Vojnotehnicki glasnik/Military Technical Courier
ISSN: 0042-8469
ISSN: 2217-4753
vojnotehnicki.glasnik@mod.gov.rs
University of Defence
Serbia

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Fabiano, Nicola
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Vojnotehnicki glasnik/Military Technical Courier, vol. 70, no. 4, 2022
University of Defence, Serbia
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Review papers

Path integral in quantum field theories

Интеграл по траектории в квантовой теориии поля

Интеграл пута у квантној теоријама поља

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Received: 14 January 2022
Revised document received: 13 October 2022
Accepted: 14 October 2022

Abstract:

Introduction/purpose: Starting from the Hamiltonian an alternative description of quantum mechanics has been given, based on the sum of all possible paths between an initial and a final point.
Methods: Theoretical methods of mathematical physics. Integral method based on the path integral.
Results: The method and concepts of the path integral could be applied to other branches of physics, not limited to quantum mechanics.
Conclusions: The Path Integral approach gives a global description of fields, unlike the usual Lagrangian approach which is a local description of fields.
Keywords: path integral, quantum mechanics, quantum field theory.

Р е з ю м е :

Введение / цель: Исходя из гамильтониана в настоящей статье дано альтернативное описание квантовой механики, основанное на сумме всех возможных траекторий между начальной и конечной точками.
Методы: Теоретические методы математической физики. Интегральный метод на основе интеграла по траекториям.
Результаты: Метод и концепции интеграла по траекториям могут применятся в других областях физики, не ограничиваясь квантовой механикой.
Выводы: Подход интеграла по траекториям дает всестороннее описание полей в отличие от обычного лагранжевого подхода, который представляет локальное описание полей.
К л ю ч е в ы е с л о в а : интеграл по траекториям, квантовая механика, квантовая теория поля.

Abstract:

Увод / циљ: Полазећи од Хамилтонијана, дат је алтернативни опис квантне механике, заснован на збиру свих могућих путева између почетне и финалне тачке.
Методе: Теоријске методе математичке фиизике. Интегрални метод заснован на интегралу пута.
Резултати: Методе и концепти интеграла пута могу бити примењени и на друге гране физике, нису ограничени на квантну механику.
Закључак: Приступ заснован на интегралу пута даје глобални опис поља, за разлику од уобичајеног приступа заснованог на Лагранжијану који представља локални опис поља.
Keywords: интеграл пута, квантна механика, квантна теорија поља.

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Path integral

The standard formulations of quantum mechanics, developed by Schrödinger, Heisenberg and others in the 20-ies, have shown to be equivalent to one another soon thereafter.

In 1933, Dirac (Dirac, 1933) made the observation that the action plays a central role in classical mechanics – he considered the Lagrangian formulation of classical mechanics to be more fundamental than the Hamiltonian one, but it seemed to have no important role in quantum mechanics as it was known at the time. He arrived at the conclusion that the propagator in quantum mechanics "corresponds to" $e^{(i/\#)S}$, where S is the classical action evaluated along the classical path.

In 1948, Feynman developed Dirac's suggestion (Feynman, 1948), and succeeded in deriving the third formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths, not just the classical one, between the initial and final points (Feynman, 1950, 1951). Each path contributes $e^{(i/\#)S}$ to the propagator. So while Dirac considered only the classical path, Feynman showed that all paths contribute: in a sense, the quantum particle takes all paths and the amplitudes for each path add according to the usual quantum mechanical rule for combining amplitudes.

This discovery remains valid even for relativistic quantum mechanics, represented by quantum field theory. While the usual Lagrangian approach is a local description, the path integral approach corresponds to a global description of fields, being integrated over all possible configurations.

Young's experiment

Suppose to create a Gedankenexperiment inspired by the original Young's two slit diffraction experiment (Feynman & Hibbs, 1965). A source S emits non classical particles (for instance, electrons) that end on a detector sited in O. In between, there is a screen with two slits, A_1 and A_2. The source emits particles at the time t = 0 that are detected at the time t = T. From quantum mechanics, we know that because of the superposition principle, the amplitude of particle detection is obtained by summing over all possible amplitudes, that is, the amplitude of traveling through the slit A_1 to O, and the amplitude of traveling through the slit A_2, namely

$$\mathcal{A}(\text{Starting from } S,\ \text{detected at } \mathcal{O}) = \sum_{i=1}^{2} \mathcal{A}(S \to A_i \to O)\ , \tag{1}$$

and of course one sums over different A_is when having more slits than two.

Add now another screen between A and O, with slits B_i. Then another one between B and O with slits C_i and so on. We have to add all these intermediate steps, so in the limit of infinite screens with the infinite number of slits we have the relation

$$\mathcal{A}(\text{From } S \text{ detected at } \mathcal{O} \text{ travelling in the time } T) = \sum_{path} \mathcal{A}(S \to O \text{ in the time } T \text{ for a particular path})\ , \tag{2}$$

so we have to sum over all possible paths that start from S and end in O in the time T.

We shall now fully translate eq. (2) in the quantum mechanics language. Remember that the Hamiltonian H is the generator of time translations, so the amplitude to propagate from an initial point q_I to a final point q_F in a time T is given by

$$\langle q_F \left| e^{-iTH} \right| q_I \rangle \ . \tag{3}$$

Dirac suggested, and Feynman first used eq. (3) to obtain an expression for eq. (2) by splitting each path into infinitesimal elements and then taking the continuum limit.

