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Article

On the stability of a laminated beam with structural damping and Gurti–Pipkin thermal law

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Abstract: In this paper, we investigate the stabilization of a one-dimensional thermoelastic laminated beam with structural damping coupled with a heat equation modeling an expectedly dissipative effect through heat conduction governed by Gurtin–Pipkin thermal law. Under some assumptions on the relaxation function ., we establish the well-posedness of the problem by using Lumer–Phillips theorem. Furthermore, we prove the exponential stability and lack of exponential stability depending on a stability number by using the perturbed energy method and Gearhart– Herbst–Prüss–Huang theorem, respectively.

Keywords: laminated beam, Gurtin–Pipkin thermal law, well-posedness, exponential stability, lack of exponential stability.

1 Introduction

In this paper, we investigate the well-posedness and asymptotic stability of a thermoelastic laminated beam with structural damping and Gurtin–Pipkin thermal law, i.e., for $(x, t) \in (0, 1) \times (0, +\infty)$,

$$\begin{aligned} &\rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ &I_\rho(3w - \psi)_{tt} - D(3w - \psi)_{xx} - G(\psi - \varphi_x) + \delta\theta_x = 0, \\ &I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ &k\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \delta(3w - \psi)_{tx} = 0 \end{aligned} \tag{1}$$

with the initial and boundary conditions

$$\begin{aligned} &\varphi(x,0)=\varphi_0(x), && \psi(x,0)=\psi_0(x), && x\in[0,1],\\ &w(x,0)=w_0(x), && \theta(x,0)=\theta_0(x), && x\in[0,1],\\ &\varphi_t(x,0)=\varphi_1(x), && \psi_t(x,0)=\psi_1(x), && x\in[0,1],\\ &w_t(x,0)=w_1(x), && \theta(-s)|_{s>0}=\theta_0(s), && x\in[0,1], \end{aligned} \tag{21}$$

$$\begin{aligned} &\varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta_x(0,t)=0, && t\in[0,+\infty),\\ &\varphi(1,t)=\psi_x(1,t)=w_x(1,t)=\theta(1,t)=0, && t\in[0,+\infty), \end{aligned} \tag{22}$$

where the functions $\varphi(x,t),\ \psi(x,t),\ 3w(x,t)-\psi(x,t),\ \theta(x,t),\ g(s)$ denote the trans- verse displacement of the beam, which departs from its equilibrium position, rotation angle, effective rotation angle, relative temperature, and the memory kernel, respectively; . *w (x, t)* is proportional to the amount of slip along the interface at time *t* and longitudinal spatial variable *x; g(s)* is the heat conductivity relaxation kernel, whose properties will be specified later; (1). describes the dynamics of the slip; *ρ, G, I., D, γ, β* are the density of the beams, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness of the beams, and adhesive damping parameter, respectively. Moreover, *ρ, G, I., D, δ, γ, α, k, β* are positive constant.

Problem (1) is closely related to 1D thermoelastic Timoshenko beam model in the sense that (1) reduces to the Timoshenko system with Gurtin–Pipkin thermal law [12] studied by Dell'Oro and Pata [9] if the slip . is assumed to be identically zero. When there is no thermal effect, problem (1) is called laminated beam. Hansen [13] derived a model for a two-layered plate in which slip could occur along the interface. Concerned with the beam analog, with strain-rate damping as in the above described plate model [13, Eq. (3.16)], the basic evolution equations for the system are given by

$$\begin{aligned} &\rho\varphi_{tt}+S_x=0, \qquad I_\rho(3w-\psi)_{tt}-M_x-S=0,\\ &I_\rho w_{tt}-Dw_{xx}+S+\frac{4}{3}\gamma w+\frac{4}{3}\alpha w_t=0, \end{aligned}$$

where . is the shear force, and . is the bending moment. The constitutive equations are $S=G(\psi-\varphi_x),\ M=D(3w-\psi)_x$. Hansen and Spies [14] derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, namely,

$$\begin{aligned}
&\rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0,\\
&I_\rho(3w - \psi)_{tt} - D(3w - \psi)_{xx} - G(\psi - \varphi_x) = 0,\\
&I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0
\end{aligned} \tag{3}$$

for *(x, t) # (0, 1)* × *(0, +∞).* Later on, Wang et al. [29] consi√dered syste√m (3) with the cantilever boundary conditions and two different wave speeds $(\sqrt{G/\rho}$ and $\sqrt{D/I_\rho})$, they pointed out that system (3) can reach the asymptotic stability, but it does not reach the exponential stability due to the action of the slip *w*. To achieve the exponential decay result, the authors in [29] added an additional boundary control such that the boundary conditions become

$$\begin{aligned}
&\varphi(0,t) = \xi(0,t) = w(0,t) = 0, \qquad w_x(1,t) = 0,\\
&3w(1,t) - \xi(1,t) - \varphi_x(1,t) = u_1(t) := k_1\varphi_t(1,t),\\
&\xi_x(1,t) = u_2(t) := -k_2\xi_t(1,t),
\end{aligned}$$

where $\xi = 3w - \psi$, and k_1 and k_2 are positive constant feedback gains. Furthermore, Cao et al. [3] proved the exponential stability for system (3) with following boundary conditions:

$$\begin{aligned}
&\psi(0,t) - \varphi_x(0,t) = u_1(t) := -k_1\varphi_t(0,t) - \varphi(0,t),\\
&3w_x(1,t) - \psi_x(1,t) = u_2(t) := -k_2\xi_t(1,t) - \xi(1,t),
\end{aligned}$$

provided . $k_1 \neq \sqrt{\rho/G}$ and $k_2 \neq \sqrt{I_\rho/D}$. More importantly, the authors proved that the dominant part of the system is itself exponentially stable.

Concerning a laminated beam with thermoelastic dissipation effective in the bending moment, we have

$$\begin{aligned}
&\rho\varphi_{tt} + S_x = 0, \qquad I_\rho(3w - \psi)_{tt} - \widetilde{M}_x - S = 0,\\
&I_\rho w_{tt} - Dw_{xx} + S + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0,\\
&k\theta_t + q_x + \delta(3w - \psi)_{tx} = 0,
\end{aligned} \tag{4}$$

where 0 is the temperature difference, *q* denotes the heat flux, $S = G(\psi - \phi_x)$, and $M = D(3w - \psi)_x\ \delta\vartheta$. Derivative of the heat flux term in the formulation of the rate equation

$$\tau q_t + \kappa q + \theta_x = 0 \tag{5}$$

was introduced independently by Cattaneo [4] and Vernotte [28] with a fixed constant $\varkappa > 0$ and small $\tau > 0$. Combining (4) and (5), Apalara [1] considered a laminated beam with structural damping and second sound

$$\begin{aligned} &\rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ &I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi-\varphi_x) + \delta\theta_x = 0, \\ &I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0, \\ &k\theta_t + q_x + \delta(3w-\psi)_{tx} = 0, \\ &\tau q_t + \kappa q + \theta_x = 0 \end{aligned} \tag{6}$$

for $(x,t) \in (0,1) \times (0,+\infty)$. The stabilization of system (6) has been analyzed in [1], where Apalara obtained the well-posedness and uniform stability results depending on the following stability number:

$$\chi_\tau = \left(1 - \frac{\tau k G}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau G \delta^2}{\rho I_\rho}.$$

Mukiawa et al. [23] studied a thermoelastic laminated beam system without structural damping, but with a finite memory acting on the bending moment and established a gen- eral and optimal decay estimate. If we assume Gurtin–Pipkin thermal law [12] of heat conduction

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)\,\mathrm{d}s = 0, \tag{7}$$

where g is called the memory kernel, we can obtain equation (1).. The aim of this paper is to study the well-posedness and asymptotic stability of a thermoelastic laminated beam with structural damping and Gurtin–Pipkin thermal law, i.e., (1)–(2). In fact, Cattaneo law (5) can be reduced as a particular instance of (7), which have been proved in [9]. For other asymptotic behavior results to laminated beams, we refer the reader to [6, 14, 16, 21, 29] and the references therein.

