Existence theorem for integral inclusions by a fixed point theorem for multivalued implicit-type contractive mappings

Usman Ali, Muhammad; Pitea, Ariana

Existence theorem for integral inclusions by a fixed point theorem for multivalued implicit-type contractive mappings

Muhammad Usman Ali muh_usman_ali@yahoo.com
COMSATS University Islamabad, Pakistán
Ariana Pitea arianapitea@yahoo.com
University Politehnica of Bucharest, Rumania

Abstract: In this article, we introduce fixed point theorems for multivalued mappings satisfying implicit-type contractive conditions based on a special form of simulation functions. We also provide an application of our result in integral inclusions. Our outcomes generalize/extend many existing fixed point results.

Keywords: fixed point, implicit simulation function, integral inclusions, multivalued mappings.

1 Introduction

The contraction mapping principle proposed by Banach is an important and significant inception in functional analysis due to its applicability in other areas of mathematics and applied sciences. With the passage of time, this result has become a solid base for metric fixed point theory. Thus, several researchers generalized it by stating fixed point, common fixed point, coincidence point, couple fixed point theorems regarding mappings satisfying certain type of contractive conditions on metric spaces or on various abstract spaces. In [31], nonlinear contractions in ordered metric spaces are studied in their cyclic form. In [13], metric spaces endowed with partial order proved to be a suitable setting to develop fixed point theorems for adequate mappings and also an application regarding the existence and uniqueness of a solution to a periodic boundary value problem. Generalized weak Berinde contractions on partial metric spaces are studied in [30]. In [15], a quasicontractivity-type condition, which entails the conclusions from Banach principle, is presented. In [34], a Banach-type condition is studied in connection with the completeness of the underlying metric space. [16] and [32] refer to some Prešic´-type generalization of the Banach principle. Fixed point properties in the context of α ψ-contractions are developed in [25]. In [8], closed multivalued mappings are in view with respect to α ϕcontractive conditions. In [9], weakly compatible mappings, which satisfy an implicit relation, are studied. In [12] and [14], it is stated a generalization of some weak contraction principle. In [10], b-metric-like spaces feature common fixed point properties. In [29], cone b-metric spaces are an adequate setting to develop generalized Hardy–Rogers-type contractions. [20] focuses on Mizoguchi–Takahashi-type fixed point theorems. In [23], approximate fixed point theorems are

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

334

stated in the context of α-contractive mappings. The α-admissibility was used to prove interesting fixed point results in [2, 26, 28] or [27]. In [35], implicit contraction mappings are studied. Later on, the combination of metric fixed point theory and optimization theory enabled discussions on best proximity points of nonself mappings satisfying certain types of proximal contractive conditions on metric spaces or on abstract spaces. Browder theorems are extended in [17]. In [1], multivalued operators are studied from the point of view of their best proximity points. [3] has in view Kakutani multimappings, while [4] equilibrium pairs for finite families of multivalued mappings are presented. Work [7] is dedicated to the study of proximal contractions by means of suitable simulation functions. Controlled contractions are used in [5] in order to obtain best proximity properties. [6] is devoted to best proximity results for Prešic´-type operators. Generalized proximal contractive mappings are developed in [11], while [18] refers to global optimal solutions. The existence of best proximity points for generalized classes of contractions is performed in [21]. Applications of best proximity points associated with α ψ-proximal contractions are presented in [19]. Hyperconvex spaces proved to be an adequate framework to develop best proximity theorems for mappings endowed with suitable continuity properties in [22].

2 Preliminaries

In this article, we will use the following type of implicit functions presented in [7]. Here κψ represents the set of functions κ : (R+)4 → R+ = [0,∞) endowed with the following properties:

(K1) κ is continuous and nondecreasing in each coordinate; (K2) If l ≥ j and l ≤ κ(l, j, l, l), then l = 0;

(K3) If l < j and l ≤ κ(j, j, l, j), then l ≤ ψ(j), where ψ : : [0, ∞) → [0, ∞) is a nondecreasing mapping with $\sum_{n=1}^{\infty} \#_n (t) < \infty$ for all t ≥ 0.

Definition 1. (See [7].) A mapping χ : R+ x R+ → R is known as an implicit simulation function with respect to κψ if the following conditions hold:

(C1) χ(al, κ(m, n, o, p)) ≤ κ(m, n, o, p) − al for any a, m, n, o, p ≥ 0;

(C2) χ(j, κ(0, 0, j, j/2)) ≥ a if and only if κ(0, 0, j, j/2) - j ≥ a for any real number a;

(C3) χ(j, κ(0, 0, j, j/2)) ≥ 0 implies j = 0.

The next example illustrates such a kind of mappings.

Example 1. (See [7].) Define the mappings χ : R+ × R+ → R and κ : (R +)4 → R+ = [0, ∞),

$$\chi(l, j) = j - l \quad \text{and} \quad \kappa(m, n, o, p) = \zeta \max\{m, n, o, p\},$$

where ζ # [0, 1). One can check that the above defined χ is an implicit simulation function with respect to the above defined function κ.

