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Articles

Effective bounds of the variance of statistics on multisets of necklaces

Vėrinių multiaibių statistikos dispersijos efektyvūs įverčiai

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Abstract: The variance of a linear statistics on multisets of necklaces is explored. The upper and lower bounds with optimal constants are obtained.

Keywords: Tur´an–Kubilius inequality, polynomials over finite field, additive function.

Summary: Nagrinėjama tiesinės statistikos, apibrėžtos atsitiktinių vėrinių multiaibėje, dispersija. Gauti tikslūs viršutinieji ir apatinieji įverčiai.

Keywords: Turanas–Kubiliaus nelygybė, daugianariai virš baigtinio lauko, priedų funkcija.

1 Introduction and results

Let $(P, \# \cdot \#)$ be an initial set of weighted objects and

$$\pi(j) := |\{p \in P : \|p\| = j\}| < \infty$$

for every $j = 1, 2, \ldots$. Examine the set G with the extended weight function $\# \cdot \#$ of multisets comprised of $p \in P$. Namely, $a \in G$ if $a = \{p_1, \ldots, p_r\}$ and $\#a\# = \#p_1\# + \cdots + \#p_r\#$ including the empty multiset # of weight 0. Then

$$m(n) := |\varsigma_n| := |\{a \in \varsigma : \|a\| = n\}| = \sum_{l(\bar{k})=n} \prod_{j=1}^{n} \binom{\pi(j) + k_j - 1}{k_j},$$

where $l(\bar{k}) = 1k_1 + + nk_n$ if $\bar{k} = (k_1, \ldots, k_n) \in N_0^n$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the present paper, we deal with the multisets for which $m(n) = q^n$, where $q \geq 2$ is an arbitrary natural number. If q is a prime power, then may be interpreted as $F_q^*[t]$, the set of monic polynomials over a finite field F_q. Then P is the subset of irre- ducible polynomials. For an arbitrary such q, there exist combinatorial constructions, called multisets of necklaces satisfying $m(n) = q^n$ (see, [1, Example 2.12, p. 43]). For multisets, we have the following relations

$$\pi(n) = \frac{1}{n}\sum_{d|n} q^{n/d}\mu(d), \qquad q^n = \sum_{d|n} d\pi(d), \tag{1}$$

where in the summations, d runs over natural divisors of n and $\mu(d)$ stands for the Möbius function. The equalities are equivalent to the formal power series relation

$$\sum_{n=0}^{\infty} q^n x^n = \frac{1}{1-qx} = \prod_{j=1}^{\infty}(1-x^3)^{-\pi(j)}.$$

Take an a $\in$ $\mathbf{G_n}$ uniformly at random, that is, sample it with probability $\nu_n(\{a\}) = q^{-n}, n \# N$ *and* $\nu_0(\{\#\}) = 1$. *If* $k_j(a) \geq 0$ is the number of elements pi in a $\in$ $\mathbf{G_n}$ of weight j , then $\mathbf{k(a) = k_1(a), \ldots, k_n(a)}$ is the structure vector of $a \# G_n$ satisfying $\mathbf{l(k(a)) = n}$. Its distribution is

$$\nu_n\left(\bar{k}(a) = \bar{s}\right) = 1\left\{\ell\left(\bar{s}\right) = n\right\}q^n\prod_{j=1}^{\infty}\binom{\pi(j) + s_j - 1}{s_j}. \tag{2}$$

where $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{N}_0^n$ and $\mathbf{1}.\{.\}$ stands for the indicator function.

We are interested in the distribution with respect to $\boldsymbol{\nu_n}$ of the linear statistics

$$h(\bar{c}) := h(\bar{c}, a) = c_1 k_1(a) + \cdots + c_n k_n(a), \qquad \bar{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n. \tag{3}$$

The number of components in a is such a function, namely, it equals $\mathbf{k_1(a)}+ +\mathbf{k_n(a)}$. We refer to [1] for more sophisticated examples.

The present paper is devoted to the variance of $h(\bar{c})$ which is a sum of dependent random variables (r.vs) as the relation $h(\bar{j}, a) = \ell(\bar{k}(a)) = n$ for each a $\in G_n$ shows. Estimating it, we propose an approach to overcome technical obstacles stemming from dependence.

In the sequel, the expectations and variances with respect to νn will be denoted by $\mathbf{E_n}$ and $\mathbf{V_n}$ while, when the probability space (Ω, F, P) is not specified, we will respectively use the notation $\mathbf{E}$ and $\mathbf{V}$. The summation indexes i, j, l, k, m, m^1 and m^2 will be natural numbers.