For the case of the beams with Gurtin–Pipkin thermal law [12], a large number of interesting decay results depending on the stability number have been established. Re- cently, Dell'Oro and Pata [9] considered Timoshenko system with Gurtin–Pipkin thermal law, i.e., for *(x, t) # (0, L)* × *(0, +∞),*

$$\begin{aligned}
&\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0,\\
&\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0,\\
&\rho_3\theta_t - \frac{1}{\beta}\int\limits_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0,
\end{aligned}$$

where $\rho_1, \kappa, \rho_2, b, \delta, \rho_3, \beta$ are positive constants. The authors obtained the exponential stability depending on the stability number

$$\xi_g = \left(\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right)\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right) - \frac{\beta}{g(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b}.$$

Later on, Dell'Oro [8] considered the thermoelastic Bresse–Gurtin–Pipkin system, i.e., for *(x, t) # (0, L)* × *(0, +∞)* ,

$$\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = 0,\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \delta\theta_x = 0,\\
&\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = 0,\\
&\rho_3\theta_t - k_1\int\limits_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s + \delta\psi_{tx} = 0,
\end{aligned}$$

and obtained that the system is exponentially stable if and only if

$$\alpha_g := \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k_1}\frac{\rho_1\delta^2}{\rho_3 bk} = 0 \quad \text{and} \quad k = k_0.$$

For other related results, we refer the reader to [5, 17–20, 26].

In this paper, we first prove the well-posedness by using Lumer–Phillips theorem. And then, by using the perturbed energy method, we establish an exponential stability result depending on the stability number

$$\alpha_g := \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{1}{g(0)k_1}\frac{\rho_1\delta^2}{\rho_3 bk} = 0 \quad \text{and} \quad k = k_0.$$

To overcome the difficulty brought by Gurtin–Pipkin thermal law, we use some appropri- ated multipliers to construct a Lyapunov functional.

For the case $\chi_g \neq 0,$ we prove the lack of exponential stability by using Gearhart–Herbst–Prüss–Huang theorem.

The remaining part of this paper is organized as follows. In Section 2, we introduce some hypotheses and present our main results. In Section 3, we prove the well-posedness for problem (1)–(2). In Section 4, we establish an exponential decay result to prob- lem (1)–(2). In Section 5, we prove the lack of exponential stability for problem (1)–(2). Section 6 is devoted to the conclusion and open problem. Throughout this paper, we use . to denote a generic positive constant.

2 Preliminaries and main results

In this section, we first introduce some notation and present our hypotheses. Then we give some lemmas, which will be used in the proof of main results.

To deal with the memory, following [7], we introduce a new auxiliary variable $\eta = 0, \eta^t(x,s)$ by (see also [9, 10])

$$\eta = \eta^t(x,s) = \int_0^s \theta(x, t-\sigma)\,\mathrm{d}\sigma, \quad (x,t,s) \in [0,1] \times [0,\infty) \times \mathbb{R}^+,$$

which satisfies the boundary conditions $\eta^t(1,s) = 0,\ \eta^t_x(0,s) = 0.$ Then . satisfies $\eta_t + \eta_s = \theta(t),$ where . $\eta^t(x,0) = 0,\ t \in [0,\infty)$ and $\eta^0(x,s) = \eta_0(s) = \int_0^s \theta_0(\sigma)\,\mathrm{d}\sigma,\ s \in \mathbb{R}^+.$ Assume $g(\infty) = 0$, a change of variable and a formal integration by parts yield

$$\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s = -\int_0^\infty g'(s)\eta_{xx}(s)\,\mathrm{d}s.$$

Now, we denote $\mu(s) = -g'(s),$ then $\int_0^\infty g(s)\theta_{xx}(t-s)\,\mathrm{d}s = \int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s.$ Hence system (1)–(2) can be written

$$\begin{aligned}
&\rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0,\\
&I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi - \varphi_x) + \delta\theta_x = 0,\\
&I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\alpha w_t = 0,\\
&k\theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\,\mathrm{d}s + \delta(3w-\psi)_{tx} = 0,\\
&\eta_t + \eta_s = \theta
\end{aligned} \tag{8}$$

for *(x, t) # (0, 1)* × *(0, +∞)* with initial and boundary conditions

$$\begin{aligned}
&\varphi(x,0)=\varphi_0(x),\quad \psi(x,0)=\psi_0(x),\quad w(x,0)=w_0(x),\quad x\in[0,1],\\
&\theta(x,0)=\theta_0(x),\quad x\in[0,1],\\
&\varphi_t(x,0)=\varphi_1(x),\quad \psi_t(x,0)=\psi_1(x),\quad w_t(x,0)=w_1(x),\quad x\in[0,1],\\
&\eta(x,0)=0,\quad \eta_0(x,s)=\int_0^s \theta_0(x,\sigma)\,\mathrm{d}\sigma,\quad x\in[0,1],
\end{aligned} \tag{91}$$

$$\begin{aligned}
\varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta_x(0,t)=\eta_x^t(0,s)=0,\quad t\in[0,+\infty),\\
\varphi(1,t)=\psi_x(1,t)=w_x(1,t)=\theta(1,t)=\eta^t(1,s)=0,\quad t\in[0,+\infty).
\end{aligned} \tag{92}$$

For the memory kernel ., we assume $g\in C^2(\mathbb{R}^+)\cap W^{1,1}(\mathbb{R}^+)$ and
(G1) *g* is a bounded convex summable function on [0, ∞);
(G2) *g* has a total mass $\int_0^\infty g(s)\,\mathrm{d}s=1;$
(G3) *g´*. is an absolutely continuous function on *R+* so that

$$g'(s)\leqslant 0,\qquad g''(s)\geqslant 0,\qquad g'(0)=\lim_{s\to 0} g'(s)\in(-\infty,0);$$

(G4) There exists a positive constant ξ so that, for almost every *s > 0*,

$$g''(s)+\xi g'(s)\geqslant 0.$$

Remark 1.

In particular, *u* is summable on **R+** with $\int_0^\infty \mu(s)\,\mathrm{d}s=g(0).$ Furthermore, noting that *g(s)* has total mass 1 , we have $\int_0^\infty s\mu(s)\,\mathrm{d}s=1.$

Next, we introduce the vector function $U=(\varphi,\,u,\,3w-\psi,\,3v-u,\,w,\,v,\,\theta,\,\eta)^{\mathrm{T}}$ with $u=\varphi_t$ and $v=w_t.$ Then system (8)–(9) can be written as

$$\begin{aligned}
&\partial_t U=\mathcal{A}U,\\
&U(x,0)=U_0(x)=(\varphi_0,\ \varphi_1,\ 3w_0-\psi_0,\ 3w_1-\psi_1,\ w_0,\ w_1,\ \theta_0,\ \eta_0)^{\mathrm{T}},
\end{aligned} \tag{10}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} u \\ -\frac{G}{\rho}(\psi - \varphi_x)_x \\ 3v - u \\ \frac{D}{I_\rho}(3w - \psi)_{xx} + \frac{G}{I_\rho}(\psi - \varphi_x) - \frac{\delta}{I_\rho}\theta_x \\ v \\ \frac{D}{I_\rho} w_{xx} - \frac{G}{I_\rho}(\psi_- \varphi_x) - \frac{4\gamma}{3I_\rho} w - \frac{4\alpha}{3I_\rho} v \\ \frac{1}{k\beta} \int\limits_0^\infty \mu(s) \eta_{xx}(s)\,\mathrm{d}s - \frac{\delta}{k}(3v - u)_x \\ -\eta_s + \theta \end{pmatrix}.$$