The following result was presented in [7] (along the paper, N = {1, 2, . . . }).

redalyc.org

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

335

Theorem 1. (See [7].) Let T be a mapping from a complete metric space (X, dm) into itself, which satisfies

$$\chi\left(\alpha(j,l)d_m(Tj,Tl), \kappa\left(d_m(j,l), d_m(j,Tj), d_m(l,Tl), \frac{d_m(l,Tj)+d_m(j,Tl)}{2}\right)\right) \geqslant 0$$

for all j, l # X, where χ is an implicit simulation function with respect to $\kappa\psi$. Further, assume that the following conditions hold:

(i) T is α-admissible, that is, for j, l $\in$ X, $\alpha(j,l) \geq 1$ implies $\alpha(Tj, Tl) \geq 1$;
(ii) There exists j1 $\in$ X satisfying $\alpha(j1, Tj1) \geq 1$;
(iii) For all sequences {jn} in X with $\alpha(jn, jn+1) \geq 1$, n $\in$ N and jn → j, we have
$\alpha(jn, j) \geq 1$ for each n $\in$ N.

Then T has a fixed point.

Consider (X, dm) a metric space and CL(X) the collection of all nonempty closed subsets of X. For J $\in$ CL(X) and l $\in$ X, dm(l, J) = inf{dm(l, j), j $\in$ J}. For each J, L $\in$ CL(X), let

$$H_m(J,L) = \begin{cases} \max\{\sup_{j\in J} d_m(j,L), \sup_{l\in L} d_m(l,J)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

The function Hm is known as the generalized Hausdorff metric induced by dm.

3 Main results

Here Γ represents the set of all functions $\eta : (R+)5 → R+ = [0, \infty)$ having the following properties:
(E1) $\eta(m, n, o, p, q) = 0$ if and only if at least one of m, n, o, p, q is zero;
(E2) η is continuous.

We now present the first result of this section.

Theorem 2. Let X be a nonempty set and dm a distance on it. Assume that T is a mapping from X into CL(X) that satisfies

$$\chi\left(\alpha(j,l)H_m(Tj,Tl), \kappa\left(d_m(j,l), d_m(j,Tj), d_m(l,Tl), \frac{d_m(l,Tj)+d_m(j,Tl)}{2}\right)\right)$$
$$+ L\eta\big(d_m(j,l), d_m(j,Tj), d_m(l,Tl), d_m(j,Tl), d_m(l,Tj)\big) > 0 \tag{1}$$

for all j, l $\in$ X with j ≠ l, where χ is an implicit simulation function with respect to $\kappa\psi$, $\eta \in$ Γ and L $\geq$ 0. Further, assume that the following conditions hold:

(i) T is α-admissible, that is, for j, l $\in$ X, $\alpha(j, l) \geq 1$ implies $\alpha(a, b) \geq 1$ for all a $\in$ Tj, b $\in$ Tl;
(ii) There exist j1 $\in$ X and j2 $\in$ Tj1 satisfying $\alpha(j1, j2) \geq 1$;

(iii) For all sequences $\{jn\}$ in X with $\alpha(jn, jn+1) \geq 1$, $n \in N$ and $jn \to j$, we have $\alpha(jn, j) \geq 1$ for each $n \in N$.

Then T has a fixed point.

Proof. Hypothesis (ii) ensures the existence of two elements $j1, j2 \in X$ with $\alpha(j1, j2) \geq 1$ and $j2 \# Tj1$. Without loss of generality, we may presume that $j1$ $j2$. Inequality (1) and condition (C1) imply that

$$\kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right)$$
$$- \alpha(j_1, j_2)H_m(Tj_1, Tj_2)$$
$$+ L\eta\big(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), d_m(j_1, Tj_2), d_m(j_2, Tj_1)\big) > 0. \tag{2}$$

As $dm(j2, Tj1) = 0$, we get by the use of (E1) that

$$\eta\big(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), d_m(j_1, Tj_2), d_m(j_2, Tj_1)\big) = 0.$$

Thus, inequality (2) implies

$$H_m(Tj_1, Tj_2)$$
$$\leqslant \alpha(j_1, j_2)H_m(Tj_1, Tj_2)$$
$$< \kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right).$$

There is $\varepsilon1 > 0$ such that

$$H_m(Tj_1, Tj_2) + \varepsilon_1$$
$$\leqslant \kappa\left(d_m(j_1, j_2), d_m(j_1, Tj_1), d_m(j_2, Tj_2), \frac{d_m(j_2, Tj_1) + d_m(j_1, Tj_2)}{2}\right). \tag{3}$$

Since $\varepsilon1 > 0$, then we have an element $j3 \in Tj2$ satisfying

$$d_m(j_2, j_3) \leqslant H_m(Tj_1, Tj_2) + \varepsilon_1. \tag{4}$$

If $j2 = j3$, there is nothing left to be proved. Therefore, we may consider $j2 = j3$. By using relations (3), (4) and the nondecreasing behavior of κ we get

$$d_m(j_2, j_3) \leqslant \kappa\left(d_m(j_1, j_2), d_m(j_1, j_2), d_m(j_2, j_3), \frac{d_m(j_1, j_2) + d_m(j_2, j_3)}{2}\right). \tag{5}$$

We now claim that dm(j2, j3) < dm(j1, j2). Suppose this inequality does not hold; then we have dm(j2, j3) ≥ dm(j1, j2). By using this in inequality (5) it follows

$$d_m(j_2, j_3) \leqslant \kappa\big(d_m(j_2,j_3), d_m(j_1,j_2), d_m(j_2,j_3), d_m(j_2,j_3)\big).$$

From this inequality and condition (K2) we get dm(j2, j3) = 0, which is not possible due to our assumption. Thus, dm(j2, j3) < dm(j1, j2). Hence, inequality (5) yields

$$d_m(j_2, j_3) \leqslant \psi\big(d_m(j_1, j_2)\big). \tag{6}$$

Note that hypothesis (i) compels that α(j2, j3) ≥ 1.