Divide the time T in N parts each lasting $\delta t = T/N$, then eq. (3) could be rewritten as

$$\langle q_F \left| e^{-iTH} \right| q_I \rangle = \langle q_F \left| e^{-i\delta tH} e^{-i\delta tH} \ldots e^{-i\delta tH} \right| q_I \rangle \ , \tag{4}$$

the term $e^{-i\delta tH}$ being repeated N times. Now use the fact that $|q\#$ is a complete set of states, that is, $\int dq/(2\pi)^{1/2} \, |q\rangle\langle q| = 1$, and insert 1 between every exponential factor $\exp(-i\delta tH)$:

$$\begin{gathered} \langle q_F \left| e^{-iTH} \right| q_I \rangle = \\ \left(\prod_{j=1}^{N-1} \int \frac{dq_j}{\sqrt{2\pi}} \right) \langle q_F \left| e^{-i\delta tH} \right| q_{N-1} \rangle \langle q_{N-1} \left| e^{-i\delta tH} \right| q_{N-2} \rangle \cdots \\ \langle q_2 \left| e^{-i\delta tH} \right| q_1 \rangle \langle q_1 \left| e^{-i\delta tH} \right| q_I \rangle \ . \end{gathered} \tag{5}$$

Feynman's formulation of quantum mechanics

The key ingredient of eq. (5) is the factor $\#q_j{+}1 \, |e^{-i\delta tH}| q_j \#$. From quantum mechanics we know the explicit form of the Hamiltonian function,

$$H = \frac{\hat{p}^2}{2m} + V(\hat{q}) \ , \tag{6}$$

where $\hat{p}, \hat{q}$ are the usual operators with eigenspace $\hat{p}|p\rangle = p\,|p\rangle$, $\hat{q}|q\rangle = q|q\rangle$. Since the spaces $_q$ and $_p$ are connected via a Fourier transformation, they have the property that $\#q|p\# = e^{ipq}$, $\#p|q\# = e^{-ipq}$, and the $_p$ space is complete as well as the $_q$ space: $\int dp/(2\pi)^{1/2} \, |p\rangle\langle p| = 1$. From the explicit form of the Hamiltonian (6),

$$e^{-i\delta tH} = e^{-i\delta t\hat{p}^2/2m} e^{-i\delta tV(\hat{q})} \ , \tag{7}$$

and by a judicious insertion of factors 1 coming from the completeness of the $_q$ and $_p$ spaces we find

$$\langle q_{j+1} \left| e^{-i\delta t H} \right| q_j \rangle = \frac{1}{2\pi} \int \mathrm{d}q \int \mathrm{d}p \, \langle q_{j+1} \left| e^{-i\delta t \hat{p}^2/2m} \right| p \rangle \langle p \left| e^{-i\delta t V(\hat{q})} \right| q \rangle \langle q \mid q_j \rangle \, . \tag{8}$$

It is clear that for any function f, $f(\hat{q})|q\rangle = f(q)|q\rangle$ and $f(\hat{p})|p\rangle = f(p)|p\rangle$, because it is acting on eigenstates. Therefore, we could drop the symbol of the operator in eq. (8) and write

$$\begin{aligned} \langle q_{j+1} \left| e^{-i\delta t H} \right| q_j \rangle = \\ \frac{1}{2\pi} \int \mathrm{d}q \int \mathrm{d}p \, e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} \langle q_{j+1} \mid p \rangle \langle p \mid q \rangle \langle q \mid q_j \rangle = \\ \int \mathrm{d}q \int \mathrm{d}p \, e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{ipq_{j+1}} e^{-ipq} \delta(q - q_j) = \\ e^{-i\delta t V(q_j)} \int \mathrm{d}p \, e^{-i\delta t p^2/2m} e^{ip(q_{j+1} - q_j)} \, . \end{aligned} \tag{9}$$

We could readily recognize that the last integral over $_p$ is Gaussian and can be solved with the aid of eq. (57) of (Fabiano, 2021a):

$$\begin{aligned} \langle q_{j+1} \left| e^{-i\delta t H} \right| q_j \rangle = \\ e^{-i\delta t V(q_j)} \int \mathrm{d}p \, e^{-i\delta t p^2/2m} e^{ip(q_{j+1} - q_j)} = e^{-i\delta t V(q_j)} \left(\frac{-2\pi i m}{\delta t} \right)^{1/2} \times \\ e^{[im(q_{j+1} - q_j)^2]/2\delta t} = e^{-i\delta t V(q_j)} \left(\frac{-2\pi i m}{\delta t} \right)^{1/2} e^{i\delta t (m/2)[(q_{j+1} - q_j)/\delta t]^2} \, . \end{aligned} \tag{10}$$

Putting this result into eq. (5) gives us

$$\langle q_F \left| e^{-iTH} \right| q_I \rangle = \left(\frac{-2\pi i m}{\delta t} \right)^{N/2} \prod_{j=0}^{N-1} \int \mathrm{d}q_j \, e^{i\delta t \left\{ (m/2) \sum_{j=0}^{N-1} [(q_{j+1} - q_j)/\delta t]^2 - V(q_j) \right\}} \, , \tag{11}$$

where $q_0 \equiv q_I$ and $q_N \equiv q_F$. We can now go to the continuum limit, that is, $\delta t \to 0$ or $N \to +\infty$, so we can replace $[(q_{j+1} - q_j)/\delta t]^2$ with $_q{}^2$ and sums with integrals.

A very important definition is the integral over paths:

$$\int \mathcal{D}q(t) \equiv \lim_{N\to+\infty} \left(\frac{-2\pi i m}{\delta t}\right)^{N/2} \prod_{j=0}^{N-1} \int \mathrm{d}q_j \ , \tag{12}$$

where the D symbol means that one has to integrate over all possible paths $_q(t)$ with fixed start and ending points, $q(0) = q_I$ and $q(T) = q_F$. It is a functional integration.

We have thus obtained the so called path integral representation for the amplitude:

$$\langle q_F \left| e^{-iTH} \right| q_I \rangle = \int \mathcal{D}q(t)\ e^{i\int_0^T \mathrm{d}t\ \frac{1}{2}m\dot{q}^2 - V(q)} = \int \mathcal{D}q(t)\ e^{i\int_0^T \mathrm{d}t\ \mathcal{L}(q,\dot{q})} \ . \tag{13}$$

Comparing both sides of eq. (13), one could notice that starting from the Hamiltonian we have naturally ended up with the Lagrangian. In classical mechanics, the action S is defined starting from the Lagrangian as $S(q) = \int_0^T \mathrm{d}t\ \mathcal{L}(q,\dot{q}),$ and is a functional of q(t). By restoring Planck's constant # and by dropping the explicit t notation for the functional measure, we could rewrite eq. (13) as

$$\langle q_F \left| e^{-(i/\hbar)TH} \right| q_I \rangle = \int \mathcal{D}q\ e^{(i/\hbar)S(q)} \ . \tag{14}$$

It is worth noticing that the quantum mechanical amplitude of eq. (14) involves the explicit calculation of the classical action S. The path integral is the only occurrence where the action is explicitly needed, where in all other cases only the extremisation of the action, that is, the equations of motion, are required.