We consider the following spaces:

$$H_*^1(0,1) = \{\eta \mid \eta \in H^1(0,1)\colon \eta(0) = 0\},$$
$$\widetilde{H}_*^1(0,1) = \{\eta \mid \eta \in H^1(0,1)\colon \eta(1) = 0\},$$
$$H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \qquad \widetilde{H}_*^2(0,1) = H^2(0,1) \cap \widetilde{H}_*^1(0,1)$$

and the energy space

$$\mathcal{H} = \widetilde{H}_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathcal{M},$$

where

$$\mathcal{M} = L_\mu^2(\mathbb{R}^+, \widetilde{H}_*^1(0,1)) = \left\{\eta : \mathbb{R}^+ \to \widetilde{H}_*^1(0,1) \;\middle|\; \int\limits_0^\infty \mu(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s < \infty\right\}$$

equipped with the norm $\|\varphi\|_{\mathcal{M}}^2 = \int_0^\infty \mu(s)\|\varphi_x(s)\|_2^2\,\mathrm{d}s$ and inner product $\langle\varphi, \psi\rangle_{\mathcal{M}} = \int_0^\infty \mu(s)\int_0^1 \varphi_x(s)\psi_x(s)\,\mathrm{d}x\,\mathrm{d}$. In particular $\langle -\eta_s, \eta\rangle_{\mathcal{M}} = (1/2)\int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\,\mathrm{d}s.$

Moreover, in light of (G4) on *u*, we deduce

$$\xi \int\limits_0^\infty \mu(s)\|\eta_x(s)\|_2^2\,\mathrm{d}s \leqslant -\int\limits_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\,\mathrm{d}s. \tag{11}$$

Besides, H is the Hilbert space equipped with the norm

$$\|U\|_{\mathcal{H}}^2 = \rho\|u\|_2^2 + I_\rho\|3v-u\|_2^2 + 3I_\rho\|v\|_2^2 + G\big\|(\psi-\varphi_x)\big\|_2^2 + D\big\|(3w-\psi)_x\big\|_2^2 + 4\gamma\|w\|_2^2 + 3D\|w_x\|_2^2 + k\|\theta\|_2^2 + \frac{1}{\beta}\|\eta\|_{\mathcal{M}}^2$$

and the inner product

$$(U,\widetilde{U})_{\mathcal{H}} = \rho\int_0^1 u\tilde{u}\,\mathrm{d}x + I_\rho\int_0^1 (3v-u)(3\tilde{v}-\tilde{u})\,\mathrm{d}x + 3I_\rho\int_0^1 v\tilde{v}\,\mathrm{d}x + k\int_0^1 \theta\tilde{\theta}\,\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)(\tilde{\psi}-\tilde{\varphi}_x)\,\mathrm{d}x + D\int_0^1 (3w-\psi)_x(3\tilde{w}-\tilde{\psi})_x\,\mathrm{d}x + 4\gamma\int_0^1 w\tilde{w}\,\mathrm{d}x + 3D\int_0^1 w_x\tilde{w}_x\,\mathrm{d}x + \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^1 \eta_x\tilde{\eta}_x\,\mathrm{d}x\,\mathrm{d}s$$

for . =(*ϕ, u,* 3.−*ψ,* 3.−*u, w, v, ϑ, η*). and .˜ =(.˜, *u*˜, 3.˜−.˜, 3.˜−.˜, *w*., *v*., *ϑ*., *η*˜).

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \Big\{ U\in\mathcal{H} \;\Big|\; \varphi\in\widetilde{H}_*^2(0,1),\ u\in\widetilde{H}_*^1(0,1),\ 3w-\psi\in H_*^2(0,1),\ 3v-u\in H_*^1(0,1),\ w\in H_*^2(0,1),\ v\in H_*^1(0,1),\ \theta\in\widetilde{H}_*^1(0,1),\ \eta\in\mathcal{N},\ \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s\in\widetilde{H}_*^2(0,1),\ \varphi_x(0,t)=\psi_x(1,t)=w_x(1,t)=\theta_x(0,t)=\eta_x(0,s)=0 \Big\},$$

where $\mathcal{N}=\{\eta\in\mathcal{M}\mid \eta_x\in\mathcal{M}, \eta(0)=0\}$. Clearly, $D(\mathcal{A})$ is dense in H.

The energy associated with problem (8)–(9) is defined by

$$E(t) = \frac{1}{2}\bigg(\rho\int_0^1 \varphi_t^2\,\mathrm{d}x + I_\rho\int_0^1 (3w_t-\psi_t)^2\,\mathrm{d}x + 4\gamma\int_0^1 w^2\,\mathrm{d}x + 3I_\rho\int_0^1 w_t^2\,\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x + D\int_0^1 (3w_x-\psi_x)^2\,\mathrm{d}x + 3D\int_0^1 w_x^2\,\mathrm{d}x + k\int_0^1 \theta^2\,\mathrm{d}x + \frac{1}{\beta}\int_0^\infty \mu(s)\big\|\eta_x(s)\big\|_2^2\,\mathrm{d}s\bigg). \tag{12}$$

Now, we give our main results in this paper as follows.

Theorem 1.

Let U_0 #H , then problem (10) *admits a unique weak solution U # C(R +; H) . Moreover, if U_0 # D(A) , then* $U \in C(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H})$..

Theorem 2.

Assume that $\chi_g = 0$. *Let* $U_0 \in \mathcal{H}$, *then there exist positive constants a, b such that the energy E (t) associated with problem* (8).(9) *satisfies*

$$E(t) \leqslant a e^{-bt}, \quad t \geqslant 0. \tag{13}$$

Theorem 3.

Assume that $\chi_g \neq 0$. *Let* $U_0 \in \mathcal{H}$, *then problem* (8).(9) *is not exponentially stable.*

Based on two propositions from [9, Props. 11, 12], we give the full equivalence between Cattaneo law and Gurtin–Pipkin thermal law.

Theorem 4.

If the laminated beam with structural damping and Cattaneo law is exponen- tially stable, then so is the laminated beam with structural damping and Gurtin–Pipkin thermal law, and vice versa.

3 Well-posedness: proof of Theorem 1

To obtain the well-posedness, we need to prove that $\mathcal{A}: D(\mathcal{A}) \to \mathcal{H}$ is a maximal mono- tone operator. To achieve this goal, we need to prove that *A* is dissipative and *Id - A* is surjective.

Using the inner product and integration by parts, we can easily obtain

$$(\mathcal{A}U, U)_{\mathcal{H}} = -4\alpha \int_0^1 w_t^2 \, \mathrm{d}x + \frac{1}{\beta} \int_0^\infty \mu'(s) \left\| \eta_x(s) \right\|_2^2 \mathrm{d}s \leqslant 0$$

for any *U # D(A)* . Hence A is dissipative.