Continuing the proof after the above pattern, we get a sequence {jn} satisfying jn+1 ∈ Tjn, α(jn, jn+1) ≥ 1 and dm(jn+1, jn+2) < ψ(dm(jn, jn+1)) < dm(jn, jn+1) for n ∈ N. Having in mind the monotone behavior of ψ and inequality (6), we are led to

$$d_m(j_{n+1}, j_{n+2}) \leqslant \psi^n\big(d_m(j_1, j_2)\big) \quad \text{for each } n \in \mathbb{N}. \tag{7}$$

Next, we will prove that {jn} is a Cauchy sequence. Consider the natural numbers q, p, q > p. By using the triangular inequality and relation (7) we obtain

$$d_m(j_q, j_p) \leqslant \sum_{e=p}^{q-1} d_m(j_e, j_{e+1}) \leqslant \sum_{e=p}^{q-1} \psi^e\big(d_m(j_1, j_2)\big)$$
$$\leqslant \sum_{e=p}^{\infty} \psi^e\big(d_m(j_1, j_2)\big) \to 0.$$

This shows that {jn} is a Cauchy sequence in X. By the completeness of X there is an element j# ∈ X so that jn → j#. Hypothesis (iii) implies α(jn, j#) ≥ 1 for each n ∈ N.

Without loss of generality, we may assume that jn all n ∈ N, it follows that jn ≠

j#, n ∈ N. From inequality (1), for all n # N, it follows that.

$$\chi\Big(\alpha(j_n, j^*)H_m(Tj_n, Tj^*), \kappa\Big(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*),$$
$$\frac{d_m(j^*, Tj_n) + d_m(j_n, Tj^*)}{2}\Big)\Big)$$
$$+ L\eta\big(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n)\big) > 0.$$

By using hypothesis (C1) and the fact that $\alpha(jn, j\#) \geq 1$ in the above inequality we get

$$
\begin{aligned}
d_m(j_{n+1}, Tj^*) \\
\leqslant\ & \alpha(j_n, j^*) H_m(Tj_n, Tj^*) \\
<\ & \kappa\left(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), \frac{d_m(j^*, Tj_n) + d_m(j_n, Tj^*)}{2} \right) \\
& + L\eta\big(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n) \big) \\
\leqslant\ & \kappa\left(d_m(j_n, j^*), d_m(j_n, j_{n+1}), d_m(j^*, Tj^*), \frac{d_m(j^*, j_{n+1}) + d_m(j_n, Tj^*)}{2} \right) \\
& + L\eta\big(d_m(j_n, j^*), d_m(j_n, Tj_n), d_m(j^*, Tj^*), d_m(j_n, Tj^*), d_m(j^*, Tj_n) \big).
\end{aligned}
$$

Taking the limit n → ∞ leads us to

$$
d_m(j^*, Tj^*) \leqslant \kappa\left(0, 0, d_m(j^*, Tj^*), \frac{d_m(j^*, Tj^*)}{2} \right).
$$

From the above inequality and (C2) we get

$$
\chi\left(d_m(j^*, Tj^*), \kappa\left(0, 0, d_m(j^*, Tj^*), \frac{d_m(j^*, Tj^*)}{2} \right) \right) \geqslant 0.
$$

By using condition (C3) it follows dm(j#, Tj#) = 0, that is, j# $\in$ Tj#.

The next result can be obtained from the above one by considering $\alpha(j, l) = 1$ for all $j, l \in X$.

Corollary 1. Let T be a mapping from a complete metric space (X, dm) into CL(X), which satisfies

$$
\begin{aligned}
& \chi\left(H_m(Tj, Tl), \kappa\left(d_m(j, l), d_m(j, Tj), d_m(l, Tl), \frac{d_m(l, Tj) + d_m(j, Tl)}{2} \right) \right) \\
& + L\eta\big(d_m(j, l), d_m(j, Tj), d_m(l, Tl), d_m(j, Tl), d_m(l, Tj) \big) > 0
\end{aligned}
\tag{8}
$$

for all $j, l \in X$ with $j \ /= l$, where χ is an implicit simulation function with respect to $\kappa\psi$, $\eta \in \Gamma$ and $L \geq 0$. Then T has a fixed point.

Example 2. Let X = C[0, 1] be the collection of all continuous functions from [0, 1] into R with the metric $d(u, l) = \max_{p \in [0, 1]} |u(p) - l(p)|$. Define an operator T as

$$
Tu = \left\{ \int_0^1 (1 - p)u(p)\, \mathrm{d}p \right\} \quad \text{for all } u \in X.
$$

Then we have

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

339

$$H_m(Tu, Tl) = \max_{p \in [0,1]} \left| \int_0^1 (1-p)u(p) \, \mathrm{d}p - \int_0^1 (1-p)l(p) \, \mathrm{d}p \right|$$

$$\leqslant \max_{p \in [0,1]} \left| \int_0^1 (1-p) \, \mathrm{d}p \right| \max_{p \in [0,1]} \left| u(p) - l(p) \right|$$

$$= \frac{1}{2}d(u,l) < \frac{2}{3}d(u,l), \quad u \neq l.$$

Thus, by taking $\chi(l, j) = j\,l$, $\kappa(m, n, o, p) = (2/3)m$ and $L = 0$ in inequality (8) we get the above inequality. Thus, by Theorem 2 and Corollary 1 the mapping T has a fixed point.