Theorem 1 If $\bar{c} \in \mathbb{R}^n$ and $n \in \mathbb{N}_0$, then

$$V_n h\left(\bar{c}\right) = \sum_{1\square jk\square n} c_j^2 \pi\left(j\right) k q^{-jk} - \sum_{\substack{il+jk>n \\ il\square n, jk\square n}} c_i c_j \pi\left(i\right)\pi\left(j\right) q^{il-jk}. \tag{4}$$

The sketch of the proof is given at the beginning of Section 2.

t is known [1] that, for a fixed j, the r.v. $\mathbf{k_j(a)}$ converges in distribution to the r.v. γj distributed according the negative binomial law $\mathbf{NB(\pi(j), q^{-j})}$. **If** $\{\gamma 1, \gamma 2, ..\}$. are mutually independent, define the statistics $\mathbf{Y_n} = \mathbf{c_1\gamma_1 + + n\gamma_n}$. We shall see that the first sum on the right-hand side in (4) is close to $\mathbf{VY_n}$; therefore, estimating $\mathbf{V_n h(c^-)}$, we use the following quadratic forms:

$$B_n\left(\bar{c}\right) := \sum_{1 \le jk \le n} c_j^2 \pi\left(j\right) k q^{-jk}, \qquad R_n\left(\bar{c}\right) = \sum_{m \le n} m q^{-2m} \left(\sum_{j|m} c_j \pi\left(j\right)\right)^2.$$

Theorem 2 *If* $n \ge 2$, *then*

$$V_n h\left(\bar{c}\right) \le B_n\left(\bar{c}\right) + \frac{1}{2} R_n\left(\bar{c}\right). \tag{5}$$

The inequality becomes an equality for

$$c_j = c_j^* := \frac{3}{\pi(j)} \sum_{d|j} d q^d \mu\left(\frac{j}{d}\right) - \left(2n+1\right) j, \qquad 1 \le j \le n. \tag{6}$$

Corollary 1 If $n \ge 2$ and $\bar{c} \ne \bar{0}$, , then

$$V_n h\left(\bar{c}\right) < \frac{3}{2} B_n\left(\bar{c}\right) < \left(\frac{3}{2} - \frac{q-1}{q} n q^{-n}\right) V Y n. \tag{7}$$

The inequalities are trivial for functions proportional to $h(\bar{j}, a) = n$ if $a \in G_n$ because of $V_n h(j) = 0$ then. A shift of $\bar{c}$ eliminates this inconvenience. Observe that either of $B_n(\bar{c} - t\bar{j})$ and $R_n(\bar{c} - t\bar{j})$ attain their minimums in $t \in \mathbb{R}$ at

$$t = t_c := \frac{2}{(n+1)n} \sum_{m \le n} m q^{-m} \sum_{j|m} c_j \pi\left(j\right).$$

Theorem 3 *If* $n \ge 3$, *then*

$$B_n\left(\bar{c} - t_c\bar{j}\right) - \frac{1}{3} R_n\left(\bar{c} - t_c\bar{j}\right) \le V_n h\left(\bar{c}\right) \le B_n\left(\bar{c} - t_c\bar{j}\right) - \frac{1}{2} R_n\left(\bar{c} - t_c\bar{j}\right).$$

Both inequalities are sharp.

Corollary 2 *If* $n \ge 3$ *and* $\bar{c} \ne \alpha\bar{j}$ *for every* $\alpha \in \mathbb{R}$

$$\frac{3}{2} B_n\left(\bar{c} - t_c\bar{j}\right) \le V_n h\left(\bar{c}\right) < \frac{3}{2} B_n\left(\bar{c} - t_c\bar{j}\right).$$

The proofs of the last two theorems presented in Section 2 are built upon the ideas and auxiliary results obtained in [4], [2] and [5].

2 Proofs

We firstly recall known facts about random multisets which can be found in [3] and [1, Section 2.3]. Let $\bar{y}^{(x)} = \left(y_1^{(x)}, y_2^{(x)}, \ldots\right)$ be the infinite dimensional vector of independent r.vs having the negative binomial distributions $\mathbf{NB}(\pi(j), x^j)$, namely,

$$P\left(\gamma_j^{(x)} = m\right) = \binom{\pi(j) + m - 1}{m}\left(1 - x^j\right)^{\pi(j)} x^{jm}, \qquad m = 0, 1, \ldots$$