Next, we turn to prove *Id – A* is surjective, i.e., for any $F = (f_1, f_2, \ldots, f_8) \in \mathcal{H}$, there exists $V = (v_1, v_2, \ldots, v_8) \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})V = F, \tag{14}$$

that is,

$$v_1 - v_2 = f_1, \qquad v_3 - v_4 = f_3, \qquad v_5 - v_6 = f_5, \qquad v_8 + \partial_s v_8 - v_7 = f_8,$$
$$\rho v_2 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho f_2,$$
$$I_\rho v_4 + G\partial_x v_1 + G v_3 - D\partial_{xx} v_3 - 3G v_5 + \delta\partial_x v_7 = I_\rho f_4,$$
$$\left(I_\rho + \frac{4}{3}\alpha\right) v_6 - G\partial_x v_1 - G v_3 + \left(3G + \frac{4\gamma}{3}\right) v_5 - D\partial_{xx} v_5 = I_\rho f_6,$$
$$k v_7 - \frac{1}{\beta}\int_0^\infty \mu(s)\partial_{xx} v_8 \,\mathrm{d}s + \delta\partial_x v_4 = k f_7. \tag{15}$$

From (15) ₁and $v_S(0) = 0$ we have

$$v_2 = v_1 - f_1, \qquad v_4 = v_3 - f_3, \qquad v_6 = v_5 - f_5,$$
$$v_8 = \left(1 - \mathrm{e}^{-s}\right) v_7 + \int_0^s \mathrm{e}^{\tau - s} f_8(\tau)\,\mathrm{d}\tau. \tag{16}$$

Inserting (16) into (15)2, (15)3, (15)4 and (15)5, we obtain

$$\rho v_1 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho(f_1 + f_2),$$
$$(I_\rho + G) v_3 + G\partial_x v_1 - D\partial_{xx} v_3 - 3G v_5 + \delta\partial_x v_7 = I_\rho(f_3 + f_4),$$
$$\left(I_\rho + 3G + \frac{4\gamma}{3} + \frac{4\alpha}{3}\right) v_5 - G\partial_x v_1 - G v_3 - D\partial_{xx} v_5 = I_\rho(f_5 + f_6) + \frac{4}{3}\alpha f_5,$$
$$k v_7 - \frac{1}{\beta}\int_0^\infty \left(1 - \mathrm{e}^{-s}\right)\mu(s)\partial_{xx} v_7 \,\mathrm{d}s + \delta\partial_x v_3$$
$$= k f_7 + \frac{1}{\beta}\int_0^\infty \mu(s)\int_0^s \mathrm{e}^{\tau - s}\partial_{xx} f_8(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \delta\partial_x f_3. \tag{17}$$

Multiplying (17) by $\bar{v}_1$, $\bar{v}_3$, $3\bar{v}_5$, and $\bar{v}_7$, respectively, and integrating over (0, 1), we can obtain

$$
\begin{aligned}
&\int_0^1 \rho v_1 \tilde{v}_1 \,\mathrm{d}x - \int_0^1 G\partial_{xx} v_1 \tilde{v}_1 \,\mathrm{d}x - \int_0^1 G\partial_x v_3 \tilde{v}_1 \,\mathrm{d}x + \int_0^1 3G\partial_x v_5 \tilde{v}_1 \,\mathrm{d}x \\
&\quad = \int_0^1 \rho(f_1 + f_2)\tilde{v}_1 \,\mathrm{d}x, \\
&\int_0^1 (I_\rho + G) v_3 \tilde{v}_3 \,\mathrm{d}x + \int_0^1 G\partial_x v_1 \tilde{v}_3 \,\mathrm{d}x - \int_0^1 D\partial_{xx} v_3 \tilde{v}_3 \,\mathrm{d}x - \int_0^1 3G v_5 \tilde{v}_3 \,\mathrm{d}x + \int_0^1 \delta\partial_x v_7 \tilde{v}_3 \,\mathrm{d}x \\
&\quad = \int_0^1 I_\rho f_4 \tilde{v}_3 \,\mathrm{d}x, \\
&\int_0^1 (3I_\rho + 9G + 4\gamma + 4\alpha) v_5 \tilde{v}_5 \,\mathrm{d}x - \int_0^1 3G\partial_x v_1 \tilde{v}_5 \,\mathrm{d}x - \int_0^1 3G v_3 \tilde{v}_5 \,\mathrm{d}x - \int_0^1 3D\partial_{xx} v_5 \tilde{v}_5 \,\mathrm{d}x \\
&\quad = \int_0^1 3I_\rho (f_5 + f_6)\tilde{v}_5 \,\mathrm{d}x + \int_0^1 4\alpha f_5 \tilde{v}_5 \,\mathrm{d}x, \\
&\int_0^1 k v_7 \tilde{v}_7 \,\mathrm{d}x - \frac{1}{\beta}\int_0^1 \tilde{v}_7 \int_0^\infty (1 - \mathrm{e}^{-s})\mu(s)\partial_{xx} v_7 \,\mathrm{d}s\,\mathrm{d}x + \int_0^1 \delta\partial_x v_3 \tilde{v}_7 \,\mathrm{d}x \\
&\quad = \int_0^1 \delta\partial_x f_3 \tilde{v}_7 \,\mathrm{d}x + \frac{1}{\beta}\int_0^1 \tilde{v}_7 \int_0^\infty \mu(s) \int_0^s \mathrm{e}^{\tau - s}\partial_{xx} f_8(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}x + \int_0^1 k f_7 \tilde{v}_7 \,\mathrm{d}x.
\end{aligned}
\tag{18}
$$

From (18) we have the following variational formulation:

$$
B\big((v_1, v_3, v_5, v_7)^{\mathrm{T}}, (\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^{\mathrm{T}}\big) = F\big((\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^{\mathrm{T}}\big) \tag{19}
$$

for all $(\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^{\mathrm{T}} \in \widetilde{H}_*^1(0,1) \times H_*^1(0,1) \times H_*^1(0,1) \times L^2(0,1)$, where

$$
\begin{aligned}
&B\big((v_1, v_3, v_5, v_7)^{\mathrm{T}}, (\tilde{v}_1, \tilde{v}_3, \tilde{v}_5, \tilde{v}_7)^{\mathrm{T}}\big) \\
&\quad = \int_0^1 G(-\partial_x v_1 - v_3 + 3v_5)(-\partial_x \tilde{v}_1 - \tilde{v}_3 + 3\tilde{v}_5)\,\mathrm{d}x + \int_0^1 \rho v_1 \tilde{v}_1 \,\mathrm{d}x + \int_0^1 I_\rho v_3 \tilde{v}_3 \,\mathrm{d}x \\
&\qquad + \int_0^1 (3I_\rho + 4\gamma + 4\alpha) v_5 \tilde{v}_5 \,\mathrm{d}x + \int_0^1 k v_7 \tilde{v}_7 \,\mathrm{d}x + \int_0^1 D\partial_x v_3 \partial_x \tilde{v}_3 \,\mathrm{d}x + \int_0^1 3D\partial_x v_5 \partial_x \tilde{v}_5 \,\mathrm{d}x \\
&\qquad + \int_0^1 \frac{1}{\beta}\left(g(0) - \int_0^\infty \mathrm{e}^{-s}\mu(s)\,\mathrm{d}s\right)\partial_x v_7 \partial_x \tilde{v}_7 \,\mathrm{d}x + \delta\int_0^1 (\partial_x v_7)\tilde{v}_3 \,\mathrm{d}x + \delta\int_0^1 (\partial_x v_3)\tilde{v}_7 \,\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
F\big((\tilde v_1,\tilde v_3,\tilde v_5,\tilde v_7)^{\mathrm T}\big)
&= \int_0^1 \big[\rho(f_1+f_2)\tilde v_1 + I_\rho(f_3+f_4)\tilde v_3 + 3I_\rho(f_5+f_6)\tilde v_5 \\
&\qquad + 4\alpha f_5\tilde v_5 + \delta\partial_x f_3\tilde v_7 + k f_7\tilde v_7\big]\,\mathrm{d}x \\
&\quad + \frac{1}{\beta}\int_0^1 \tilde v_7 \int_0^\infty \mu(s)\int_0^s \mathrm{e}^{\tau-s}\partial_{xx} f_8(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Now, we introduce the Hilbert space $V = \widetilde H^1_*(0,1)\times H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)$ equipped with the norm

$$
B\big((v_1,v_3,v_5,v_7)^{\mathrm T},(v_1,v_3,v_5,v_7)^{\mathrm T}\big) \geqslant c\big\|(v_1,v_3,v_5,v_7)\big\|_V^2.
$$

Then $B(\cdot,\cdot)$ and $F(\cdot)$ are bounded. Furthermore, we obtain that there exists a positive constant . such that

$$
v_2\in \widetilde H^1_*(0,1),\qquad v_4\in H^1_*(0,1),\qquad v_6\in H^1_*(0,1).
$$

Hence $B(\cdot,\cdot)$ is coercive.