Schrödinger equation

Our next step is to derive the Schrödinger equation by means of path integral formalism. Since it is a differential equation we need only to find out the infinitesimal evolution of the wave function in time and space. Setting the initial conditions as $t_I = 0$, $q_I = q'$, $t_F = t$, $q_F = q$; $\delta t = t$ and $\eta = q' - q$ are infinitesimal. The time and space evolution for the wave equation from the point $(0, q')$ to the point (q, t) is given by

$$\psi(q,t) = \int_{-\infty}^{+\infty} \mathrm{d}q'\ K(q,t;q',0)\psi(q',0) \ , \tag{15}$$

where K is the evolution amplitude with proper normalisation, as $|\psi|^2 = 1$. From eq. (10), we have the explicit form for a propagation amplitude between two points, so restoring # we can write

$$K(q,\delta t;q',0) = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2} e^{i\delta t/\hbar\left\{(m/2)[(q-q')/\delta t]^2 - V(q')\right\}} \ . \tag{16}$$

By changing the integration variable to $\eta = q' - q$ and reinserting eq. (16) into eq. (15), we obtain

$$\psi(q,\delta t)=\left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2}\int_{-\infty}^{+\infty}\mathrm{d}\eta e^{i\delta t/\hbar\left\{(m/2)[\eta/\delta t]^2-V(q+\eta)\right\}}\psi(q+\eta,0)\ . \tag{17}$$

Now, we have two infinitesimal quantities, η and δt. Because of the speed of light, we have the limit η/δt<1 and both are infinitesimals of the same order. So we can expand the potential and the wave function at the same time

$$\begin{gathered}e^{-i\delta t/\hbar V(q+\eta)}=1-i\frac{\delta t}{\hbar}\left[V(q)+\eta V'(q)+\mathcal{O}(\eta^2)\right]= \\ 1-i\frac{\delta t}{\hbar}V(q)-i\frac{\delta t}{\hbar}\eta V'(q)+\mathcal{O}(\delta t^2,\eta^2)\ ,\end{gathered} \tag{18}$$

and

$$\psi(q+\eta)=\psi(q,0)+\eta\psi'(q,0)+\frac{1}{2}\eta^2\psi''(q,0)+\mathcal{O}(\eta^3)\ . \tag{19}$$

Plugging Taylor expansions back in eq. (17) yields

$$\begin{gathered}\psi(q,\delta t)=\left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2}\int_{-\infty}^{+\infty}\mathrm{d}\eta e^{im\eta^2/(2\hbar\delta t)}\times \\ \left[\psi(q,0)-i\frac{\delta t}{\hbar}V(q)\psi(q,0)+\eta\psi'(q,0)+\frac{1}{2}\eta^2\psi''(q,0)+\mathcal{O}(\delta t^2,\eta^3)\right]\ .\end{gathered} \tag{20}$$

By inspection, the integral in η is reduced to Gaussian momenta given in eq. (58) of (Fabiano, 2021a), where linear terms vanish because of symmetry. By resolving integrals, we obtain

$$\begin{gathered}\psi(q,\delta t)=\left(\frac{m}{2\pi i\hbar\delta t}\right)^{1/2}\left[\left(\frac{2\pi i\hbar\delta t}{m}\right)^{1/2}\left(\psi(q,0)-i\frac{\delta t}{\hbar}V(q)\psi(q,0)\right)+\right. \\ \left.\left(\frac{2\pi i\hbar\delta t}{m}\right)^{1/2}\frac{i\hbar\delta t}{2m}\psi''(q,0)+\mathcal{O}(\delta t^3)\right]= \\ \psi(q,0)+\delta t\left[\frac{i\hbar}{2m}\psi''(q,0)-i\frac{1}{\hbar}V(q)\psi(q,0)\right]+\mathcal{O}(\delta t^2)\ .\end{gathered} \tag{21}$$

After moving the first term ψ(q, 0) to lhs and dividing it by δt, we obtain

$$\frac{\psi(q,\delta t)-\psi(q,0)}{\delta t}=-\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}+V(q)\right]\psi(q,0)+\mathcal{O}(\delta t^2)\ ,$$

(22)

and by taking the limit δt → 0 we obtain the time dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q,t)\ . \tag{23}$$

Relativistic field theory

Instead of dealing with fixed initial and final positions q_I and q_F we are often faced with specifying more general initial and final states |I# and |F #. Then we are interested in calculating #F|e^{-iTH}|I#, that can be obtained from eq. (13) by inserting two complete sets of states

$$\langle F|e^{-iTH}|I\rangle = \int\int \mathrm{d}q_F\ \mathrm{d}q_I\ \langle F|q_F\rangle\langle q_F\left|e^{-iTH}\right|q_I\rangle\langle q_I|I\rangle\ . \tag{24}$$

Almost always initial and final states are the same, that is, the ground state |0#. The amplitude #0|e^{-iTH}|0# is denoted by Z,

$$Z \equiv \langle 0|e^{-iTH}|0\rangle\ , \tag{25}$$

because Zustandssumme, that is, the "sum over states" was the original German term for the partition function.

The path integral formalism can be extended from quantum mechanics to continuum field theories that describe physical systems with an infinite numbers of degrees of freedom. Starting from q(t), a 0 + 1 dimensional case for quantum mechanics (we have just discretised the time coordinate in section Young's experiment) to a field theory in 1 + 1 dimensions for simplicity, ϕ(x, t), the procedure is completely analogue. The new step is the space discretisation - the length L of space has to be divided in infinitesimal parts δx such that

$$\delta x = \frac{L}{N'}\ , \tag{26}$$

and by denoting the coordinate as x_m=mδx, with $\phi(x_m)=\phi_m$ for 0≤m≤N′ we can define the functional integral over the field ϕ like:

$$\int\mathcal{D}\phi\ \equiv \lim_{L\to+\infty}\lim_{N'\to+\infty}\prod_{m=0}^{N'}\int\mathrm{d}\phi_m\ , \tag{27}$$

in complete analogy to eq. (12). The action is now of course a function of ϕ and $\partial_\mu\phi$, $S(\phi) = \int \mathrm{d}^D x_\mu \, \mathcal{L}(\phi, \partial_\mu \phi)$.