where $0 < x \leq q^{-1}$. Then $\gamma_j^{(q^{-1})} = \gamma_j$ which has been introduced in Introduction. For convenience, we extend $\mathbf{k(a)}$ to $\mathbf{k(a)} := (\mathbf{k_1(a)}, \ldots, \mathbf{k_n(a)}, 0, \ldots)$ and use infinite dimensional vectors.*Set* $\theta^{(x)} = 1y_1^{(x)} + \cdots + n, y_n^{(x)} + (n+1)y_{n+1}^{(x)} +, \ldots$ The latter r.v. is well defined if $0 < x < q^{-1}$, since the condition of the Boreli–Cantelli lemma is satisfied:

$$\sum_{j=1}^{\infty} p\left(\gamma_j^{(x)} \neq 0\right) = \sum_{j=1}^{\infty}\left(1 - \left(1 - x^j\right)^{\pi(j)}\right) < \infty.$$

Lemma 1 *If* $\bar{s} = (s_1, \ldots, s_j, s_j + 1, \ldots) \in \mathbb{N}_0^{\infty}$ *and* $0 < x < q^{-1}$*, then*

$$v_n\left(\bar{k}(a) = s\right) = p\left(\gamma^{(x)} = s \middle| \theta^{(x)} = n\right).$$

Proof. Actually, this is Lemma 2.2 in [3] stated there for $\mathbf{F_q[t]}$. The details remain the same in the more general case.

Lemma 2 For a functional $\Psi : \mathbb{N}_0^{\infty} \to \mathbb{R}$ such that $E\left|\Psi\left(\bar{\gamma}^{(x)}\right)\right| < \infty$, , we have

$$E\Psi\left(\gamma^{(x)}\right) = (1 - qx)\left(\Psi(0) + \sum_{n=1}^{\infty} E_n \Psi\left(\bar{k}(a)\right) q^n x^n\right), \qquad 0 < x < q^{-1}.$$

Proof. Apply Lemma 1 in the double averaging as follows:

$$E\Psi\left(\gamma^{(x)}\right) = \sum_{n=0}^{\infty} E\left(\Psi\left(\gamma^{(x)}\right)\middle|\theta^{(x)} = n\right) P\left(\theta^{(x)} = n\right) \qquad = \sum_{n=0}^{\infty} E_n \Psi\left(\bar{k}(a)\right) P\left(\theta^{(x)} = n\right).$$

Proof of Theorem 1. It is straightforward. Applying the last lemma for the relevant Ψ, one can easily find the needed mixed moments of $\mathbf{k_j(a)}$, $1 \leq j \leq n$, and further, the variance of the linear combination *h(a)*.

To prove Theorems 2 and 3, we will apply the following lemmas concerning par- ticular matrices and quadratic forms.

Lemma 3 *Let* $U = ((u_{ij}))$, $i, j \leq n$ *, be the symmetric matrix with the entries*

$$u_{ij} = 1\{i + j > n\}(ij)^{-1/2}.$$

The spectrum of U *is the set* $\{1, 1/2, 1/3, \ldots, (-1)^{n-1}/n\}$ *. The eigenvectors corresponding to the first three eigenvalues are proportional to* $\bar{c}_r = (c_{r1}, \ldots, c_{rn})$*, where* $r = 1, 2, 3$ *and, for* $j \le n$ *,*

$$e_{1j} = \sqrt{j}, \qquad e_{2j} = (3j - 2n - 1)\sqrt{j}, \qquad e_{3j} = \left(10j^2 - 6(2n+1)j + 3n^2 + 3n + 2\right)\sqrt{j}.$$

Proof. This is the byproduct of works [4] and [2].

Afterwards, let $\bar{e}_r$, $1 \le m \le n$, be the orthogonal basis of R^n comprised of the eigenvectors of U and $\bar{x}'$ means the transposed vector $\bar{x}$.

Lemma 4 If $b_m \in \mathrm{R}$ and $1 \le m \le n$ and $n \ge 2$, then

$$-\frac{1}{2}\sum_{1\le m\le n} mb_m^2 \le \sum_{\substack{1\le m1,m2\le n\\ m1+m2>n}} b_{m1}b_{m2} \le \sum_{1\le m\le n} mb_m^2. \tag{9}$$

If $n \ge 3$ *and*

$$\sum_{m\le n} mb_m = 0, \tag{10}$$

then

$$\sum_{\substack{1\le m1,m2\le n\\ m1+m2>n}} b_{m1}b_{m2} \le \frac{1}{3}\sum_{1\le m\le n} mb_m^2. \tag{11}$$

Moreover, each bound in (9) and (11) are achieved, respectively, for $b_m = e_{rm}/\sqrt{m}$, *, where* $r = 2, 1, 3$ *and* e_{rm} *have been defined in Lemma 3.*

Proof. Inequalities (9) are seen from Lemma 3 after the substitution $b_m = x_m/\sqrt{m}$, , $m<n$ since the extreme eigenvalues are 1 and 1/2.