As a consequence, by applying Lax–Milgram lemma [24] we can obtain that (18) has a unique solution $(v_1,v_3,v_5,v_7)^{\mathrm T}\in V$. Then, substituting v_1, v_3, v_5 into (16)1, we obtain

$$
\int_0^\infty \mu(s)\big\|\partial_x v_8(s)\big\|_2^2\,\mathrm{d}s \leqslant 2g(0)\|\partial_x v_7\|_2^2 + 2\|f_8\|_{\mathcal M}^2,
$$

Using (16)2 and the method in [30, Prop. 2.2], we have

$$
\begin{gathered}
v_1\in \widetilde H^2_*(0,1),\qquad v_3\in H^2_*(0,1),\qquad v_5\in H^2_*(0,1),\qquad v_7\in \widetilde H^1_*(0,1),\\
\partial_x v_1(0)=\partial_x v_3(1)=\partial_x v_5(1)=\partial_x v_7(0)=0.
\end{gathered}
$$

which gives us $v_8\in\mathcal M$. Then from (15). we can obtain $\partial_s v_8 = v_7 - v_8 + f_8 \in \mathcal M$.. Hence, $v_8\in\mathcal M$.. Next, we turn to prove that

$$
\begin{gathered}
v_1\in \widetilde H^2_*(0,1),\qquad v_3\in H^2_*(0,1),\qquad v_5\in H^2_*(0,1),\qquad v_7\in \widetilde H^1_*(0,1),\\
\partial_x v_1(0)=\partial_x v_3(1)=\partial_x v_5(1)=\partial_x v_7(0)=0.
\end{gathered}
$$

Now, if $(\tilde v_3,\tilde v_5,\tilde v_7)\equiv(0,0,0)\in H^1_*(0,1)\times H^1_*(0,1)\times L^2(0,1)$, then (19) reduces to

$$\int_0^1 G(\partial_x v_1 - v_3 + 3v_5)\partial_x \tilde{v}_1 \, \mathrm{d}x = \int_0^1 \rho v_1 \tilde{v}_1 \, \mathrm{d}x - \int_0^1 \rho(f_1 + f_2)\tilde{v}_1 \, \mathrm{d}x \tag{20}$$

for all $\tilde{v}_1 \in \widetilde{H}^1_*(0,1)$, which implies

$$G\partial_{xx} v_1 = \rho v_1 - G\partial_x v_3 + 3G\partial_x v_5 - \rho(f_1 + f_2) \in L^2(0,1). \tag{21}$$

From the regularity theory for the linear elliptic equations, we obtain $v_1 \in \widetilde{H}^2_*(0,1)$.

Moreover, (20) is also true for any $\phi \in C^1([0,1]) \subset H^1_*(0,1)$ $(\phi(1) = 0)$. Thus, we get

$$\int_0^1 G\partial_x v_1 \partial_x \phi \, \mathrm{d}x + \int_0^1 \rho v_1 \phi \, \mathrm{d}x - \int_0^1 G\partial_x v_3 \phi \, \mathrm{d}x + \int_0^1 3G\partial_x v_5 \phi \, \mathrm{d}x = \int_0^1 \rho(f_1 + f_2)\phi \, \mathrm{d}x$$

for all $\phi \in C^1([0,1])$, $\varphi(1) = 0$. Using (21) and the integration by parts, we have

$$\partial_x v_1(0)\phi(0) = 0, \quad \phi \in C^1\big([0,1]\big), \ \phi(1) = 0.$$

Hence, $\partial_x v_1(0) = 0$. In the same way, we get

$$v_3 \in H^2_*(0,1), \qquad v_5 \in H^2_*(0,1), \qquad v_7 \in \widetilde{H}^1_*(0,1),$$
$$\partial_x v_3(1) = \partial_x v_5(1) = \partial_x v_7(0) = 0.$$

From the classical regularity theory for the linear elliptic equations we know that there exists a unique solution $U \in D(\mathcal{A})$ such that (14) is satisfied. So the operator *Id* is surjective.

As a consequence, $\mathcal{A}$ is a maximal monotone operator. Therefore, we established the well-posedness result stated in Theorem 1 by using Lumer–Phillips theorem (see [2]).

4 Exponential decay: proof of Theorem 2

In this section, we prove the exponential stability for system (8)–(9) when $\chi_g = 0$. It will be achieved by using the perturbed energy method. Before we prove our result, we need some useful lemmas.

Lemma 1.

Let $(\phi, \psi, w, \vartheta)$ *be the solution of problem* (8).(9). *Then the energy function* **E(t)** *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -4\alpha \int_0^1 w_t^2 \,\mathrm{d}x + \frac{1}{\beta} \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2 \,\mathrm{d}s \leqslant 0, \quad t \geqslant 0. \tag{22}$$

Proof. Multiplying (8)1 by φ_t,, (8)2 by $(3w\ \psi)_t$, (8)3 by $3w_t$, (8)4 by 0 and integrat- ing over (0, 1), using integration by parts and the boundary conditions in (9), we can obtain (22). This completes the proof.

Lemma 2.

Let $(\phi, \psi, w, \vartheta)$ *be the solution of* (8).(9). *Then the functional* $F_1(t) = -(k/g(0)) \int_0^\infty \mu(s) \int_0^1 \theta\eta(s) \,\mathrm{d}x \,\mathrm{d}s$ *satisfies the estimate*

$$F_1'(t) \leqslant -\frac{k}{2} \int_0^1 \theta^2 \,\mathrm{d}x - c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2 \,\mathrm{d}s + \varepsilon_1 \int_0^1 (3w_t - \psi_t)^2 \,\mathrm{d}x \tag{23}$$

for any ε_1 **> 0.**

Proof. Taking the derivative of $F1(t)$ with respect to t , using (8)4, (8)5 and integrating by parts, we get

$$F_1'(t) = -k \int_0^1 \theta^2 \,\mathrm{d}x + \frac{k}{g(0)} \int_0^\infty \mu(s) \int_0^1 \theta\eta_s(s) \,\mathrm{d}x \,\mathrm{d}s + \frac{1}{\beta g(0)} \left\| \int_0^\infty \mu(s)\eta_x(s) \,\mathrm{d}s \right\|_2^2 - \frac{\delta}{g(0)} \int_0^\infty \mu(s) \int_0^1 (3w - \psi)_t \eta_x(s) \,\mathrm{d}x \,\mathrm{d}s.$$

Using integration by parts, Poincaré's inequality [27, Lemma 2.2], and Young's inequality with $\varepsilon > 0$ and $\varepsilon_1 > 0$, , we infer that

$$\frac{k}{g(0)} \int_0^\infty \mu(s) \int_0^1 \theta \eta_s(s)\,\mathrm{d}x\,\mathrm{d}s$$

$$= -\frac{k}{g(0)} \int_0^\infty \mu'(s) \int_0^1 \theta \eta(s)\,\mathrm{d}x\,\mathrm{d}s \leqslant \varepsilon \int_0^1 \theta^2\,\mathrm{d}x - \frac{c}{\varepsilon} \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s,$$

$$\frac{1}{\beta g(0)} \left\| \int_0^\infty \mu(s) \eta_x(s)\,\mathrm{d}s \right\|_2^2$$

$$\leqslant c \int_0^\infty \mu(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s, -\frac{\delta}{g(0)} \int_0^\infty \mu(s) \int_0^1 (3w-\psi)_t \eta_x(s)\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \varepsilon_1 \int_0^1 (3w_t - \psi_t)^2\,\mathrm{d}x + \frac{c}{\varepsilon_1} \int_0^\infty \mu(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s.$$

Here we take $\varepsilon = k/2$, then we can get (23) by using above inequalities and (11). This completes the proof.