In the following theorem, we will use b-metric spaces, where the triangle inequality of metric spaces is replaced by one in which the right-hand side of the classical triangle inequality is multiplied by a constant s, s $\geq$ 1. Here we denote by $\kappa\psi s$ the collection of functions $\kappa : (R+)4 \rightarrow R+ = [0, \infty)$, which satisfy (K1), (K2) and (K3) with a nondecreasing mapping $\psi : [0,\infty) \rightarrow [0,\infty)$ such that $\sum \infty n=1$ sn ψn $< \infty$ for all t $\geq$ 0, where s $\geq$ 1.

Theorem 3. Let X be a nonempty set and bm a continuous b-metric so that (X, bm) is complete. Let T be a mapping from X into CL(X) that satisfies

$$\chi\left(\alpha(j,l)H_{b_m}(Tj, Tl), \kappa\left(b_m(j,l), b_m(j,Tj), b_m(l,Tl), \frac{b_m(l,Tj) + b_m(j,Tl)}{2s} \right) \right)$$
$$+ L\eta\big(b_m(j,l), b_m(j,Tj), b_m(l,Tl), b_m(j,Tl), b_m(l,Tj) \big) \geqslant 0 \tag{9}$$

for all j, l # X, where χ is an implicit simulation function with respect to $\kappa\psi s$, η # Γ and L $\geq$ 0. Further, assume that the following conditions hold:

(i) T is αs-admissible, that is, for j, l $\in$ X, $\alpha(j, l) \geq$ s implies $\alpha(a, b) \geq$ s for each a $\in$ Tj, b $\in$ Tl;

(ii) There exist j1 $\in$ X and j2 $\in$ Tj1 satisfying $\alpha(j1, j2) \geq$ s;

(iii) For each sequence {jn} in X with $\alpha(jn, jn+1) \geq$ s, n $\in$ N and jn $\rightarrow$ j, we have $\alpha(jn, j) \geq$ s for all n $\in$ N.

Then T has a fixed point.

Proof. Hypothesis (ii) ensures that there are two elements j1, j2 in X with $\alpha(j1, j2) \geq$ s and j2 # Tj1. Without loss of generality, we may assume that j1 $\neq$ j2. Inequality (9) and condition (C1) imply that

$$\kappa\left(b_m(j_1, j_2), b_m(j_1, Tj_1), b_m(j_2, Tj_2), \frac{b_m(j_2, Tj_1) + b_m(j_1, Tj_2)}{2s} \right)$$
$$- \alpha(j_1, j_2)H_{b_m}(Tj_1, Tj_2)$$
$$+ L\eta\big(b_m(j_1, j_2), b_m(j_1, Tj_1), b_m(j_2, Tj_2), b_m(j_1, Tj_2), b_m(j_2, Tj_1) \big) \geqslant 0.$$

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

340

Taking account of property (E1) of the functions from Γ, as bm(j2, Tj1) = 0, we get that

$$\eta\big(b_m(j_1, j_2), b_m(j_1, Tj_1), b_m(j_2, Tj_2), b_m(j_1, Tj_2), b_m(j_2, Tj_1)\big) = 0.$$

Thus, relation (10) compels

$$
\begin{aligned}
sH_{b_m}&(Tj_1, Tj_2)\\
&\leqslant \alpha(j_1, j_2)H_{b_m}(Tj_1, Tj_2)\\
&\leqslant \kappa\bigg(b_m(j_1, j_2), b_m(j_1, Tj_1), b_m(j_2, Tj_2), \frac{b_m(j_2, Tj_1) + b_m(j_1, Tj_2)}{2s}\bigg).
\end{aligned}
$$

(11)

As s ≥ 1, there exists an element j3 $\in$ Tj2 such that

$$b_m(j_2, j_3) \leqslant sH_{b_m}(Tj_1, Tj_2).$$

(12)

Again, we may assume j2 = j3. By taking advantage of relations (11), (12) and the nondecreasing behavior of κ it follows

$$b_m(j_2, j_3) \leqslant \kappa\bigg(b_m(j_1, j_2), b_m(j_1, j_2), b_m(j_2, j_3), \frac{b_m(j_1, j_2) + b_m(j_2, j_3)}{2}\bigg).$$

(13)

We now claim bm(j2, j3) < bm(j1, j2). By reductio ad absurdum we presume that bm(j2, j3) ≥ bm(j1, j2). Having in mind also inequality (13), we obtain

bm(j2, j3) ≤ κ bm(j2, j3), bm(j1, j2), bm(j2, j3), bm(j2, j3) .

This inequality, jointly with condition (K2), imposes bm(j2, j3) = 0, which is a contradiction to the assumption that j2 ≠ j3. Thus, bm(j2, j3) < bm(j1, j2). Relation (13) yields

bm(j2, j3) ≤ κ bm(j1, j2), bm(j1, j2), bm(j2, j3), bm(j1, j2) .

This inequality combined with property (K3) compel

bm(j2, j3) ≤ ψ(bm(j1, j2)).

Note that hypothesis (i) ensures that α(j2, j3) ≥ s.

