and its properties. *Journal of Mathematical Analysis and Applications*, 336(2), pp.797-811. Available at: <https://doi.org/10.1016/j.jmaa.2007.03.018>.
- Singh, D.K. & Rawat, R.A.H.U.L. 2013. Integrals involving generalized Mittag-Leffler function. *Journal of Fractional Calculus and Applications*, 4(2), pp.234-244 [online]. Available at: [http://math-frac.org/Journals/JFCA/Vol4\(2\)_July_2013/Vol4\(2\)_Papers/07_Vol.%204\(2\)%20July%202013,%20No.%207,%20pp.%20234-244..pdf](http://math-frac.org/Journals/JFCA/Vol4(2)_July_2013/Vol4(2)_Papers/07_Vol.%204(2)%20July%202013,%20No.%207,%20pp.%20234-244..pdf) [Accessed: 20 August 2022].
- Srivastava, H.M. & Manocha, H.L. 1984. *A Treatise on Generating Functions*. New York: Hasted Press.
- Srivastava, H.M. & Tomovski, Ž. 2009. Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Applied Mathematics and Computation*, 211(1), pp.198-210. Available at: <https://doi.org/10.1016/j.amc.2009.01.055>.
- Wiman, A., 1905. Über den Fundamentalsatz in der Theorie der Funktionen $E_v(x)$. *Acta Mathematica*, 29, pp.191-201. Available at: <https://doi.org/10.1007/BF02403202>.
- Wright, E.M. 1935a. The Asymptotic Expansion of the Generalized Bessel Function. *Proceedings of the London Mathematical Society*, s2-38(1), pp.257-270. Available at: <https://doi.org/10.1112/plms/s2-38.1.257>.
- Wright, E.M. 1935b. The Asymptotic Expansion of the Generalized Hypergeometric Function. *Journal of the London Mathematical Society*, s1-10(4), pp.286- 293. Available at: <https://doi.org/10.1112/jlms/s1-10.40.286>.

ADDITIONAL INFORMATION

FIELD: Mathematics

ARTICLE TYPE: Original scientific paper

EDITORIAL NOTE: The fourth author of this article, Nicola Fabiano, is a current member of the Editorial Board of the Military Technical Courier. Therefore, the Editorial Team has ensured that the double blind reviewing process was even more transparent and more rigorous. The Team made additional effort to maintain the integrity of the review and to minimize any bias by having another associate editor handle the review procedure independently of the editor – author in a completely transparent process. The Editorial Team has taken special care that the referee did not recognize the author's identity, thus avoiding the conflict of interest.

КОММЕНТАРИЙ РЕДКОЛЛЕГИИ: Четвертый автор данной статьи Никола Фабиано является действующим членом редколлегии журнала «Военно-технический вестник». Поэтому редколлегия провела более открытое и более строгое двойное слепое рецензирование. Редколлегия приложила дополнительные усилия для того чтобы сохранить целостность рецензирования и свести к минимуму предвзятость, вследствие чего второй редактор-сотрудник управлял процессом рецензирования независимо от редактора-автора, таким образом процесс рецензирования был абсолютно прозрачным. Редколлегия во избежание конфликта интересов позаботилась о том, чтобы рецензент не узнал кто является автором статьи.

РЕДАКЦИЈСКИ КОМЕНТАР: Четврти аутор овог чланка Никола Фабиано је актуелни члан Уредничког одбора Војнотехничког гласника. Због тога је уредништво спровело транспарентнији и ригорознији двоструко слепи процес рецензије. Уложило је додатни напор да одржи интегритет рецензије и необјективност сведе на најмању могућу меру тако што је други уредник сарадник водио процедуру рецензије независно од уредника аутора, при чему је тај процес био апсолутно транспарентан. Уредништво је посебно водило рачуна да рецензент не препозна ко је написао рад и да не дође до конфликта интереса.

ALTERNATIVE LINK

<https://scindeks.ceon.rs/article.aspx?artid=0042-84692204797H> (html)

<https://aseestant.ceon.rs/index.php/vtg/article/view/40296> (pdf)

<https://doaj.org/article/a68c3fe3b5f74461a07057a45b0d5743> (pdf)

<https://elibrary.ru/item.asp?id=49550391> (pdf)

<http://www.vtg.mod.gov.rs/archive/2022/military-technical-courier-4-2022.pdf> (pdf)