Lemma 3.

Let $(\phi, \psi, w, \vartheta)$ *be the solution of* (8).(9). *Then the functional* $F_2(t) = (kI_\rho/\delta) \int_0^1 (3w-\psi)_t \int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x$ *satisfies the estimate*

$$F_2'(t) \leqslant -\frac{I_\rho}{2} \int_0^1 (3w_t - \psi_t)^2\,\mathrm{d}x + \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2\,\mathrm{d}x + \varepsilon_2 \int_0^1 (3w_x - \psi_x)^2\,\mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 \theta^2\,\mathrm{d}x - c \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s \tag{24}$$

for any $\varepsilon_2 > 0$.

Proof. Taking the derivative of $F2(t)$ with respect to t, using (8)2, (8)4 and integrating by parts, we get

$$F_2'(t) \leqslant -\frac{I_\rho}{2} \int_0^1 (3w_t - \psi_t)^2\,\mathrm{d}x + \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2\,\mathrm{d}x + \varepsilon_2 \int_0^1 (3w_x - \psi_x)^2\,\mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_2}\right) \int_0^1 \theta^2\,\mathrm{d}x - c \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2\,\mathrm{d}s$$

Using (11), Young's and Cauchy–Schwarz inequalities with $\varepsilon_2 > 0,$ we establish esti- mate (24).

Lemma 4.

Let $(\phi, \psi, w, \vartheta)$ *be the solution of* (8).(9). *Then the functional*

$$F_3(t) = \rho D \int_0^1 \varphi_t (3w-\psi)_x \,\mathrm{d}x - I_\rho G \int_0^1 (3w-\psi)_t(\psi-\varphi_x)\,\mathrm{d}x + \frac{\rho k I_\rho}{\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^1 \theta\varphi_t\,\mathrm{d}x - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty \mu(s)\int_0^1 (\psi-\varphi_x)\eta_x(s)\,\mathrm{d}x\,\mathrm{d}s$$

satisfies the estimat

$$F_3'(t) \leqslant -\frac{G^2}{2}\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x + c\int_0^1 \left[(3w_t-\psi_t)^2 + w_t^2\right]\mathrm{d}x - c\int_0^\infty \mu'(s)\left\|\eta_x(s)\right\|_2^2\,\mathrm{d}s. \tag{25}$$

Proof. By (8)1, (8)2, (8)4 and integrating by parts, we get

$$F_3'(t) = -G^2\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x - I_\rho G\int_0^1 (3w-\psi)_t\psi_t\,\mathrm{d}x - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty \mu(s)\int_0^1 \psi_t\eta_x(s)\,\mathrm{d}x\,\mathrm{d}s - \frac{\rho I_\rho}{\beta\delta}\left(\frac{D}{I_\rho}-\frac{G}{\rho}\right)\int_0^\infty \mu'(s)\int_0^1 (\psi-\varphi_x)\eta_x(s)\,\mathrm{d}x\,\mathrm{d}s - \frac{\rho I_\rho}{\delta}\frac{g(0)}{\beta}\chi_g\int_0^1 \theta_x(\psi-\varphi_x)\,\mathrm{d}x.$$

Similarly as in [1, Lemma 2.4], using $\chi_g = 0$, Young's and Cauchy–Schwarz inequalities, and the fact that $\psi_t = -(3w_t-\psi_t)+3w_t,$ we get (25).

Lemma 5.

Let (ϕ, ψ, w, ϑ) be the solution of (8).(9). *Then the functional* $F_4(t) = -\rho\int_0^1 \varphi\varphi_t\,\mathrm{d}x$ *satisfies the estimate*

$$F_4'(t) \leqslant -\rho\int_0^1 \varphi_t^2\,\mathrm{d}x + \varepsilon_4\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + \varepsilon_4\int_0^1 w_x^2\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x \tag{26}$$

for any $\varepsilon_4 > 0$.

Proof. By differentiating $F4(t)$ with respect to t, using (8). and integrating by parts, we obtain

$$F_4'(t) = -\rho\int_0^1 \varphi_t^2\mathrm{d}x + G\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x - G\int_0^1 \psi(\psi-\varphi_x)\,\mathrm{d}x.$$

Using Young's and Poincaré's inequalities, we obtain

$$F_4'(t) \leqslant -\rho\int_0^1 \varphi_t^2\,\mathrm{d} + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 (\psi-\varphi_x)^2\,\mathrm{d}x + \varepsilon_4\int_0^1 \psi_x^2\,\mathrm{d}x$$

for $\varepsilon_4 > 0$. Note that

$$\int_0^1 \psi_x^2\mathrm{d}x = \int_0^1 (\psi_x - 3w_x + 3w_x)^2\mathrm{d}x \leqslant 2\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + 18\int_0^1 w_x^2\mathrm{d}x.$$

Then estimate (26) is obtained.

Lemma 6.

Let (ϕ, ψ, w, ϑ) be the solution of (8).(9). *Then the functional* $F_5(t) = I_\rho\int_0^1 (3w-\psi)(3w-\psi)_t\,\mathrm{d}x$ *satisfies the estimate*

$$F_5'(t) \leqslant -\frac{D}{2}\int_0^1 (3w_x - \psi_x)^2 \,\mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \,\mathrm{d}x + c\int_0^1 (\psi - \varphi_x)^2 \,\mathrm{d}x + c\int_0^1 \theta^2 \,\mathrm{d}x. \tag{27}$$

Proof. Taking the derivative of $F_5(t)$ with respect to t, using (8)2 and integrating by parts, we get

$$F_5'(t) = -D\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + G\int_0^1 (\psi - \varphi_x)(3w - \psi)\,\mathrm{d}x + \delta\int_0^1 (3w - \psi)_x \theta \,\mathrm{d}x.$$

Then, using Poincaré's and Young's inequalities, we arrive at (27).

Lemma 7.

Let $(\phi, \psi, w, \vartheta)$ *be the solution of* (8).(9). *Then the functional* $F_6(t) = I_\rho \int_0^1 w w_t \,\mathrm{d}x$ *satisfies the estimate*

$$F_6'(t) \leqslant -\frac{2\gamma}{3}\int_0^1 w^2 \mathrm{d}x - D\int_0^1 w_x^2 \,\mathrm{d}x + c\int_0^1 w_t^2 \,\mathrm{d}x + c\int_0^1 (\psi - \varphi_x)^2 \,\mathrm{d}x. \tag{28}$$

Proof. By differentiating $F_6(t)$ with respect to t, using (8)3 and integrating by parts, then use Young's inequality to obtain (28). This completes the proof.

Now we define the following Lyapunov functional

$$\mathscr{L}(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + F_5(t) + F_6(t),$$

where N, N_1, N_2, N_3 are positive constants to be selected later. Then we have the lemma as follows.

Lemma 8.

