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Article

Stability analysis of fractional-order systems with randomly time-varying parameters

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Abstract: This paper is concerned with the stability of fractional-order systems with randomly time- varying parameters. Two approaches are provided to check the stability of such systems in mean sense. The first approach is based on suitable Lyapunov functionals to assess the stability, which is of vital importance in the theory of stability. By an example one finds that the stability conditions obtained by the first approach can be tabulated for some special cases. For some complicated linear and nonlinear systems, the stability conditions present computational difficulties. The second alternative approach is based on integral inequalities and ingenious mathematical method. Finally, we also give two examples to demonstrate the feasibility and advantage of the second approach. Compared with the stability conditions obtained by the first approach, the stability conditions obtained by the second one are easily verified by simple computation rather than complicated functional construction. The derived criteria improve the existing related results.

Keywords: fractional-order system, randomly time-varying parameters, stability, Lyapunov functional, integral inequalities.

1 Introduction

In recent years, the increasing interest of the scientific community towards fractional calculus experienced an exceptional boost and its applications can now be found in a va- riety of real world problems, for example, viscous material [6], random and disordered media [21, 23], finance [16, 26], electrical circuits [14], automatic control system [18, 30] and so on. The reason of the success of the fractional calculus is that the fractional calculus operators are nonlocal, which makes them more precisely to characterize actual evolution process than the integer-order calculus and help to model physical problems in a more realistic manner.

Fractional differential equations are now considered as useful tools as they can model many physical systems. In order to find out the essential performance of the established equations, the existence and stability of the solutions of the equations is the first pre- requisite. In the last few years, several results on this topic were presented including asymptotic stability [1, 4, 15, 24]), exponential stability [2, 25] and Mittag-Leffler stability [5, 19, 27–30]. The general method for analyzing the stability is based on the first method of Lyapunov, the second method of Lyapunov and other mathematical techniques. The idea of the first one is that the system is stable if there are some Lyapunov functional candidates for the system, while the second one only provides a sufficient condition to show the stability of the system, and one cannot find a Lyapunov functional candidate to conclude the stability. Other mathematical strategies are mainly based on the expressions of the solutions to the systems and integral inequalities.

Fractional stochastic differential systems often arise in applications [1, 3, 7, 13, 22, 26, 30]. In recent years, such systems have attracted more and more researchers' attention in the field of stochastic differential systems. A lot of important results based on the existence and uniqueness and stability of the solutions to the systems were obtained. For example, in 1959, Bertram and Sarachik [1] studied the stability of the following system:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t), \quad t \geqslant 0,$$

Where

$$A = \begin{cases} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, & \text{prob} = p, \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & \text{prob} = 1 - p. \end{cases}$$

In 1977, Ladde [15] used logarithmic norm to discuss the stability of linear systems with random parameters. However, to the authors' knowledge, the stability of solution of fractional-order differential system with randomly time-varying coefficients have yet to be reported. In this paper, we plan to investigate the stability of the following fractional- order differential system with randomly time-varying parameters:

$$({}^{C}D^{\alpha}_{0+}x)(t) = f\big(t, \omega(t), x(t)\big), \tag{1}$$

where $0 < \alpha \leqslant 1, \omega(t)$ denotes the randomly time-varying parameters, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in . and locally

Lipschitz with respect to x in mean sense on $[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$. And assume that $f(t, \omega(t), 0) = 0$, so that $x = 0$ is an equilibrium point whose stability is to be examined. *If* $w(t)$ is a constant, then system (1) becomes a classical fractional differential system.

2 Preliminaries

In this section, we recall some basic definitions and properties about fractional calculus. For more details, we can refer them to the monograph [12].

Let $\Omega = [0,T]\ (0 \leqslant T \leqslant \infty)$ be a finite or infinite interval of the real axis R. We denote by $L^p(\Omega)$ the set of all the Lebesgue-measurable functions . on . with the norm $\|f\|_p < \infty$. Denote $C^m(\Omega)$ a space of functions f, which are m-times continuously differentiable on Ω with the norm $\|f\|_m = \sum_{k=0}^{m} \max_{x\in\Omega} |f^{(k)}(x)|$.

Definition 1.

Let $\alpha > 0$ and $f \in L^1(\Omega)$. Then the Riemann–Liouville fractional integral of order . with respect to . is defined as

$$\mathcal{I}_{0+}^{\alpha} f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t \geqslant 0,$$

where $\Gamma(\cdot)$ is the gamma function, and # denotes the convolution operator.

The Riemann–Liouville fractional integral operator have the following properties [8].

Property 1.