Continuing the above pattern, we get a sequence {jn} endowed with the properties jn+1 $\in$ Tjn and α(jn, jn+1) ≥ s, bm(jn+1, jn+2) < bm(jn, jn+1) and

$$b_m(j_{n+1}, j_{n+2}) \leqslant \psi^n\big(b_m(j_1, j_2)\big) \quad \text{for all } n \in \mathbb{N}.$$

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

341

Next, we will prove that {jn} is a Cauchy sequence. Consider the natural numbers q, p, q > p. By using the triangular inequality, relation (14), and the fact that s ≥ 1 we get

$$b_m(j_q, j_p) \leqslant \sum_{e=p}^{q-1} s^e b_m(j_e, j_{e+1}) \leqslant \sum_{e=p}^{q-1} s^e \psi^e \big(b_m(j_1, j_2)\big)$$

$$\leqslant \sum_{e=p}^{\infty} s^e \psi^e \big(b_m(j_1, j_2)\big) \to 0.$$

This shows that {jn} is a Cauchy sequence in X, whose completeness ensures the existence of an element j# ∈ X so that jn → j#. Hypothesis (iii) implies α(jn, j#) ≥ s for each n ∈ N since α(jn, jn+1) ≥ s for all n ∈ N. Without loss of generality, it can be presumed that jn ≠ j# for all n ∈ N. From (9) we get

$$\chi\Big(\alpha(j_n, j^*)H_{b_m}(Tj_n, Tj^*),$$

$$\kappa\Big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), \frac{b_m(j^*, Tj_n) + b_m(j_n, Tj^*)}{2s}\Big)\Big)$$

$$+ L\eta\big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\big) \geqslant 0.$$

By the use of condition (C1) and the relation α(jn, j#) ≥ s in the above inequality, for n ∈ N, we get

$$s b_m(j_{n+1}, Tj^*)$$

$$\leqslant \alpha(j_n, j^*)H_{b_m}(Tj_n, Tj^*)$$

$$\leqslant \kappa\Big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), \frac{b_m(j^*, Tj_n) + b_m(j_n, Tj^*)}{2s}\Big)$$

$$\quad + L\eta\big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\big)$$

$$\leqslant \kappa\Big(b_m(j_n, j^*), b_m(j_n, j_{n+1}), b_m(j^*, Tj^*),$$

$$\quad \frac{b_m(j^*, j_{n+1}) + b_m(j_n, j^*) + b_m(j^*, Tj^*)}{2}\Big)$$

$$\quad + L\eta\big(b_m(j_n, j^*), b_m(j_n, Tj_n), b_m(j^*, Tj^*), b_m(j_n, Tj^*), b_m(j^*, Tj_n)\big).$$

By taking the limit n → ∞, using the continuity of bm, κ and η and the fact that sbm(jn+1, Tj#) ≥ bm(j#, Tj#) − sbm(jn+1, j#), it follows that

$$b_m(j^*, Tj^*) \leqslant \kappa\Big(0, 0, b_m(j^*, Tj^*), \frac{b_m(j^*, Tj^*)}{2}\Big).$$

From the above inequality and condition (C2) we get

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto
342

$$\chi\left(b_m\left(j^*, Tj^*\right), \kappa\left(0, 0, b_m\left(j^*, Tj^*\right), \frac{b_m\left(j^*, Tj^*\right)}{2}\right)\right) \geqslant 0.$$

By using (C3) we obtain bm(j#, Tj#) = 0. That is, j# ∈ Tj#.

By defining $\alpha : X \times X [0,\infty)$, $\alpha(j, k) = s$ for each j, k # X, we get the following corollary.

Corollary 2. Consider X a nonempty set endowed with a continuous b-metric bm so that (X, bm) is complete. Let T be a mapping from X into CL(X) that satisfies

$$\chi\left(sH_{b_m}(Tj, Tl), \kappa\left(b_m(j, l), b_m(j, Tj), b_m(l, Tl), \frac{b_m(l, Tj) + b_m(j, Tl)}{2s}\right)\right)$$
$$+ L\eta\left(b_m(j, l), b_m(j, Tj), b_m(l, Tl), b_m(j, Tl), b_m(l, Tj)\right) \geqslant 0$$

for all j, l #X, where χ is an implicit simulation function with respect to κψs , η # Γ and L ≥ 0. Then T has a fixed point.

The following result can be obtained directly from Theorem 3 by considering $\chi(l, j) = j - l$ for all $l, j \in R+$, $\kappa(m, n, o, p) = q$ max{m, n, o, p} and $\eta(m, n, o, p, q) = mnopq$ for all $m, n, o, p, q \in R+$.

Corollary 3. Consider X a nonempty set endowed with a continuous b-metric bm so that (X, bm) is complete. Let T be a mapping from X into CL(X) that satisfies

$$\alpha(j, l)H_{b_m}(Tj, Tl)$$
$$\leqslant q\max\left\{b_m(j, l), b_m(j, Tj), b_m(l, Tl), \frac{b_m(l, Tj) + b_m(j, Tl)}{2}\right\}$$
$$+ Lb_m(j, l)b_m(j, Tj)b_m(l, Tl)b_m(j, Tl)b_m(l, Tj), \quad j, l \in X,$$

where $q \in [0, 1)$ and L ≥ 0. Further, assume that the following conditions hold:

(i) T is αs-admissible, that is, for j, l ∈ X, α(j, l) ≥ s implies α(a, b) ≥ s for all a ∈ Tj, b ∈ Tl;

(ii) There exist j1 ∈ X and j2 ∈ Tj1 satisfying α(j1, j2) ≥ s;

(iii) For all {jn} in X with α(jn, jn+1) ≥ s, n ∈ N and jn → j, we have α(jn, j) ≥ s for all n ∈ N.

Then T has a fixed point.