An essential difference from the quantum mechanical case, however, is that, from a mathematically rigorous point of view, the integral just defined in eq. (27) is divergent in the continuum limit. This difficulty is obviated by absorbing the divergence into a normalisation constant N when computing quantities such as, for instance, the partition function of eq. (25):

$$Z = N \int \mathcal{D}\phi \, e^{(i/\hbar)S(\phi)} \, . \tag{28}$$

From this expression for Z, we see that the integral in the classical limit # → 0 is given by a phase S multiplied by a large quantity, that is, a rapidly oscillating quantity. Mathematically, it is clear that the major contribution to the path integral comes from fields that extremise the action, while other configurations tend to cancel each other by symmetry. Those fields are the ones that satisfy

$$\frac{\delta S(\phi)}{\delta \phi} = 0 \, , \tag{29}$$

and such fields are by definition classical fields ϕ_{cl} that solve Lagrange equations

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_{cl})} = \frac{\delta \mathcal{L}}{\delta \phi_{cl}} \, . \tag{30}$$

To prove this statement, we will use the so-called saddle point method or stationary phase method that applies when the integral could be written as some exponential function. For a review on the subject, see, for instance (Fabiano & Mirkov, 2022). We can expand the action in series to read

$$S(\phi) = S(\phi_{cl}) + \frac{1}{2}\left(\frac{\delta^2 S(\phi_{cl})}{\delta \phi^2}\right)(\phi - \phi_{cl})^2 + \mathcal{O}((\phi - \phi_{cl})^3) \, , \tag{31}$$

where the linear term is missing by definition because the action is stationary. By plugging this result back into eq. (28) we have, yet another time, a Gaussian integral in infinite dimensions that can be solved with the aid of eq. (60) of (Fabiano, 2021a):

$$Z = N e^{(i/\hbar)S(\phi_{cl})} \left[\frac{2\pi i \hbar}{\det[S''(\phi_{cl})]}\right]^{1/2} [1 + \mathcal{O}(\hbar)] \, , \tag{32}$$

and it is clear that the exponential term is the essential contribution as # →0.

Free field

We begin with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] \ , \tag{33}$$

that describes the so called free or Gaussian theory. The equation of motion is the well–known Klein–Gordon equation describing a relativistic boson particle of the mass m

$$(\Box + m^2)\phi = 0 \ , \tag{34}$$

where $\Box = \partial^\mu\partial_\mu$ is the d'Alembert operator, with a plane wave solution$\phi(x, t) = e^{i(\omega t - \# \ k.\#x \ )}$ and a dispersion relation $\omega^2 = \vec{k}^2 + m^2$. Before writing the amplitude, it is customary to add a term like $J(x)\phi(x)$ in the Lagrangian,where J(x) is the so-called source function whose actual form is not relevant, provided integrals are convergent, as will be clear later. We have

$$Z = N \int \mathcal{D}\phi \ e^{i\int \mathrm{d}x^4 \ \left\{\frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] + J\phi\right\}} \ , \tag{35}$$

and focussing on the action integrating by parts, and provided the fields ϕfall off sufficiently rapidly at infinity, we could rewrite it as

$$\int \mathrm{d}x^4 \ \frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] + J\phi = \int \mathrm{d}x^4 \ \left[-\frac{1}{2}\phi(\Box^2 + m^2)\phi^2 + J\phi\right] \ . \tag{36}$$

By putting this new form back into eq. (35)

$$Z = N \int \mathcal{D}\phi \ e^{i\int \mathrm{d}x^4 \ \left\{-\frac{1}{2}\phi(\Box^2 + m^2)\phi^2 + J\phi\right\}} \ , \tag{37}$$

one obtain once again a Gaussian integral, quite similar to the one of eq. (57) of (Fabiano, 2021a). This time, however, a and b are not numbers, but matrices. Consider the generalisation of the Gaussian integral to matrices, then we have

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{N}\mathrm{d}x_i \ e^{-\frac{1}{2}\mathbf{x}\cdot A\cdot\mathbf{x} + J\cdot\mathbf{x}} = \left[\frac{(2\pi)^N}{\det(A)}\right]^{\frac{1}{2}} e^{\frac{1}{2}J\cdot A^{-1}\cdot J} \ , \tag{38}$$

where $x \cdot A \cdot x = x_i A_{ij} x_j$ and $J \cdot x = J_{ij} x_j$, with repeated indices summed over. To prove eq. (38), diagonalise A by an orthogonal transformation O : A = O−1 · D · O, where D is a diagonal matrix with elements given by all eigenvalues of A. This operation is always possible because A is a definite positive matrix, otherwise the integral would not converge. Define a new variable $y = O \cdot x$, that is, $y_i = O_{ij} x_j$, then the exponential will reduce to a sum of squares:

$$\mathbf{x} \cdot A \cdot \mathbf{x} = x_i A_{ij} x_j = \mathbf{y} \cdot O^{-1} \cdot A \cdot O \cdot \mathbf{y} = \mathbf{y} \cdot D \cdot \mathbf{y} = y_i D_{ii} y_i \ . \tag{39}$$

The Jacobian of such transformation is 1 by definition, so eq. (38) reduces to a product of one dimensional Gaussian integrals, which proves the formula for J = 0. If not a further step is needed, a variable translation defined as $y' = y + A^{-1} J$, which again does not change the integration measure, $dy' = dy$.

Coming back to eq. (37), the role of A is here played by the differential operator $-(\# + m^2)$. Its inverse is given by the function D that obeys

$$-(\Box + m^2) D(x - y) = \delta^{(4)}(x - y) \ , \tag{40}$$

because, since we are dealing with the continuum, Kronecker's delta δ_{ij} for the definition of inverse operators A^{-1}_{jk} have to be replaced by Dirac's delta functions δ(4)(x − y). The resulting function D is the well–known free propagator for a scalar relativistic particle of the mass m, here written as a less familiar function of the coordinates x instead of its more popular Fourier transform

We end up with

$$Z(J) = C e^{-(i/2) \int \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, J(x) D(x-y) J(y)} \equiv C e^{iW(J)} \ , \tag{41}$$

where D(x − y) obeys eq. (40). The overall normalisation factor C clearly does not depend on J, but on the determinant of D which has no interest.Observe that C = Z(J = 0) so that

$$Z(J) \equiv Z(J = 0) e^{iW(J)} \ , \tag{42}$$

and

$$W(J) = -\frac{1}{2} \int \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, J(x) D(x - y) J(y) \tag{43}$$

is only quadratic in J, while Z(J) depends on arbitrarily high powers of J.