Let α, β > 0 and $f \in L^1(\Omega)$. *Then*

(i) $\mathcal{I}_{0+}^{\alpha} f$ *is nondecreasing with respect to f. In particular,* if $f \geqslant 0$, then $\mathcal{I}_{0+}^{\alpha} f \geqslant 0$;

(ii) The operator $\mathcal{I}_{0+}^{\alpha}$ *is compact, and* $\sigma(\mathcal{I}_{0+}^{\alpha}) = \{0\}$, *where* $\sigma(\cdot)$ *denotes the spectral set of* $\mathcal{I}_{0+}^{\alpha}$;

(iii) $(\mathcal{I}_{0+}^{\alpha}\mathcal{I}_{0+}^{\beta} f)(t) = (\mathcal{I}_{0+}^{\beta}\mathcal{I}_{0+}^{\alpha} f)(t) = (\mathcal{I}_{0+}^{\alpha+\beta} f)(t)$; ;

(iv) *For the real-valued continuous function f, it has* $\|\mathcal{I}_{0+}^{\alpha} f\| \leqslant \mathcal{I}_{0+}^{\alpha}\|f\|$, *where #·# denotes an arbitrary norm.*

For the fractional derivatives, there are two types that are commonly used: the Riemann– Liouville fractional derivative and the Caputo derivative.

Definition 2.

Let $f \in L^1(\Omega)$ and $m-1<\alpha \leqslant m, m \in \mathbb{N}^+$. The Riemann–Liouville fractional derivative and Caputo fractional derivative of order . with respect to . are defined, respectively, as

$$D_{0+}^{\alpha} f(t)=\frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d} t^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\, \mathrm{d} \tau,$$

$$\left({}^C D_{0+}^{\alpha} f\right)(t)=D_{0+}^{\alpha}\left(f(t)-\sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k\right), \quad t \geqslant 0.$$

Further, if $f(t) \in C^m(\Omega)$, then Caputo fractional derivative can also be defined as

$$\left({}^C D_{0+}^{\alpha} f\right)(t)=\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, \mathrm{d} \tau, \quad t \geqslant 0,$$

.$f(t) \in C^m(\Omega)$.which is known as a smooth fractional derivative.

Note that if $f^{(k)}(0)=0,\ k=0,1,\ldots,m-1$., then $\left({}^C D_{0+}^{\alpha} f\right)(t)$ coincides with $\left(D_{0+}^{\alpha} f\right)(t)$.)

The Caputo fractional derivative shares many similar properties with the ordinary derivative, and it is suitable for initial value problems, and so it can be applied into a lot of engineering and physical problems in real world.

Property 2.

Let $m-1<\alpha \leqslant m, m \in \mathbb{N}^+$. *The following formulas hold*:

$$\left({}^C D_{0+}^{\alpha} \mathcal{I}_{0+}^{\alpha} f\right)(t)=f(t), \qquad \left(\mathcal{I}_{0+}^{\alpha}\, {}^C D_{0+}^{\alpha} f\right)(t)=f(t)-\sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k .$$

Definition 3

The generalized Mittag-Leffler function is defined by

$$E_{\beta, \gamma}^{\rho}(z)=\sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(\beta k+\gamma) k!}, \quad \beta, \gamma, \rho>0,\ z \in \mathbb{C},$$

where $(\rho)_0=1, (\rho)_k=\rho(\rho+1) \cdots(\rho+k-1), k=1,2, \ldots$.

In particular, when $p = 1$, it becomes the two-parameter Mittag-Leffler function, i.e. $E^1_{\beta,\gamma}(z) = E_{\beta,\gamma}(z)$; when $\rho = \gamma = 1$, it becomes the one-parameter Mittag-Leffler function, i.e., $E^1_{\beta,1}(z) = E_\beta(z)$..

Definition 4.

[20] The Wright function is defined by

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(n\lambda + \mu)}, \quad \lambda > -1,\ \mu > 0,\ z \in \mathbb{C}.$$

Note that the case $\lambda = 0$ is trivial since .$W_{0,\mu}(z) = e^z/\Gamma(\mu)$. In particular, for the case $0 < \nu < 1$, $W_{-\nu,1-\nu}(-z) = M_\nu(z)$, where *Mv(z)* is the Mainardi's function defined as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-n\nu + 1 - \nu)}.$$

The Laplace transform of the Mainardi's function is

$$\int_0^\infty e^{-st} M_\nu(t)\,dt = E_\nu(-s). \tag{2}$$

On the other hand, *Mv(z)* satisfies the following two equalities:

$$\int_0^\infty \frac{\nu}{t^{\nu+1}} M_\nu\left(\frac{1}{t^\nu}\right) e^{-st}\,dt = e^{-s^\nu}, \qquad \int_0^\infty t^\delta M_\nu(t)\,dt = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)},$$

where $\delta > -1$, $0 < \nu < 1$.