4 Application to integral inclusions

Here we apply our result to prove the existence of a solution to the integral inclusion having the following form:

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

343

$$j(u) \in \int_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u), \quad u \in [a,b],$$

(15)

where l, c, v : [a, b] → R are continuous functions, c(u) ≤ v(u) for all u # [a, b], and W : [a, b] x [a, b] x R → Pcv(R), Pcv(R) is a collection of nonempty, convex and compact subsets of R such that W (., ., j) is a lower semicontinuous operator for each j # C([a, b], R), where C([a, b], R) represents the space of all continuous functions from [a, b] into R.

Consider X = C([a, b], R); this space is a complete b-metric space with dm(j, l) = supu [a,b] j(u) - l(u) 2 and s = 2.

We now define an operator T : C([a, b], R) → CL(C([a, b], R)) for the integral inclusion (15) as

$$Tj(u) = \left\{ e \in C\big([a,b],\mathbb{R}\big)\colon\ e \in \int_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u),\ u \in [a,b] \right\}.$$

We denote by Wj(u, p) := W (u, p, j(p)) for all u, p ∈ [a, b], j ∈ C([a, b], R). The Michael's selection theorem [24] ensures that for Wj : [a, b] × [a, b] → Pcv(R), there exists a continuous operator wj : [a, b] × [a, b] → R with wj(u, p) ∈ Wj(u, p) for each u, p ∈ [a, b]. By this fact we get $\int_{c(u)}^{v(u)} w_j(u,p)\,\mathrm{d}p + l(u) \in Tj(u)$. Thus, Tj is nonempty.

Moreover, the arguments provided in [33] confirm that Tj is also a closed operator.

We now state and prove an existence theorem for the integral inclusion (15).

Theorem 4. Let X = C([a, b], R), and let the operator T : X → CL(X),

$$Tj = \left\{ e \in C\big([a,b],\mathbb{R}\big)\colon\ e(u) \in \int_{c(u)}^{v(u)} W\big(u,p,j(p)\big)\,\mathrm{d}p + l(u),\ u \in [a,b] \right\},$$

where l, c, v, j : [a, b] → R are continuous functions, c(u) ≤ v(u) for all u # [a, b], and W : [a, b] x [a, b] x R → Pcv(R) is such that W (. , ., j) is a lower semicontinuous operator for all j. Further, assume that the following conditions hold:

(i) There exists a mapping $\alpha : X \times X \to (0, \infty)$ satisfying the following:

(i-a) There exist j1 ∈ X and j2 ∈ Tj1 with α(j1, j2) ≥ 2;

(i-b) For j, k ∈ X with α(j, k) ≥ 2, we have α(a, b) ≥ 2 for each a ∈ Tj, b ∈ Tk;

(i-c) For all {jn} in X with α(jn, jn+1) ≥ 2, n ∈ N and jn → j, we have α(jn, j) ≥ 2 for all n ∈ N.

(ii) There exists a continuous mapping q : X × X → [0, ∞) such that

$$T j = \left\{ e \in C([a,b], \mathbb{R}): e(u) \in \int_{c(u)}^{v(u)} W(u, p, j(p))\, \mathrm{d}p + l(u),\ u \in [a, b] \right\},$$

for each u,p # and j,k # X, where ψ:$[0,\infty) \to [0,\infty)$ is a nondecreasing continuous mapping such that $\sum_{n=1}^{\infty} 2^n \psi^n(t) < \infty$ for all $t \geq 0$. Moreover, the mapping q : $X \times X \to [0, \infty)$ satisfies

$$\int_{c(u)}^{v(u)} q(j(p), k(p))\, \mathrm{d}p \leqslant \sqrt{\frac{1}{\alpha(j, k)}}, \quad u \in [a, b].$$

Then the integral inclusion (15) has a solution.

Proof. Our aim is to prove the existence of a fixed point for the above defined operator T by using Theorem 3. For this, we focus on relation (9). Let j, k # X and e # Tj. Let

w (u, p) ∈ W (u, p) for u, p ∈ [a, b] $e(u) = \int_{c(u)}^{v(u)} w_j(u, p)\, \mathrm{d}p + l(u),\ u \in [a, b].$

By using hypothesis (ii) we have r(u, p) ∈ Wk(u, p) such that

$$\left| w_j(u, p) - r(u, p) \right| \leqslant q(j(p), k(p)) \sqrt{\psi\left(\left| j(p) - k(p) \right|^2 \right)} \quad \text{for all } u, p \in [a, b].$$

Now, consider the operator S,

$$S(u, p) = W_k(u, p)$$

$$\cap \left\{ m \in \mathbb{R}:\ \left| w_j(u, p) - m \right| \leqslant q(j(p), k(p)) \sqrt{\psi\left(\left| j(p) - k(p) \right|^2 \right)} \right\},$$

where u, p ∈ [a, b]. The lower semicontinuity of the operator S yields that there exists wk : [a, b] × [a, b] → R with wk(u, p) ∈ S(u, p) for each u, p ∈ [a, b]. Thus, we get

$$h(u) = \int_{c(u)}^{v(u)} w_k(u, p)\, \mathrm{d}p + l(u) \in \int_{c(u)}^{v(u)} W(u, p, k(p))\, \mathrm{d}p + l(u), \quad u \in [a, b],$$

and, for each u ∈ [a, b], we have

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

345

$$
\begin{aligned}
\left| e(u) - h(u) \right|^2 &\leqslant \left(\int_{c(u)}^{v(u)} \left| w_j(u,p) - w_k(u,p) \right| \mathrm{d}p \right)^2 \\
&\leqslant \left(\int_{c(u)}^{v(u)} q\big(j(p), k(p)\big) \sqrt{\psi\big(\left|j(p) - k(p)\right|^2\big)}\, \mathrm{d}p \right)^2 \\
&\leqslant \left(\sqrt{\psi\Big(\sup_{p \in [a,b]} \left|j(p) - k(p)\right|^2 \Big)} \int_{c(u)}^{v(u)} q\big(j(p), k(p)\big)\, \mathrm{d}p \right)^2 \\
&= \psi\big(d_m(j,k)\big) \left(\int_{c(u)}^{v(u)} q\big(j(p), k(p)\big)\, \mathrm{d}p \right)^2 \leqslant \frac{1}{\alpha(j,k)} \psi\big(d_m(j,k)\big).
\end{aligned}
$$