Green functions

By going in momentum space, eq. (40) is easily solvable (Schwinger,1951). Remembering the Dirac delta function in momentum space

$$\delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik(x-y)}\ , \tag{44}$$

one obtains

$$D(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}\ . \tag{45}$$

With the help of eqs. (59) and (61) of (Fabiano, 2021a), it is possible to obtain the explicit form for D(x) in Euclidean space. By rewriting the denominator with eq. (61) of (Fabiano, 2021a) and computing the Gaussian integral, we obtain

$$D(x) = \frac{1}{2^D \pi^{D/2}} \int_0^{+\infty} \mathrm{d}t\ t^{-D/2} e^{-\frac{x^2}{4t} - tm^2} = \frac{1}{(2\pi)^{(D/2)}} \frac{K_{(D-2)/2}(|x|)}{|x|^{(D-2)/2}} m^{(D-2)/2}\ , \tag{46}$$

where $K_\nu(x)$ is a Bessel function. For a half integer argument, Bessel functions reduce to elementary functions, for example in D = 1

$$D(x) = \frac{1}{2m} e^{-m|x|}\ , \tag{47}$$

while for D = 3

$$D(x) = \frac{1}{4\pi|x|} e^{-m|x|}\ . \tag{48}$$

Equation (46) can be used to obtain asymptotic behaviours for D(x); one finds for $|x| \to +\infty$

$$D(x) = \left(\frac{\pi}{2}\right)^{1/2} (2\pi)^{-D/2} |x|^{(-D/2+1/2)} m^{(D/2-3/2)} e^{-m|x|}\ , \tag{49}$$

while for $|x| \to 0$ we have

$$D(x) = \frac{1}{4}\pi^{-D/2}\Gamma\left(\frac{D}{2} - 1\right)|x|^{-D+2} \quad \text{for } D > 2\ ,$$

(50)

and

$$D(x) = (4\pi)^{-D/2}\Gamma\left(1 - \frac{D}{2}\right) m^{D-2} \quad \text{for } D < 2 \tag{51}$$

respectively.

We shall see next the importance of the source function. From eq. (35),we see how a functional derivative in J(x) will furnish us the expectation value of the field. Using the fact that

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y) \ , \tag{52}$$

as per definition of a functional derivative, from Z(J) we could obtain the propagator, or the time ordered two point function as

$$-iG(x-y) = -iG(x,y) = \langle 0\,|T\phi(x)\phi(y)|\,0\rangle = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)} Z(J)\Big|_{J=0} \tag{53}$$

It is straightforward to generalise this expression to an n–point function:

$$G(x_1, x_2, \ldots, x_n) = i^n \langle 0\,|T\phi(x_1)\phi(x_2)\ldots\phi(x_n)|\,0\rangle = \frac{\delta^n Z(J)}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\Big|_{J=0} \ . \tag{54}$$

Explicitly calculating the four point function yelds:

$$\begin{aligned} -\langle 0\,|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\,0\rangle &= -\frac{\delta^4 Z(J)}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0} \\ &= G(x_1-x_2)G(x_3-x_4) + G(x_1-x_3)G(x_2-x_4) + \\ &\quad G(x_1-x_4)G(x_2-x_3) \ , \end{aligned} \tag{55}$$

the sum of all possible combinations of x_i comes out because of the functional derivative that sports also a Dirac's delta. In this manner, we have derived Wick's theorem on contractions starting purely with *c*–numbers expressions.

Z(J) can also be written as a power series in J. Calling

$$Z^{(n)}(x_1, x_2, \ldots, x_n) \equiv \left. \frac{\delta^n Z(J)}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} \right|_{J=0} , \tag{56}$$

and noting besides the equivalence with eq. (54), we can write

$$Z(J) = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \ldots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \; J(x_1)\ldots J(x_n) Z^{(n)}(x_1, \ldots, x_n) \; . \tag{57}$$

In this manner, we have shown that path integral formalism can rederive all the expressions earlier known of canonical formalism without using operators algebra.

Connected graphs

When analyzing Feynman graphs, there are two distinct types of diagrams: connected and disconnected graphs (Coleman, 198 5): the latter can be separated into two, or more, distinct parts without cutting a line; not so for the former. For instance, a propagator is a connected graph.

Z(J) is also known as the generating functional, and it generates both types of Feynman diagrams described above. However, in a variety of physical problems, for example renormalisation theory, and statistical mechanics, it is useful to generate only connected graphs. Also, the scattering amplitude receives contribution only from connected diagrams. We have already defined such generating functional in eq. (42), called W (J). By neglecting the normalisation constants, we have the relation [1]

$$W(J) = -i \log Z(J) \tag{58}$$

among two generating functionals. By taking repeated derivatives with respect to J, we find

$$\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} = \frac{i}{Z^2} \frac{\delta Z}{\delta J(x_1)} \frac{\delta Z}{\delta J(x_1)} - \frac{i}{Z^2} \frac{\delta^2 Z}{\delta J(x_1)\delta J(x_1)} , \tag{59}$$

and

$$\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} = \left(\frac{i}{Z^2} \frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)} \frac{\delta^2 Z}{\delta J(x_3)\delta J(x_4)} + \text{permutations} \right) - \frac{i}{Z} \frac{\delta^4 Z}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} . \tag{60}$$

Following the Taylor expansion of eq. (57), we could write an analogous series for W :

$$W(J) = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \dots \int \mathrm{d}x_1 \dots \mathrm{d}x_n \; J(x_1) \dots J(x_n) W^{(n)}(x_1, \dots, x_n) \; . \tag{61}$$

By taking J = 0 and comparing the two series, we arrive at:

$$iW^{(2)}(x_1, x_2) = Z^{(2)}(x_1, x_2) \; , \tag{62}$$

rather unsurprising as the propagator is connected. To higher orders, however, the relations becomes non–trivial:

$$W^{(4)}(x_1, x_2, x_3, x_4) = i \left[Z^{(2)}(x_1, x_2) Z^{(2)}(x_3, x_4) + \text{ permutations } \right] + Z^{(4)}(x_1, x_2, x_3, x_4) \; . \tag{63}$$

It is possible to prove that W generates only connected graphs to all orders, that is, that W (n) is the n–point connected Green function.