The special functions listed as above play an important role in the investigation of fractional differential equations. In the following, we give the solutions of some kinds of fractional differential equations with help of the special functions.

Lemma 1.

(See [12].*) Let* $0 < \alpha \leqslant 1$, $\lambda \in \mathbb{R}$. *The solution to the initial value problem*

$$\left({}^C D^\alpha_{0+} x\right)(t) - \lambda x(t) = f(t), \qquad x(0) = x_0 \in \mathbb{R}$$

has the form

$$x(t) = E_\alpha(\lambda t^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda(t-\tau)^\alpha\big) f(\tau)\, \mathrm{d}\tau.$$

Lemma 2.

(*See* [12].) *Let* $0 < \alpha \leqslant 1, \gamma > -\alpha, \lambda \in \mathbb{R}$. *The solution to the initial value problem*

$$\left({}^C D_{0+}^\alpha x\right)(t) - \lambda t^\gamma x(t) = 0, \qquad x(0) = x_0 \in \mathbb{R}$$

has the form

$$x(t) = x_0 E_{\alpha,1+\gamma/\alpha,\gamma/\alpha}\left(\lambda t^{\alpha+\gamma}\right),$$

where $E_{\nu,m,l}(z)$ *is defined by the following series*:

$$E_{\nu,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \nu, m, l > 0,\ z \in \mathbb{C},$$

with

$$c_0 = 1, \qquad c_k = \prod_{j=0}^{k-1} \frac{\Gamma(\nu(jm+1))}{\Gamma(\nu(jm+l+1)+1)}, \quad k = 1, 2, \ldots .$$

Lemma 3.

(*See* [10].) *Let* $0 < \alpha \le 1$*, and let* $a(t)$ *be a bounded and continuous function on* Ω*. The solution to the initial value problem*

$$({}^C D_{0+}^\alpha x)(t) - a(t)x(t) = 0, \qquad x(0) = x_0 \in \mathbb{R}$$

has the form $x(t) = \sum_{k=0}^{\infty} \mathcal{R}_a^k x_0$, *where* $\mathcal{R}_a$, *is a bounded linear operator defined on* $C(\Omega)$:

$$(\mathcal{R}_a \varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} a(\tau)\varphi(\tau)\, \mathrm{d}\tau,$$

R0 *is an identity operator, and .k denotes the k-times composition operator of…*

Lemma 4.

(See [9].) *Let* $0 < \alpha \leqslant 1,\ 0 < \beta < 1,\ \mu_1, \mu_2 \in \mathbb{R}$ *with* $\mu_2 \neq 0.$ *with μ. unique solution of the following initial value problem*

$$(\mathcal{R}_a\varphi)(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-\tau)^{-\alpha} a(\tau)\varphi(\tau)\,\mathrm{d}\tau,$$

has the form

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} (-1)^k \mu^k t^{(\alpha+\beta)k} E^{k+1}_{\alpha,(\alpha+\beta)k+1}\bigl(-\mu_1 t^\alpha\bigr) x_0 \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \mu^k \int\limits_0^t t^{(\alpha+\beta)k+\alpha-1} E^{k+1}_{\alpha,(\alpha+\beta)k+\alpha}\bigl(-\mu_1 (t-\tau)^\alpha\bigr) f(\tau)\,\mathrm{d}\tau, \end{aligned}$$

where $\mu = \mu_2\Gamma(\beta).$.

Next, we list some properties about the special functions and two integral inequalities,

which will be used in the latter discussion.

Lemma 5.

(See [11].) *Let* $\rho, \mu, \gamma, \nu, \sigma > 0$ *, and let* $t > 0$*, then*

$$\int\limits_0^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}\bigl(\lambda(t-\tau)^\rho\bigr) \tau^{\nu-1} E^{\sigma}_{\rho,\nu}\bigl(\lambda\tau^\rho\bigr)\,\mathrm{d}\tau = t^{\mu+\nu-1} E^{\gamma+\sigma}_{\rho,\mu+\nu}\bigl(\lambda t^\rho\bigr).$$

In particular

$$\int\limits_0^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}\bigl(\lambda(t-\tau)^\rho\bigr) \tau^{\nu-1} E_{\rho,\nu}\bigl(\lambda\tau^\rho\bigr)\,\mathrm{d}\tau = t^{\mu+\nu-1} E^{\gamma+1}_{\rho,\mu+\nu}\bigl(\lambda t^\rho\bigr),$$

$$\int\limits_0^t (t-\tau)^{\mu-1} E^{\gamma}_{\rho,\mu}\bigl(\lambda(t-\tau)^\rho\bigr) E_{\rho}\bigl(\lambda\tau^\rho\bigr)\,\mathrm{d}\tau = t^{\mu} E^{\gamma+1}_{\rho,\mu+1}\bigl(\lambda t^\rho\bigr).$$

Lemma 6.