Consequently, it follows that

$\alpha(j, k)\, dm(e, h) \leq \psi\, dm(j, k)$.

By replacing the role of j and k we conclude that

$\alpha(j, k)\, Hbm\, (Tj, Tk) \leq \psi\, dm(j, k)$ for each $j, k \in X$.

By taking $\chi(l, j) = j - l$ for all l, j # R+, $\kappa(m, n, o, p) = \psi(m)$, $\eta(m, n, o, p, q) = mnopq$ for all m, n, o, p, q R+ and L = 0, inequality (9) reduces to the above inequality. Moreover, hypotheses (i-a), (i-b), (i-c) of the result imply hypotheses (i), (ii) and (iii) of Theorem 3. Thus, Theorem 3 ensures that a fixed point of the operator T does exist, that is, the integral inclusion (15) has a solution.

The following existence theorem is obtained by defining $\alpha : X \times X \to [0, \infty)$, $\alpha(j, k) = 2$ for each j, k X in Theorem 4. Also, note that the following result can be proved by using Corollary 2.

Theorem 5. Let X = C([a, b], R), and let the operator $T : X \to CL(X)$,

$$
Tj(u) = \left\{ e \in C\big([a, b], \mathbb{R}\big) : e \in \int_{c(u)}^{v(u)} W\big(u, p, j(p)\big)\, \mathrm{d}p + l(u),\ u \in [a, b] \right\},
$$

where l, c, v, j : [a, b] → R are continuous functions, $c(u) \leq v(u)$ for all u # [a, b], and $W : [a, b] \times [a, b]\ R \to Pcv(R)$ is such that W (, , j) is a lower semicontinuous operator for any j. Further, assume that there exists a continuous mapping $q : X \times X \to [0, \infty)$ such that

$$
H_{b_m}\big(W\big(u, p, j(p)\big), W\big(u, p, k(p)\big)\big) \leqslant q\big(j(p), k(p)\big) \sqrt{\psi\big(\left|j(p) - k(p)\right|^2\big)},
$$

u; p # [a; b], j; k 2 X, where $\Psi : [0;\infty) \to [0;\infty)$ is a nondecreasing continuous mapping such that $\sum_{n=1}^{\infty} 2^n \Upsilon^n(t) < \infty$ for all $t \geq 0$. Moreover, the mapping $q : X \times X \to [0;1)$ satisfies

$$\int\limits_{c(u)}^{v(u)} q\big(j(p), k(p)\big)\, \mathrm{d}p \leqslant \frac{1}{2}, \quad u \in [a, b].$$

Then the integral inclusion (15) has a solution.

5 Conclusion

Fixed point results with regard to multivalued mappings endowed with implicit-type contractive conditions of a special form of simulation functions are stated and proved. An application of our result in integral inclusions is presented. As further development, we intend to design numerical schemes based on our outcomes.

References

1 A. Abkar, M. Gabeleh, The existence of best proximity points for multivalued non-self mappings, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **107**(2):319–325, 2013, https://doi.org/10.1007/s13398-012-0074-6

2 A. Al-Rawashdeh, H. Aydi, F. Abdelbasset, S. Sahmim, W. Shatanawi, On common fixed points for α F-contractions and applications, *J. Nonlinear Sci. Appl.*, .(5):3445–3458, 2016, https://doi.org/10.22436/jnsa.009.05.128

3 M. A. Al-Thagafi, N. Shahzad, Best proximity pairs and equilibrium pairs for Kakutani multimaps, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **70**(3):1209–1216, 2009, https://doi.org/10.1016/j.na.2008.02.004.

4 M.A. Al-Thagafi, N. Shahzad, Best proximity sets and equilibrium pairs for a finite family of multimaps, *Fixed Point Theory Appl.*, **2008**:457069, 2008, https://doi.org/10.1155/2008/457069.

5 M.U. Ali, B. Alqahtani, T. Kamran, E. Karapınar, Best proximity point results for .-controlled proximal contraction, *IEEE Access*, .:128009–128013, 2019, https://doi.org/10.1109/ACCESS.2019.2937160.

6 M.U. Ali, M. Farheen, T. Kamran, G. Maniu, Prešic′ type nonself operators and related best proximity results, *Mathematics*, .(5):394, 2019, https://doi.org/10.3390/math7050394.

7 M.U. Ali, H. Houmani, T. Kamran, New type of proximal contractions via implicit simulation functions, *J. Nonlinear Convex Anal.*, **20**(3):435–445, 2019.