Effective action

Besides connected and disconnected diagrams, there is another important class of Feynman graphs, the one particle irreducible (1PI) diagrams.These diagrams cannot be disconnected by cutting any internal line. In other terms, one cannot obtain two Feynman diagrams by cutting a line of the 1PI diagram. Sometimes they are also known as strongly connected diagrams, because they are basically diagrams connected by more than one line.

They have a generating functional called effective action, defined by a Legendre transformation (Coleman, 1985)

$$\Gamma(\phi) = W(J) - \int \mathrm{d}^4 x \; J(x) \phi(x) \; . \tag{64}$$

The fields ϕ and J have a duality relation among them, like p and $\dot{q}$ coordinates in Hamiltonian and Lagrangian formalism. The inverse transformation gives the relation

$$W(J) = \Gamma(\phi) + \int \mathrm{d}^4 x \; J(x) \phi(x) \; . \tag{65}$$

Deriving eq. (64) with respect to ϕ gives us

$$\frac{\delta\Gamma(\phi)}{\delta\phi(x)} = -J(x) \ , \tag{66}$$

while the derivative of eq. (65) with respect to J furnishes us with the result

$$\frac{\delta W(J)}{\delta J(x)} = \phi(x) \ . \tag{67}$$

By comparing eqs. (54) and (67) we also see that

$$-i\frac{\delta \log Z(J)}{\delta J(x)} = \frac{\langle 0\,|\phi(x)|\,0\rangle}{\langle 0|0\rangle} = \phi_{cl} \ , \tag{68}$$

that is, the classical field, defined as the vacuum expectation value (VEV) of the quantum field, could be obtained by deriving the generator of connected graphs W with respect to the source field J.

By taking repeated differentials of eqs. (66) and (67) we find

$$G(x,y) = -\frac{\delta^2 W}{\delta J(x)\delta J(y)} = -\frac{\delta\phi(x)}{\delta J(y)} \ , \tag{69}$$

and

$$\Gamma(x,y) = \frac{\delta^2\Gamma}{\delta\phi(x)\delta\phi(y)} = -\frac{\delta J(x)}{\delta\phi(y)} \ . \tag{70}$$

Γ(x, y) and G(x, y) are inverse of each other. Treating them as matrices with continuous indices, we could write

$$\int \mathrm{d}^4y \ G(x,y)\Gamma(y,z) = -\int \mathrm{d}^4y \ \frac{\delta^2 W}{\delta J(x)\delta J(y)}\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)} = \int \mathrm{d}^4y \ \frac{\delta\phi(x)}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(z)} = \frac{\delta\phi(x)}{\delta\phi(z)} = \delta^{(4)}(x-z) \ . \tag{71}$$

About the third derivatives of functionals, it is clear from the last line of eq. (71) that $\int d^4y\ G(x, y)\Gamma(y, z)$ does not depend on J. In fact, by taking the derivative with respect to J(u) we find

$$\frac{\delta}{\delta J(u)}\int \mathrm{d}^4y\, G(x,y)\Gamma(y,z)=0=\int \mathrm{d}^4y\, \frac{\delta^3 W}{\delta J(x)\delta J(y)\delta J(u)}\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)}+ \int \mathrm{d}^4y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\frac{\delta^3\Gamma}{\delta\phi(y)\delta\phi(z)\delta J(u)}\,. \tag{72}$$

Now for the second term we can write

$$\int \mathrm{d}^4y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\frac{\delta}{\delta J(u)}\left[\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)}\right]= \int \mathrm{d}^4y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\int \mathrm{d}^4y'\, \frac{\delta\phi(y')}{\delta J(u)}\frac{\delta}{\delta\phi(y')}\left[\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)}\right]= -\int \mathrm{d}^4y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\int \mathrm{d}^4y'\, G(u,y')\frac{\delta^3\Gamma}{\delta\phi(y)\delta\phi(z)\delta\phi(y')}\,, \tag{73}$$

because $\delta\phi(y')/\delta J(u) = -G(u,y')$. By combining eqs. (72) and (73) one obtains

$$\int \mathrm{d}^4y\, \frac{\delta^3 W}{\delta J(x)\delta J(y)\delta J(u)}\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(z)}= \int \mathrm{d}^4y\, \frac{\delta^2 W}{\delta J(x)\delta J(y)}\int \mathrm{d}^4y'\, G(u,y')\frac{\delta^3\Gamma}{\delta\phi(y)\delta\phi(z)\delta\phi(y')}\,. \tag{74}$$

To summarise, every derivative of Γ with respect to J could be swapped with a derivative in ϕ and an integration with the Green function G, that is,

$$\frac{\delta}{\delta J(u)}=\int \mathrm{d}^4y'\, \frac{\delta\phi(y')}{\delta J(u)}\frac{\delta}{\delta\phi(y')}=-\int \mathrm{d}^4y'\, G(u,y')\frac{\delta}{\delta\phi(y')}\,. \tag{75}$$

As with Z and W , it is possible to expand Γ as a power series in ϕ:

$$\Gamma(\phi)=\sum_{n=0}^{+\infty}\frac{1}{n!}\int\ldots\int \mathrm{d}x_1\ldots\mathrm{d}x_n\, \phi(x_1)\ldots\phi(x_n)\Gamma^{(n)}(x_1,\ldots,x_n)\,. \tag{76}$$

It is possible to show that $\Gamma^{(n)}(x_1, \ldots, x_n)$ is the sum of all 1PI Feynman graphs with $_n$ external lines.