(*See* [31].) *Let* $0 < \alpha \leq 1$, $a(t)$ *and* $l(t)$ *be continuous, nonnegative functions on* Ω, *and* $u(t)$ *be a continuous, nonnegative function on* Ω *with*

$$u(t) \leqslant a(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} l(s) u(s)\, \mathrm{d}s.$$

Then it has

$$u(t) \leqslant \left[A(t) + \int_0^t L(s) A(s) \exp\left(\int_s^t L(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s \right]^{\beta}, \quad t \in \Omega.$$

If $a(t)$ *is nondecreasing on* Ω, *then the inequality is reduced to*

$$u(t) \leqslant \left[A(t) + \int_0^t L(s) A(s) \exp\left(\int_s^t L(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s \right]^{\beta}, \quad t \in \Omega.$$

If $a(t) \equiv 0$ *on* Ω, *then* $u(t) \equiv 0$, *where* $0 < \beta < \alpha \leq 1$, *and*

$$A(t) = 2^{1/\beta - 1} a^{1/\beta}(t),$$
$$L(t) = \frac{2^{1/\beta-1}}{(\Gamma(\alpha))^{1/\beta}} \left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right) \Gamma\left(\frac{1-\alpha}{1-\beta}\right) \right]^{(1-\beta)/\beta} t^{(\alpha-\beta)/\beta} l^{1/\beta}(t).$$

Lemma 7.

(*See* [8].) *Suppose* $\beta > 0$, $a(t)$ *and* $u(t)$ *are nonnegative, locally integrable functions on* Ω, $g(t)$ *is a nonnegative, nondecreasing continuous function on* Ω, *and they satisfy the following relationship*:

$$u(t) \leqslant a(t) + g(t) \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(\lambda(t-\tau)^{\beta}) u(\tau)\, \mathrm{d}\tau, \quad \lambda > 0,\ t \in \Omega. \tag{3}$$

Then, for any $t \in \Omega$, *we have*

$$u(t) \leqslant a(t) + \sum_{k=1}^{\infty} g^k(t) \int_0^t (t-\tau)^{k\beta-1} E^k_{\beta,k\beta}(\lambda(t-\tau)^{\beta}) a(\tau)\, \mathrm{d}\tau.$$

Finally, we introduce some concepts of stability in the case of a fractional-order sys- tem with randomly time-varying parameters. Stability studies about differential systems are essentially problems of convergence. For the system with randomly time-varying parameters, it is only possible to investigate the convergence in some stochastic sense such as convergence in mean sense, convergence in probability, or convergence with probability one. In this paper, we will mainly consider the stability of system in mean sense.

Definition 5.

A constant vector $x^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}} \in \mathbb{R}^n$ is an equilibrium solution of system (1) if and only if $f(t, \omega(t), x^*) = 0.$.

Remark 1.

For convenience, we state all definitions and theorems for the case when the equilibrium solution of system (1) is the origin of $\mathbb{R}^n$, i.e., $x^* = 0.$ There is no loss of generality in doing so because any equilibrium solution can be changed to the origin via a change of variables. Suppose the equilibrium solution for system (1) is x* = 0 and consider the change of variable y = *x* - *x**. Then system (1) with respect to the new variable *y* is

$$\begin{aligned}\left({}^{C}D_{0+}^{\alpha}y\right)(t) &= \left({}^{C}D_{0+}^{\alpha}(x-x^*)\right)(t) = f\left(t,\omega(t),x(t)\right)\\ &= f\left(t,\omega(t),y(t)+x^*\right) = g\left(t,\omega(t),y(t)\right),\end{aligned} \tag{4}$$

where *g(t, ω(t), 0) = 0* , and the new system (4) has equilibrium solution at the origin.

Definition 6.

The equilibrium solution *x(t) = 0* of system (1) is said to be stable in mean sense if for any # > 0, there exists a $\delta(\epsilon) > 0$ such that for any initial condition satisfying $\|x_0\| < \delta(\epsilon).$, the expected value of the norm of the solution *x(t)* satisfies $E(\|x(t)\|) < \epsilon$ for any *t* ≥ *0* .

Definition 7.

The equilibrium solution *x(t) = 0* of system (1) is said to be asymptotically stable in mean sense if, in addition, to being stable in mean sense, it is true that for each x_0, there exists a *δ* > 0 such that $\lim_{t\to\infty} E(\|x(t)\|) = 0$ whenever $\|x_0\| < \delta.$.