Let (ϕ, ψ, w, ϑ) be the solution of (8).(9). *For N large enough, there exists a positive c such that, for any t* ≥ 0,

$$(N-c)E(t) \leqslant \mathscr{L}(t) \leqslant (N+c)E(t).$$

Proof. Using Young's, Poincaré's and Cauchy–Schwarz inequalities, and the fact that (see [22])

$$\int_0^1 \varphi^2 \, dx \leqslant \int_0^1 \varphi_x^2 \, dx \leqslant 2\int_0^1 (\psi-\varphi_x)^2 \, dx + 2\int_0^1 (3w_x-\psi_x)^2 \, dx + 18\int_0^1 w_x^2 \, dx,$$

we can easily obtain that

$$\begin{aligned} &\left|\mathscr{L}(t) - NE(t)\right| \\ &\leqslant \alpha_1 \int_0^1 \varphi_t^2 \, dx + \alpha_2 \int_0^1 (3w_t-\psi_t)^2 \, dx + \alpha_3 \int_0^1 w_t^2 \, dx \\ &\quad + \alpha_4 \int_0^1 (\psi-\varphi_x)^2 \, dx + \alpha_5 \int_0^1 (3w_x-\psi_x)^2 \, dx + \alpha_6 \int_0^1 w^2 \, dx \\ &\quad + \alpha_7 \int_0^1 w_x^2 \, dx + \alpha_8 \int_0^1 \theta^2 \, dx + \alpha_9 \int_0^\infty \mu(s)\left\|\eta_x(s)\right\|_2^2 \, ds, \end{aligned} \quad (29)$$

where α_i $(\mathbf{i} = \mathbf{1,2}, \ldots, \mathbf{9})$ are positive constants. It follows from (12) and (29) that there exists a positive constant . such that $|\mathscr{L}(t) - NE(t)| \leqslant cE(t)$, which completes the proof.

Now, we are ready to prove the main result in this section.

Proof of Theorem 2. From (23)–(27) and (28) we can obtain

$$\begin{aligned}\mathscr{L}'(t) \leqslant{} & -\rho \int_0^1 \varphi_t^2 \,\mathrm{d}x - \left[\frac{I_\rho}{2} N_2 - N_1 \varepsilon_1 - cN_3 - I_\rho\right] \int_0^1 (3w_t - \psi_t)^2 \,\mathrm{d}x \\ & - (4\alpha N - cN_3 - c) \int_0^1 w_t^2 \,\mathrm{d}x \\ & - \left[\frac{G^2}{2} N_3 - N_2 \varepsilon_2 - c\left(1 + \frac{1}{\varepsilon_4}\right) - 2c\right] \int_0^1 (\psi - \varphi_x)^2 \,\mathrm{d}x \\ & - \left(\frac{D}{2} - N_2 \varepsilon_2 - \varepsilon_4\right) \int_0^1 (3w_x - \psi_x)^2 \,\mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \,\mathrm{d}x \\ & - (D - \varepsilon_4) \int_0^1 w_x^2 \,\mathrm{d}x - \left[\frac{k}{2} N_1 - cN_2\left(1 + \frac{1}{\varepsilon_2}\right) - c\right] \int_0^1 \theta^2 \,\mathrm{d}x \\ & + \left[\frac{1}{\beta} N - cN_1\left(1 + \frac{1}{\varepsilon_1}\right) - cN_2 - cN_3\right] \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2 \,\mathrm{d}s.\end{aligned}$$

At this point, we need to choose our constants very carefully. First, we choose $\varepsilon_1 = I_\rho N_2/(4N_1), \varepsilon_2 = \min\{G^2 N_3/(4N_2), D/(8N_2)\}, \varepsilon_4 = D/8,$ so that

$$\begin{aligned}\mathscr{L}'(t) \leqslant{} & -\rho \int_0^1 \varphi_t^2 \,\mathrm{d}x - \left[\frac{I_\rho}{4} N_2 - cN_3 - I_\rho\right] \int_0^1 (3w_t - \psi_t)^2 \,\mathrm{d}x \\ & - (4\alpha N - cN_3 - c) \int_0^1 w_t^2 \,\mathrm{d}x \left[\frac{G^2}{4} N_3 - \frac{8}{D} c - 3c\right] \int_0^1 (\psi - \varphi_x)^2 \,\mathrm{d}x \\ & - -\frac{D}{4} \int_0^1 (3w_x - \psi_x)^2 \,\mathrm{d}x - \frac{2\gamma}{3} \int_0^1 w^2 \,\mathrm{d}x \\ & - \frac{7D}{8} \int_0^1 w_x^2 \,\mathrm{d}x - \left[\frac{k}{2} N_1 - cN_2\left(1 + \frac{1}{\varepsilon_2}\right) - c\right] \int_0^1 \theta^2 \,\mathrm{d}x \\ & + \left[\frac{1}{\beta} N - cN_1\left(1 + \frac{N_1}{N_2}\right) - cN_2 - cN_3\right] \int_0^\infty \mu'(s) \|\eta_x(s)\|_2^2 \,\mathrm{d}s.\end{aligned} \tag{30}$$

Then, we select .. large enough so that $(G^2/4)N_3 - (8/D)c - 3c > 0.$ Next, we select .. large enough so that $(I_\rho/4)N_2 - cN_3 - I_\rho > 0.$. Furthermore, we select .. large enough so that $(k/2)N_1 - cN_2(1 + 1/\varepsilon_2) - c > 0.$ Finally, we select . large enough so that $4\alpha N - cN_3 - c > 0, (1/\beta)N - cN_1(1 + N_1/N_2) - cN_2 - cN_3 > 0.$ Using (12), we obtain that there exist positive constants *M1* and *M2* such that (30) becomes

$$\mathscr{L}'(t) \leqslant -M_1E(t) + M_2 \int_0^\infty \mu'(s)\|\eta_x(s)\|_2^2\,\mathrm{d}s \leqslant -M_1E(t), \quad t \geqslant 0.$$

From Lemma 8 we obtain

$$\mathscr{L}'(t) \leqslant -b\mathscr{L}(t), \qquad t \geqslant 0, \tag{31}$$

where $b = M_1/(N+c)$. Then, a simple integration of (31) over (0, *t*) yields

$$\mathscr{L}(t) \leqslant \mathscr{L}(0)\mathrm{e}^{-bt}, \quad t \geqslant 0. \tag{32}$$

At last, estimate (32) gives exponential stability result (13) when be combined with

Lemma 8. This completes the proof.

5 Lack of exponential stability: proof of Theorem 3

Our result is achieved by using Gearhart–Herbst–Prüss–Huang theorem to dissipative systems (see Prüss [25] and Huang [15]).

Lemma 9.

Let $S(t) = \mathrm{e}^{\mathcal{A}t}$ be a C_0- -semigroup of contractions on Hilbert space H Then S(t) is exponentially stable if and only if

$$\rho(\mathcal{A}) \supset \{\mathrm{i}\lambda\colon\ \lambda \in \mathbb{R}\} \equiv \mathrm{i}\mathbb{R} \quad \textit{and} \quad \overline{\lim_{|\lambda|\to\infty}} \|(\mathrm{i}\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

hold, where ρ(A) is the resolvent set of the differential operator A .