8 M.U. Ali, C. Vetro, Fixed point theorems for multivalued maps via new auxiliary function, *Nonlinear Anal. Model. Control*, **22**(1):84–98, 2017, https://doi.org/10.15388/NA.2017.1.6

9 I. Altun, D. Turkoglu, Some fixed point theorems for weakly compatible mappings satisfying an implicit relation, *Taiwanese J. Math.*, **13**(4):1291–1304, 2009,

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

347

10 H. Aydi, A. Felhi, S. Sahmim, Common fixed points via implicit contractions on b-metric-like spaces, *J. Nonlinear Sci. Appl.*, **10**(4):1524–1537, 2017, https://doi.org/10.22436/jnsa.010.04.20.S.S

11 Basha, N. Shahzad, Best proximity point theorems for generalized proximal contractions, *Fixed Point Theory Appl.*, **2012**:42, 2012, https://doi.org/10.1186/1687-1812-2012-42.

12 A. Bejenaru, M. Postolache, On Suzuki mappings in modular spaces, *Symmetry*, **11**(3):319, 2019, https://doi.org/10.3390/sym11030319.

13 G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*, **65**(7):1379–1393, 2006, https://doi.org/10.1016/j.na.2005.10.017.

14 B.S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, *Fixed Point Theory Appl.*, **2013**:152, 2013, https://doi.org/10.1186/1687-1812-2013-152.

15 L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**:267–273, 1974, https://doi.org/10.2307/2040075.

16 L.B. Ćirić, S.B. Prešić, On Prešić type generalisation of Banach contraction principle, *Acta Math. Univ. Comen., New Ser.*, **LXXVI**(2):143–147, 2007.

17 K. Fan, Extensions of two fixed point theorems of F.E. Browder, *Math. Z.*, **112**:234–240, 1969, https://doi.org/10.1007/BF01110225.

18 M. Gabeleh, Global optimal solutions of non-self mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.,* **75**(3):67–74, 2014.

19 M. Jleli, B. Samet, Best proximity point for α ψ-proximal contraction type mappings and applications, *Bull. Sci. Math.*, **137**(8):977–995, 2013, https://doi.org/10.1016/j.bulsci.2013.02.003.

20 T. Kamran, Mizoguchi–Takahashi's type fixed point theorem, *Comput. Math. Appl.*, **57**(3): 507–511, 2009, https://doi.org/10.1016/j.camwa.2008.10.075.

21 Q. Kiran, M.U. Ali, T. Kamran, E. Karapınar, Existence of best proximity points for controlled proximal contraction, *Fixed Point Theory Appl.*, **2015**:207, 2015, https://doi.org/10.1186/s13663-015-0455-6.

22 J. Markin, N. Shahzad, Best proximity points for relatively .-continuous mappings in Banach and hyperconvex spaces, *Abstr. Appl. Anal.*, **2013**:680186, 2013,

23 M.A. Miandaragh, M. Postolache, S. Rezapour, Some approximate fixed point results for generalized .-contractive mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.,* **75**(2):3–10, 2013.

24 E. Michael, Continuous selection I, *Annals Math.*, **63**(2):361–382, 1956.

25 B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α ψ-contractive type mappings, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **75**(4):2154–2165, 2012, https://doi.org/10.1016/j.na.2011.10.014.

26 W. Shatanawi, Common fixed points for mappings under contractive conditions of (α, β, ψ)-admissibility, *Mathematics*, .(11):261, 2018, https://doi.org/10.3390/math6110261

27 W. Shatanawi, K. Abodayeh, Common fixed point for mappings under contractive condition based on almost perfect functions and .-admissibility, *Nonlinear Funct. Anal. Appl.*, **23**(2): 247–257, 2018.

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto
348

28 W. Shatanawi, K. Abodayeh, Fixed point results for mapping of nonlinear contractive conditions of .-admissibility form, *IEEE Access*, .:50280–50286, 2019, https://doi.org/10.1109/ACCESS.2019.2910794.

29 W. Shatanawi, Z.D. Mitrovic´, N. Hussain, S. Radenovic´, On generalized Hardy–Rogers type .-admissible mappings in cone .-metric spaces over Banach algebras, *Symmetry-Basel*, **12**(1): 81, 2020, https://doi.org/10.3390/sym12010081.

30 W. Shatanawi, M. Postolache, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, *Fixed Point Theory Appl.*, **2013**:54, 2013, https://doi.org/10.1186/1687-1812-2013-54.

31 W. Shatanawi, M. Postolache, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, *Fixed Point Theory Appl.*, **2013**:60, 2013, https://doi.org/10.1186/1687-1812-2013-60.

32 S. Shukla, N. Shahzad, .-Prešic´ operators on metric spaces endowed with a graph and fixed point theorems, *Fixed Point Theory Appl.*, **2014**:127, 2014, https://doi.org/10.1186/1687-1812-2014-127

33 A. Sînta˘ma˘rian, Integral inclusions of Fredholm type relative to multivalued .-contraction,*Semin. Fixed Point Theory Cluj-Napoca, .:361–368, 2002.*

34 T. Suzuki, A generalized banach contraction principle that characterizes metric completeness, *Proc. Am. Math. Soc.*, **136**(5):1861–1869, 2008, https://doi.org/10.1090/S0002-9939-07-09055-7

35 M. Turinici, Implicit contractive maps in ordered metric spaces, in T. Rasias, L. Tóth (Eds.), *Topics in Mathematical Analysis and Applications*, Springer Optim. Appl., Vol. 94, Springer, Cham, 2014, pp. 715–746, https://doi.org/10.1007/978-3-319-06554-0_32

PDF generado a partir de XML-JATS4R por Redalyc
Proyecto académico sin fines de lucro, desarrollado bajo la iniciativa de acceso abierto

349