Definition 8.

The equilibrium solution $x(t) = 0$ of system (1) is said to be Mittag-Leffler stable in mean sense if for each x_0 , the expected value of the norm of the solution satisfies

$$\mathbf{E}\|x(t)\| \leqslant \|x_0\| E_\alpha(-\lambda t^\alpha),$$

where $0 < \alpha \leq 1$, and $\lambda > 0$.

Definition 9.

If the equilibrium solution $x(t) = 0$ of system (1) is said to be stable (asymptotically stable, Mittag-Leffler stable) in mean sense, then we also call system (1) stable (asymptotically stable, Mittag-Leffler stable) in mean sense.

3 Main results

3.1 Stability analysis based on generalized Lyapunov method

In this subsection, we will use the generalized Lyapunov functional method to analyze the stability of system (1). The method is to construct a scalar functional $V(t, x)$, which is continuous in both t and x , has first partial derivatives in these variables and equals zero only at the equilibrium solution $x(t) = 0$. Such a functional is called a Lyapunov functional. The basic idea of the Lyapunov method is that if we can construct a Lyapunov functional, which represents some tubes surrounding the equilibrium solution $x(t) = 0$ such that all solutions cross through the tubes towards $x(t) = 0$.

For any solution x, the a-order Caputo derivative of the scaler functional . is calcu- lated as

$$^{C}D^{\alpha}_{0+}V(t,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\dot{V}(\tau, x(\tau))\,\mathrm{d}\tau,$$

where

$$\dot{V}(t, x(t)) = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Theorem 1.

Let 0 < α ≤ 1*, and let* $V(t,x(t)):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ *be a continuously differential function and locally Lipschitz continuous with respect to x and satisfy the following conditions*:

(i) V (*t,* 0) = 0;

(ii) V (t, x) ≥ a#x#;

(iii)$\mathbf{E}(^CD^\alpha_{0+}V(t,x))\leqslant 0.$

Then the equilibrium solution x(.) = 0 *of system* (1) *is stable in mean sense.*

Proof. Note that V (*t, x*(*t*)) can be written as

$$V\big(t,x(t)\big)=V_0+\frac{1}{\Gamma(\alpha)}\int\limits_0^t\frac{^CD^\alpha_{0+}V\big(\tau,x(\tau)\big)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \tag{5}$$

where *V₀ = V (0, x₀)* only depends on the initial state ...

Taking the expected value on the both sides of equality (5) leads to

$$\mathbf{E}V\big(t,x(t)\big)=V_0+\frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^t(t-\tau)^{\alpha-1}\mathbf{E}\big(^CD^\alpha_{0+}V(\tau,x)\big)\,\mathrm{d}\tau.$$

Furthermore, combining condition (iii) and Property 1, we have

$$\mathbf{E}V\big(t,x(t)\big)\leqslant V_0.$$

On the other hand, from condition (ii) we get

$$a\mathbf{E}\big\|x(t)\big\|\leqslant V_0.$$

Since V (*t, x*) is locally Lipschitz continuous with respect to ., there exists a constant . such that for some *γ* > 0 and 0 < #x# ≤ y, it has V (*t, x*) ≤ L#x#. Then, for any # > 0, if $\delta(\epsilon)$ is chosen as$\delta=\min(a\epsilon/L,\gamma),$ then, for it$\|x_0\|<\delta(\epsilon),$ has

$$a\mathbf{E}\big\|x(t)\big\|\leqslant V_0\leqslant L\|x_0\|\leqslant a\epsilon.$$

This implies that the equilibrium solution x(t) = 0 of system (1) is stable in mean sense. The proof is completed.

According to Theorem 1 and the definition of Caputo fractional derivative, we can immediately obtain the following corollary.

Corollary 1.

Let $V(t,x(t)):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ *be a continuously differential function and locally Lipschitz continuous with respect to x and satisfy the conditions*

(i) *V (t, 0) = 0;*

(ii) V (t, x) ≥ a#x#;

(iii) *E(dV (t, x(t))/dt) ≤ 0.*

Then the equilibrium solution x(.) = 0 *of system* (1) *is stable in mean sense.*

Next, we will discuss the asymptotically stability of system (1). To do this, we state a lemma.

Lemma 8.

(*See* [1].) *If* g(r) *is an always increasing function defined for all r* ≥ 0 *and* g(0) = 0, *then if* E{r} ≥ *M > 0, there exists an L*(M) *> 0 such that* E{g(r)} ≥ *L.*

Theorem 2.