After the same substitution, we further examine the quadratic form with the matrix U . Condition (10) reckons the subspace of vectors $\bar{x} = (x_1, \ldots, x_n)$ satisfying $x_1 + \cdots + x_j\sqrt{j} + \cdots + x_n\sqrt{n} = \bar{x}\cdot\bar{e}_1' = 0$. This subspace is spanned over the first eigenvector. In other words, under (10), only the form values obtained in the subspace $\mathbf{L} \# \mathbf{R}^n$ spanned over the vectors $\bar{e}_2, \ldots, \bar{e}_n$ count. Hence

$$\max_{\bar{x}\in L} \|\bar{x}\|^{-2}\bar{x} U \bar{x}' = \max_{2\le r\le n} (-1)^{r-1}/r = 1/3.$$

Returning to b_m, from this we obtain inequality (11).

Proof of Theorem 2. After grouping the summands, expression (4) can be rewritten as follows:

$$V_n h(\bar{c}) = b_n(\bar{c}) - \sum_{\substack{1\le m1,m2\le n\\ m1+m2>n}} \left(q^{-m1}\sum_{1\le m\le n} c_i\pi(i)\right)\left(q^{-m2}\sum_{j|m2} c_i\pi(j)\right).$$

Now evidently estimate (5) follows from Lemma 4 with

$$b_m = q^{-m}\sum_{j|m} c_j\pi\left(j\right), \qquad m \,\square\, n.$$

Moreover, it becomes an equality if we take $c_j = c_j^*$ satisfying

$$q^{-m}\sum_{j|m} c_j^*\pi\left(j\right) = 3m - 2n - 1,$$

which by the Möbius inversion formula and (1) may be rewritten as (6).

To *prove* the first assertion of Corollary 1, it suffices to estimate the inner sum in $R_n(\bar{c})$, , namely,

$$\left(\sum_{j|m} c_j\pi\left(j\right)\right)^2 \,\square\, \sum_{j|m}\frac{c_j^3\pi\left(j\right)}{j}.\sum_{j|m} j\pi\left(j\right) = \sum_{j|m}\frac{c_j^3\pi\left(j\right)}{j}.\mathrm{q}m.$$

Further, using the expression of VY_n , we just estimate the remainder:

$$VY_n - B_n\left(\bar{c}\right) = \sum_{j\square m} c_j^3\pi\left(j\right)\sum_{k>n/j} kq^{-jk} \,\square\, nq^{-n}\sum_{j\square m}\frac{c_j^2\pi\left(j\right)}{j}\frac{q^j}{(q^j-1)^2}.\frac{q^j-1}{q^j} \qquad \square\, nq^{-n-1}\left(q-1\right)VY_n.$$

Plugging both estimates into (5), we obtain the first inequality in Corollary 1 with ≤ instead of <. In fact, we obtained the strict inequality since Cauchy's inequality applied in the last step is strict if $\bar{c}$ is not proportional to $\bar{j}$, and in this exceptional case, $Vh(\bar{c}) = 0.$.

Proof of Theorem 3. Observe that $V_nh(\bar{c}) = V_nh(\bar{c}) - tn) - V_nh(\bar{c} - t\bar{j})$ for every $t \in \mathrm{R}$. Hence the right-hand inequality follows from (5) applied for the shifted statistics.

To get the lower bound of variance, we combine (4) and (11). We start with

$$V_nh\left(\bar{c} - t_c\bar{j}\right) - B_n\left(\bar{c} - t_c\bar{j}\right) - \sum_{\substack{m1,m2\square n\\ m1,m2>n}} \bar{b}_{m1}\bar{b}_{m2},$$

where

$$\bar{b}_m = q^{-m}\sum_{j|m}\left(c_j - t_cj\right)\pi\left(j\right)$$

and $m \le n$. By the definition of tc the latter sequence satisfies condition (10). Hence by (11),

$$\sum_{\substack{m1,m2\square n\\ m1+m2>n}} \bar{b}_{m1}\bar{b}_{m2}, \,\square\, \frac{1}{3}\sum_{m\square n} m\bar{b}_m^2 = \frac{1}{3}R_n\left(\bar{c} - t_c\bar{j}\right).$$

This and (4) imply the lower bound. Moreover, the latter is sharp since Lemma 4 assures this by a choice of a particular sequence $\bar{b}_m$, $m \,\square\, n.$.

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