We can expand the effective action Γ(ϕ) in momentum space, in powers of momentum. If one considers renormalisable theory, then the effective action could be written as:

$$\Gamma(\phi) = \int \mathrm{d}x^4 \left[-V(\phi) + \frac{1}{2}(\partial\phi)^2 Z_2(\phi) + \dots \right] , \tag{77}$$

where Z2(ϕ) is the wave function renormalisation, see eq. (14) of (Fabiano,2021b). The term without derivatives, V (ϕ), is called effective potential. To express it in terms of 1PI Green functions, we have to write $\Gamma^{(n)}$ in momentum space:

$$\Gamma^{(n)}(x_1, \dots, x_n) = \int \dots \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \dots \frac{\mathrm{d}^4 k_n}{(2\pi)^4} \times$$
$$(2\pi)^4 \delta^{(4)}(k_1 + \dots + k_n) e^{i(k_1 \cdot x_1 + \dots + k_n \cdot x_n)} \Gamma^{(n)}(k_1, \dots, k_n) . \tag{78}$$

Putting this expression in eq. (76) and expanding in the powers of momenta k_i gives

$$\Gamma(\phi) = \sum_{n=0}^{+\infty} \frac{1}{n!} \int \dots \int \mathrm{d}x_1 \dots \mathrm{d}x_n \int \dots \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \dots \frac{\mathrm{d}^4 k_n}{(2\pi)^4} \times$$
$$\int \mathrm{d}^4 x \, e^{i(k_1 + \dots + k_n) \cdot x} e^{i(k_1 \cdot x_1 + \dots + k_n \cdot x_n)} \times$$
$$\left[\Gamma^{(n)}(0, \dots, 0) \phi(x_1) \dots \phi(x_n) + \dots \right] =$$
$$\int \mathrm{d}^4 x \sum_{n=0}^{+\infty} \frac{1}{n!} \left\{ \Gamma^{(n)}(0, \dots, 0) \left[\phi(x) \right]^n + \dots \right\} , \tag{79}$$

where we have used the fact that $(2\pi)^4 \delta^{(4)}(k_1 + \dots + k_n) = \int \mathrm{d}^4 x \, e^{i(k1 + \dots + kn) \cdot x}$. Comparing eqs. (79) and (77) we see that the $_n$th derivative of the effective potential V (ϕ) is the sum of all 1PI diagrams with n external lines carrying zero momenta, that is, with $k_i = 0$ for all i:

$$V(\phi) = -\sum_{n=0}^{+\infty} \frac{1}{n!} \Gamma^{(n)}(0, \dots, 0) \left[\phi(x) \right]^n . \tag{80}$$

Effective potential: an example

Suppose we have a generic Lagrangian written as

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 - V(\phi) . \tag{81}$$

In this case, a closed form for W (J) is impossible to obtain. However, it can be evaluated using the saddle point approximation described in eqs. (31)-(32). The saddle point field $\phi_s(x)$ is determined by the equation

$$\left.\frac{\delta W(J)}{\delta\phi}\right|_{\phi_s} = \left.\frac{\delta[S(\phi)+\int \mathrm{d}^4y\ J(y)\phi(y)]}{\delta\phi(x)}\right|_{\phi_s} = 0\ . \tag{82}$$

Writing the explicit form of the Lagrangian of (eq. 81) and integrating by parts the kinetic term, that is, $\int \mathrm{d}^4x\ \partial_\mu\phi\partial^\mu\phi = -\int \mathrm{d}^4x\phi\partial^2\phi$ yields

$$\partial^2\phi_s(x) + V'[\phi_s(x)] = J(x)\ . \tag{83}$$

To estimate Z, we define the integration variable as $\phi = \phi_s + \tilde{\phi}$ restore # and write

$$Z(J) = e^{(i/\hbar)W(J)} = \int \mathcal{D}\phi\ e^{(i/\hbar)[S(\phi)+J\phi]} \simeq$$
$$e^{(i/\hbar)[S(\phi_s)+J\phi_s]} \int \mathcal{D}\tilde{\phi}\ e^{(i/\hbar)\int \mathrm{d}^4x\ \frac{1}{2}\left[(\partial\tilde{\phi})^2 - V''(\phi_s)\tilde{\phi}^2\right]}\ , \tag{84}$$

having expanded in Taylor series of $\phi - \phi_s$ as in eq. (31):

$$\left.\frac{\delta^2 S}{\delta\phi^2}\right|_{\phi_s} = \partial^2 + V''(\phi_s)\ . \tag{85}$$

We observe that for any operator A, $\det A = \prod_i a_i$, where a_i are it seigenvalues. So $\prod_i a_i = e^{\sum_i \log a_i}$ and this implies $\det A = e^{\mathrm{Tr}\log A}$. The last part of eq. (84) reads

$$e^{(i/\hbar)[S(\phi_s)+J\phi_s]} \int \mathcal{D}\tilde{\phi}\ e^{(i/\hbar)\int \mathrm{d}^4x\ \frac{1}{2}\left[(\partial\tilde{\phi})^2 - V''(\phi_s)\tilde{\phi}^2\right]} =$$
$$e^{(i/\hbar)[S(\phi_s)+J\phi_s]} \left[\frac{2\pi i\hbar}{\det S''(\phi_s)}\right]^{1/2} =$$
$$e^{(i/\hbar)[S(\phi_s)+J\phi_s] + \frac{1}{2}\log(2\pi i\hbar) - \frac{1}{2}\mathrm{Tr}\log S''(\phi_s)}\ . \tag{86}$$

By dropping irrelevant constant terms, we have determined an explicit expression for W :

$$W(J) = [S(\phi_s) + J\phi_s] + \frac{i\hbar}{2}\mathrm{Tr}\log\left[\partial^2 + V''(\phi_s)\right] + \mathcal{O}(\hbar^2)\ . \tag{87}$$

The first term gives the classical contribution to the Green's function.The next term in # gives the first quantum corrections to the Green's functions. Next is the Legendre transformation, for which

$$\phi = \frac{\delta W}{\delta J} = \frac{\delta[S(\phi_s) + J\phi_s]}{\delta\phi_s}\frac{\delta\phi_s}{\delta J} + \phi_s + \mathcal{O}(\hbar) = \phi_s + \mathcal{O}(\hbar) \ , \tag{88}$$

and so for the effective action

$$\Gamma(\phi) = S(\phi) + \frac{i\hbar}{2}\mathrm{Tr}\log\left[\partial^2 + V''(\phi_s)\right] + \mathcal{O}(\hbar^2) \ , \tag{89}$$

and the effective potential

$$V_{\mathrm{eff}}(\phi) = V(\phi) + \frac{i\hbar}{2}\mathrm{Tr}\log\left[\partial^2 + V''(\phi_s)\right] + \mathcal{O}(\hbar^2) \ . \tag{90}$$