Under the hypothesis of Theorem ., further assume that V (t, x) has the property that $\mathbf{E}({}^{C}D_{0+}^{\alpha}V(t,x))$ *is negative definite, i.e.,*

$$\mathbf{E}\left({}^{C}D_{0+}^{\alpha}V(t,x)\right)\leqslant -h\left(\|x\|\right), \tag{6}$$

where h(0) = 0, *and h*(||x||) *is an increasing function. Then the equilibrium solution x*(t) = 0 *of system* (1) *is asymptotically stable in mean sense.*

Proof. We will prove it by contradiction. Assume that $\mathbf{E}\{\|x(t)\|\}$ does not tend to zero as $t\to\infty$. Then it is possible to find a *δ* > 0 such that $\mathbf{E}\{\|x(t)\|\}>\delta$ for any $t \geq t_0$. It follows from Lemma 8 that there exists a .(.) such that

$$\mathbf{E}\left\{h\left(\|x\|\right)\right\}\geqslant\kappa(\delta). \tag{7}$$

On the other hand, from inequalities (6), (7) and Property 1 we get

$$\mathbf{E}\{V(t,x(t))\} \leqslant V_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathbf{E}\{h(\|x(\tau)\|)\}\,\mathrm{d}\tau$$
$$\leqslant V_0 - \frac{\kappa(\delta)t^\alpha}{\Gamma(\alpha+1)}.$$

This implies that if we choose t such that $t^\alpha > \Gamma(\alpha+1)V_0/\kappa(\delta)$, then it has $\mathbf{E}\{V(t,\dot{x}(t))\} < 0$, which leads to a contradiction with condition (ii). It follows that $\mathbf{E}\{\|x(t)\|\}$ tends to zero as $t \to \infty$. The proof is completed.

Corollary 2.

Under the hypothesis of Theorem ., further assume that V (t, x) has the property that E(dV (t, x(t))/dt) is negative definite, i.e.

$$\mathbf{E}\left(\frac{\mathrm{d}V(t,x(t))}{\mathrm{d}t}\right) \leqslant -h(\|x\|),$$

where $h(0) = 0$*, and* $h(||x||)$ *is an increasing function. Then the equilibrium solution* $x(t) = 0$ *of system* (1) *is asymptotically stable in mean sense.*

Finally, we will consider Mittag-Leffler stability of system (1) in mean sense.

Theorem 3.

Under the hypothesis of Theorem ., further assume that there exist positive constants a., p, q (i = 1, 2, 3) and β (0 ≤ β < 1) such that **V (t, x)** *has the properties*:

$$\textbf{(iv)}\ \ \|x\|^q \leqslant V(t,x) \leqslant a_1\|x\|^p + a_2 I_{0+}^{\beta}\|x\|^q;$$
$$\textbf{(v)}\ \ \mathbf{E}({}^C D_{0+}^{\alpha} V(t,x)) \leqslant -a_3 \mathbf{E}\|x\|^p.$$

Then the equilibrium solution x (t) = 0 of system (1) *is Mittag-Leffler stable in mean sense.*

Proof. From the second inequality in condition (iv) we have

$$-a_3\|x\|^p \leqslant -\frac{a_3}{a_1}V(t,x) + \frac{a_2 a_3}{a_1} I_{0+}^{\beta}\|x\|^q. \quad ;$$

Also, from condition (v) we get

$$\mathbf{E}\big({}^{C}D_{0+}^{\alpha}V(t,x)\big) \leqslant -\frac{a_3}{a_1}\mathbf{E}V(t,x) + \frac{a_2 a_3}{a_1}\mathbf{E}\big(I_{0+}^{\beta}\|x\|^{q}\big). \tag{8}$$

On the other hand, from the first inequality in condition (iv) we have

$$\frac{a_2 a_3}{a_1} I_{0+}^{\beta}\|x\|^{q} \leqslant \frac{a_2 a_3}{a_1} I_{0+}^{\beta}V(t,x). \tag{9}$$

Then, combining inequalities (8) and (9), we can obtain

$$\mathbf{E}\big({}^{C}D_{0+}^{\alpha}V(t,x)\big) \leqslant -\frac{a_3}{a_1}\mathbf{E}V(t,x) + \frac{a_2 a_3}{a_1}\mathbf{E}\big(I_{0+}^{\beta}V(t,x)\big).$$

According to the well-known comparison principle in [17] and Lemma 4, we have

$$\mathbf{E}V(t,x) \leqslant E_{\alpha}\left(-\frac{a_2 a_3 \Gamma(\beta)}{a_1}t^{\alpha+\beta} + \frac{a_3}{a_1}t^{\alpha}\right).$$

The proof is completed.

Remark 2.