Proof of Theorem 3. We will prove that there exists a sequence of imaginary number λ. and function *Fu # H* with # F_u # $_H$ ≤ *1* such that $\|(\lambda_\mu I - \mathcal{A})^{-1}F_\mu\|_{\mathcal{H}} = \|U_\mu\|_{\mathcal{H}} \to \infty$, where

$$\lambda_\mu U_\mu - \mathcal{A}U_\mu = F_\mu \tag{33}$$

with $U_\mu = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)^{\mathrm{T}}$ not bounded. Rewriting spectral equation (33) in term of its components, we have for $\lambda_\mu = \lambda$.,

$$\begin{aligned}
&\lambda v_1 - v_2 = g_1, \quad \lambda v_3 - v_4 = g_3, \quad \lambda v_5 - v_6 = g_5, \quad \lambda v_8 + \partial_s v_8 - v_7 = g_8,\\
&\rho\lambda v_2 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2,\\
&I_\rho \lambda v_4 + G\partial_x v_1 + G v_3 - D\partial_{xx} v_3 - 3G v_5 + \delta\partial_x v_7 = I_\rho g_4,\\
&I_\rho \lambda v_6 - G\partial_x v_1 - G v_3 + \left(3G + \frac{4\gamma}{3}\right) v_5 + \frac{4\alpha}{3} v_6 - D\partial_{xx} v_5 = I_\rho g_6,\\
&k\lambda v_7 - \frac{1}{\beta}\int_0^\infty \mu(s)\partial_{xx} v_8(s)\,\mathrm{d}s + \delta\partial_x v_4 = k g_7,
\end{aligned}$$

where $\lambda \in \mathbb{R}$ and $F = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)^{\mathrm{T}} \in \mathcal{H}$. Take $g_1 = g_3 = g_5 = 0$, then the above system becomes

$$\begin{aligned}
&\rho\lambda^2 v_1 - G\partial_{xx} v_1 - G\partial_x v_3 + 3G\partial_x v_5 = \rho g_2,\\
&I_\rho \lambda^2 v_3 + G\partial_x v_1 + G v_3 - D\partial_{xx} v_3 - 3G v_5 + \delta\partial_x v_7 = I_\rho g_4,\\
&I_\rho \lambda^2 v_5 - G\partial_x v_1 - G v_3 + \left(3G + \frac{4\gamma}{3} + \frac{4\alpha}{3}\lambda\right) v_5 - D\partial_{xx} v_5 = I_\rho g_6,\\
&k\lambda v_7 - \frac{1}{\beta}\int_0^\infty \mu(s)\partial_{xx} v_8(s)\,\mathrm{d}s + \lambda\delta\partial_x v_3 = k g_7,\\
&\lambda v_8 + \partial_s v_8 - v_7 = g_8.
\end{aligned}$$

Due to the boundary conditions in (9), we can suppose that ..$v_1 = A\cos(\mu\pi x/2)$, $v_3 = B\sin(\mu\pi x/2)$, $v_5 = C\sin(\mu\pi x/2)$, $v_7 = E\cos(\mu\pi x/2)$, $v_8 = \phi(s)\cos(\mu\pi x/2)$. Choosing

$$g_2 = \frac{1}{\rho}\cos\left(\frac{\mu\pi}{2}x\right), \qquad g_4 = g_6 = g_7 = g_8 = 0,$$

then we can obtain

$$\begin{aligned}
&\left[\rho\lambda^2 + G\left(\frac{\mu\pi}{2}\right)^2\right] A - G\frac{\mu\pi}{2} B + 3G\frac{\mu\pi}{2} C = 1,\\
&-G\frac{\mu\pi}{2} A + \left[I_\rho\lambda^2 + G + D\left(\frac{\mu\pi}{2}\right)^2\right] B - 3GC - \delta\frac{\mu\pi}{2} E = 0,\\
&G\frac{\mu\pi}{2} A - GB + \left[I_\rho\lambda^2 + 3G + \frac{4\gamma}{3} + \frac{4\alpha}{3}\lambda + D\left(\frac{\mu\pi}{2}\right)^2\right] C = 0,\\
&\lambda\delta\frac{\mu\pi}{2} B + k\lambda E + \frac{1}{\beta}\left(\frac{\mu\pi}{2}\right)^2 \int_0^\infty \mu(s)\phi(s)\,\mathrm{d}s = 0,\\
&\phi'(s) + \lambda\phi(s) - E = 0. \qquad (34)
\end{aligned}$$

In the above equations, we take $\lambda = \lambda_\mu := \mathrm{i}\sqrt{G/\rho}g(\mu\pi/2)$ such that $\rho\lambda^2 + G(\mu\pi/2)^2 = 0$. Solving (34)5, we get

$$\phi(s) = \frac{E}{\lambda}\left(1 - \mathrm{e}^{-\lambda s}\right). \tag{35}$$

Then substituting (35) into (34)4, we can get

$$E = \frac{\frac{G\delta}{\rho}\frac{\mu\pi}{2}}{\frac{g(0)}{\beta}[1 - \frac{kG}{\rho}\frac{\beta}{g(0)}] - \frac{1}{\beta}\int_0^\infty \mu(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s}B.$$

The combination of (34)2 and (34)3 gives

$$I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 B + \left[\frac{4\gamma}{3} + \frac{4\alpha}{3}\lambda + I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2\right]C - \delta\frac{\mu\pi}{2}E = 0. \tag{36}$$

Substituting . into (36), we get .$C = -\Lambda_\mu/\Gamma_\mu B$, where where

$$\Lambda_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 - \frac{\frac{G\delta^2}{\rho}(\frac{\mu\pi}{2})^2}{\frac{g(0)}{\beta}[1 - \frac{kG}{\rho}\frac{\beta}{g(0)}] - \frac{1}{\beta}\int_0^\infty \mu(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s},$$

$$\Gamma_\mu = I_\rho\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right)\left(\frac{\mu\pi}{2}\right)^2 + \frac{4\alpha}{3}\lambda + \frac{4\gamma}{3}.$$

Substituting C into (34)1, we get

$$B = -\frac{\Gamma_\mu}{G\frac{\mu\pi}{2}(\Gamma_\mu + 3\Lambda_\mu)}.$$

Similarly, substituting C into (34)3, we get

$$A = \frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G\frac{\mu\pi}{2}\Gamma_\mu}B = -\frac{G\Gamma_\mu + \Lambda_\mu\Gamma_\mu + 3G\Lambda_\mu}{G^2(\Gamma_\mu + 3\Lambda_\mu)(\frac{\mu\pi}{2})^2}.$$

At this point, we introduce the number $\gamma_g = 1 - (kG/\rho)(\beta/g(0))$ and consider separately two cases.

Case $\gamma_g = 0$. Let $\mu \to \infty$, we get

$$A \to -\frac{\beta\delta^2}{\rho G\int_0^\infty \mu(s)e^{-\lambda s}\,ds}, \quad B \to 0,\ C \to 0.$$

Case $\gamma_g = 0$. Let $\mu \to \infty$, we get

$$A \to -\frac{I_\rho \chi_g(\frac{D}{I_\rho} - \frac{G}{\rho})}{G^2[(\frac{D}{I_\rho} - \frac{G}{\rho})\gamma_g + 3\chi_g]}, \quad B \to 0,\ C \to 0.$$

Thus,

$$\|U_\mu\|_{\mathcal{H}}^2 \geqslant \frac{1}{2}G\left[3C - B + \left(\frac{\mu\pi}{2}\right)A\right]^2 \to \infty \quad \mu \to \infty.$$

This implies that $\|U_\mu\|_{\mathcal{H}} \to \infty$ as $\mu \to \infty$. Therefore, there is no exponential stability. This completes the proof.

6 Conclusion and open problem

In this paper, we first prove the well-posedness for a laminated beam with Gurtin–Pipkin thermal law and structural damping. Then we prove that the system is exponentially stable if and only if that stability number is equal to zero $(\chi_g = 0)$. When the stability number is not zero $(\chi_g \neq 0)$, the problem of whether it is possible to get the polynomial stability for system (8)–(9) is still an interesting *open problem*.

Recently, Guesmia [11] considered the stability of the laminated beam with interfacial slip and infinite memory acting only on the transverse displacement, the rotation angle, and the amount of slip, respectively. He combined the energy method and the frequency domain approach to show that the infinite memory is capable alone to guarantee the strong and polynomial stability of the model, and mentioned also that "when the exponential stability is not satisfied, obtaining the optimal decay rate of solutions is, in our opinion, a very nice and hard question".

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