It is clear that in general it is not possible to obtain a closed form for the eigenvalues of the operator in eq. (85). We need to introduce some simplifications: the configurations we will study will be the ones for which ϕ is independent of x. In this case, $V''(\phi)$ becomes a constant related to a mass, $\mu(\phi)^2$. The operator $\partial^2+V''(\phi)$ becomes translationally invariant and is easily evaluable going to momentum space. After obtaining thee igenvalues of the operator, we have to calculate the logarithm and sum over for trace. Therefore

$$\begin{aligned}\mathrm{Tr}\log\left[\partial^2 + V''(\phi)\right] = \int \mathrm{d}^4x\ \langle x\left|\log\left[\partial^2 + V''(\phi)\right]\right| x\rangle = \\ \int \mathrm{d}^4x\ \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \langle x|k\rangle\langle k\left|\log\left[\partial^2 + V''(\phi)\right]\right| k\rangle\langle k|x\rangle = \\ \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \log\left[-k^2 + V''(\phi)\right] \ ,\end{aligned} \tag{91}$$

after having inserted a complete set of states. Going to Euclidean space (Fradkin, 1959) and writing the mass term, we have to deal with the expression

$$\mathcal{I}(\mu^2) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\ \log\left(k^2 + \mu^2\right) \ , \tag{92}$$

which as it stands is terribly divergent at infinity, faster than a fourth power. However, if we derive three times with respect to μ^2 we obtain a finite function,

$$\frac{d^3\mathcal{I}(\mu^2)}{(d\mu^2)^3} = \mathcal{I}^{(3)}(\mu^2) = -\frac{2}{(2\pi)^4}\int \mathrm{d}^4k\ \frac{1}{(k^2+\mu^2)^3} = \frac{2\pi^2}{(2\pi)^4}\int_0^{+\infty}\mathrm{d}k\ \frac{k^3}{(k^2+\mu^2)^3} = \frac{1}{32\pi^2\mu^2}\ . \tag{93}$$

By integrating three times in μ^2 we have

$$\mathcal{I}(\mu^2) = \frac{\mu^4\log\mu^2}{64\pi^2} + A + B\mu^2 + C\mu^4\ , \tag{94}$$

as well as three integration constants that can be reabsorbed in the original Lagrangian by renormalisation. For example, suppose that $V(\phi) = \frac{m}{2}\phi^2 + \frac{g}{4!}\phi^4$, then for the effective potential we would have obtained

$$V_{\text{eff}}(\phi) = \frac{m}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{\mu(\phi)^4\log\mu(\phi)^2}{64\pi^2}\ . \tag{95}$$

Loop expansion

We have done perturbative calculations where the expansion parameter is given by the coupling constant of the theory. Now we will organize the perturbation theory in a different form, of loop expansion, that is an expansion in increasing number of independent loops of Feynman diagrams. At first order we find the Born diagrams or tree level diagrams, that is, diagrams without loops. The next order consists of diagrams with one loop,with integration on internal momenta. Then diagrams with two loops, and so on. The loop expansion described has a small expansion parameter given by Planck's constant #.

Let I be the number of internal lines and V the number of vertices in a Feynman diagram. Then the number of independent loops L will be the number of independent internal momenta after taking into account the momentum conservation in each vertex. One combination of momenta conservation will correspond to the overall conservation of external momenta,so the number of contributing vertices will have to be diminished by 1. The number of independent loops in a given Feynman diagram will therefore be given by the expression

$$L = I - (V-1) = I - V + 1\ . \tag{96}$$

In order to relate this loop formula to the powers of #, we have to restore first its value. From the equal time commutator of canonical variables, were call that

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\hbar\delta^{(3)}(\mathbf{x}-\mathbf{y})\ , \tag{97}$$

therefore the propagator in momentum space will furnish us with a factor #:

$$G(x) = \langle 0 \,|T\phi(x)\phi(0)|\, 0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x} \frac{i\hbar}{k^2 - m^2 + i\epsilon}\ . \tag{98}$$

The other place where # appears is in the action of the path integral, $\int D\phi\, e^{(i/\#)S(\phi)}$. As this corresponds to the interaction Lagrangian in the interaction picture

$$e^{\left[\frac{i}{\hbar}\int \mathrm{d}^4x\ \mathcal{L}_{int}(\phi)\right]}\ , \tag{99}$$

this means that each vertex carries a 1/# factor. So for any given Feynman diagram, the power P of # that appears, $\#^P$, is given by

$$P = I - V = L - 1\ . \tag{100}$$

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Notes

1 Observe the similarity of W to the free energy.

Additional information

FIELD: Mathematics

ARTICLE TYPE: Review paper

EDITORIAL NOTE: The author of this article, Nicola Fabiano, is a current member of the Editorial Board of the Military Technical Courier. Therefore, the Editorial Team has ensured that the double blind reviewing process was even more transparent and more rigorous. The Team made additional effort to maintain the integrity of the review and to minimize any bias by having another associate editor handle the review procedure independently of the editor – author in a completely transparent process. The Editorial Team has taken special care that the referee did not recognize the author's identity, thus avoiding the conflict of interest.

КОММЕНТАРИЙ РЕДКОЛЛЕГИИ: Автор данной статьи Никола Фабиано является действующим членом редколлегии журнала «Военно-технический вестник». Поэтому редколлегия провела более открытое и более строгое двойное слепое рецензирование. Редколлегия приложила дополнительные усилия для того чтобы сохранить целостность рецензирования и свести к минимуму предвзятость, вследствие чего второй редактор-сотрудник управлял процессом рецензирования независимо от редактора-автора, таким образом процесс рецензирования был абсолютно прозрачным. Редколлегия во избежание конфликта интересов позаботилась о том, чтобы рецензент не узнал кто является автором статьи.

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