For the case $\alpha = 1$, Theorems 1 and 2 become Theorems 3.1 and 3.2 in [1], respectively. Theorem 3 improves and extends Theorems 5.1 and 5.4 in [19].

Example 1. Consider the stability of the following linear fractional-order system:

$$\big({}^{C}D_{0+}^{\alpha}x\big)(t) = A(t)x(t), \quad t \geqslant 0, \qquad x(0) = x_0. \tag{10}$$

Construct a Lyapunov functional as

$$V(t,x) = x^{\mathrm{T}}(t)Qx(t),$$

where Q is a constant positive definite matrix, and .. denotes the transpose of *x*.

According to Corollary 1, a sufficient condition for stability of system (10) is

$$\mathbf{E}\left(\frac{\mathrm{d}V(t,x)}{\mathrm{d}t}\right) < 0, \quad t \geqslant 0.$$

That is to say, a sufficient condition for stability of system (10) is that

$$\mathbf{E}\left(\frac{\mathrm{d}V(t,x)}{\mathrm{d}t}\right) = \mathbf{E}\big(x^{\mathrm{T}}(t)\big(A^{\mathrm{T}}(t)Q + QA(t)\big)x(t)\big) \tag{11}$$

be negative definite for all *t* ≥ *0.*

Now we consider a particular case when *A(t)* is a diagonal matrix, i.e.,

$$A(t) = \begin{bmatrix} a_1(t) & 0 & \cdots & 0 \\ 0 & a_2(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_n(t) \end{bmatrix},$$

where *a.(t) (i = 1, 2, . . . , n)* are bounded continuous functions on $[0,\infty)$ or have the form as *t.* with γ > 0. Then, according to Lemmas 2 and 3, system (10) has a unique solution, and the unique solution can be expressed in a closed form. For convenience, we denote the unique solution as

$$x(t) = \Phi(t)x_0. \tag{12}$$

Substituting (12) into (11), we obtain a sufficient condition for the stability in mean sense is that

$$\mathbf{E}\big(\Phi^{\mathrm{T}}(t)(A^{\mathrm{T}}(t)Q + QA(t))\Phi(t)\big)$$

be negative definite for all *t* ≥ *0* . Let . be an identity matrix. Then it is stable in mean sense if *ai(t)* < *0 (i = 1, 2, . . . , n)* for all *t* ≥ *0* .

In the following, we demonstrate numerical simulation. For example, we take a = 0.5, and

$$A(t) = \begin{bmatrix} -3t & 0 & 0 \\ 0 & -5t^2 & 0 \\ 0 & 0 & -t^3 \end{bmatrix}.$$

Then, according to Theorem 1, the equilibrium solution $x(t) = 0$ is stable. Figure 1 is the numerical result. From Fig. 1 one sees that the numerical results agree with the theory analysis.

Remark 3.

We obtained the stability criteria by the Lyapunov functional approach in this subsection. However, by Example 1 one can find that it is only in some special cases that a Lyapunouv function can be constructed. Zhou et al. in [30] studied the exponential stability for delayed neutral networks driven by fractional Brownian noise using some mathematical techniques and Gronwall inequality. It motivates us investigate the stability of system (1) by mathematical techniques and generalized Gronwall integral inequalities in the next subsection.

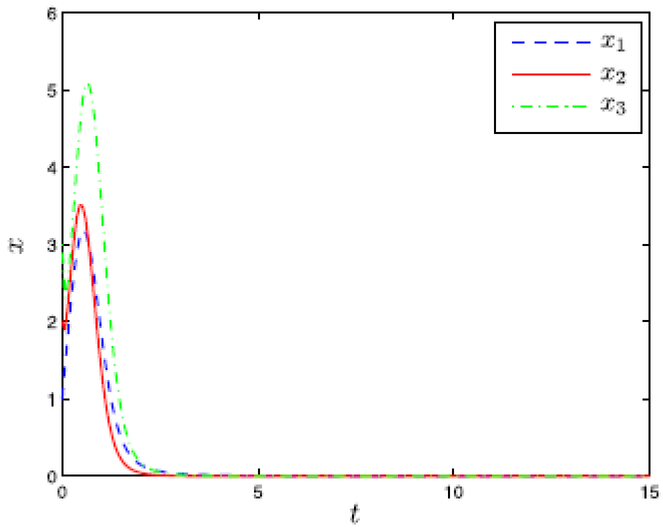

Figure 1.
Transient behavior of system (10).

3.2 Stability analysis based on integral inequalities

In this subsection, we firstly prove the boundness of the solution of system (1) by using some generalized Gronwall inequalities and then provide another alternative approach to verify the stability of system (1) in mean sense.

Lemma 9.

Let the function $f(t, \omega(t), x)$ *in system* (1) *satisfies the relation.*

$$\mathbf{E}\big\|f\big(t,\omega(t),x\big)-f\big(t,\omega(t),y\big)\big\| \leqslant L\big(t,\ \mathbf{E}\|x-y\|\big), \tag{13}$$

and let L verifies the condition

$$0 \leqslant L(t,u)-L(t,v) \leqslant M(t,v)(u-v), \quad u \geqslant v \geqslant 0, \tag{14}$$

where $M:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ *is a nonnegative continuous function. Then we have the estimate*

$$\mathbf{E}\|x(t)-x_0\| \leqslant \left[A(t)+\int_0^t \Phi(s)A(s)\exp\left(\int_s^t \Phi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{\beta}, \quad t \geqslant 0, \tag{15}$$

where 0 < β < α < 1, and

$$A(t)=2^{1/\beta-1}\left(\frac{\|x_0\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}M(\tau,0)\,\mathrm{d}\tau\right)^{1/\beta},$$

$$\Phi(t)=\frac{2^{1/\beta-1}}{(\Gamma(\alpha))^{1/\beta}}\left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{(1-\beta)/\beta} t^{(\alpha-\beta)/\beta}\big(M(t,\|x_0\|)\big)^{1/\beta}.$$

Proof. Let *x(b)* be the solution of system (1). Then we have

$$x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f\big(\tau,\omega(\tau),x(\tau)\big)\,\mathrm{d}\tau.$$

Let us define the mapping $y(t):[0,\infty)\to\mathbb{R}^n$. as

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f\big(\tau,\omega(\tau),x(\tau)\big)\,\mathrm{d}\tau.$$

Then it has .(.) = .. + .(.). Noting the fact . (*t, ω(t)*, 0) ≡ 0, we get

$$\mathbf{E}\big\|f\big(t,\omega(t),x\big)\big\| \leqslant L\big(t,\mathbf{E}\|x\|\big).$$

On the other hand, from (14) we have $L(t,0)\equiv 0$ and $L(t,u)\leqslant M(t,0)u$. Then by (13) and (14) we have

$$\begin{aligned}\mathbf{E}\|y(t)\| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\mathbf{E}\|f(\tau,\omega(\tau),x(\tau))\|\,\mathrm{d}\tau \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}L(\tau,\mathbf{E}\{\|x_0\|+\|y(\tau)\|\})\,\mathrm{d}\tau \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}L(\tau,\|x_0\|)\,\mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}M(\tau,\|x_0\|)\mathbf{E}\|y(\tau)\|\,\mathrm{d}\tau \\ &\leqslant \frac{\|x_0\|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}M(\tau,0)\,\mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}M(\tau,\|x_0\|)\mathbf{E}\|y(\tau)\|\,\mathrm{d}\tau.\end{aligned}$$

According to Lemma 4, we can obtain estimate (15). The proof is completed.

By using Lemma 9 we can formulate the following lemma, which tells us that the solution of system (1) is bounded under suitable conditions.

Lemma 10.

If the function f satisfies the condition in Lemma 9 and also there exist two positive constants **δ0** and **M** such that

$$\int_0^\infty s^{(\alpha-\beta)/\beta}\left(M(s,\delta)\right)^{1/\beta}\,\mathrm{d}s \leqslant \overline{M}$$

for all **0 ≤ δ ≤ δ0**, then there exists an **M > 0** such that for all **t ≥ 0**,

$$\mathbf{E}\|x(t)-x_0\| \leqslant \widetilde{M}.$$

Proof. With the help of Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t (t-\tau)^{\alpha-1} M(\tau,0)\,\mathrm{d}\tau \\
&\quad = \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-\alpha}\tau^{\alpha-\beta} M(\tau,0)\,\mathrm{d}\tau \\
&\quad \leqslant \left[\int_0^t \left((t-\tau)^{\alpha-1}\tau^{\beta-\alpha}\right)^{1/(1-\beta)} \mathrm{d}\tau\right]^{1-\beta} \left[\int_0^t \left(\tau^{\alpha-\beta} M(\tau,0)\right)^{1/\beta} \mathrm{d}\tau\right]^{\beta} \\
&\quad = \left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{1-\beta} \left[\int_0^t \left(\tau^{\alpha-\beta} M(\tau,0)\right)^{1/\beta} \mathrm{d}\tau\right]^{\beta}.
\end{aligned}$$

This implies that

$$\lim_{t\to\infty} \int_0^t (t-\tau)^{\alpha-1} M(\tau,0)\,\mathrm{d}\tau \leqslant \overline{M}^{\beta} \left[\Gamma\left(\frac{\alpha-\beta}{1-\beta}\right)\Gamma\left(\frac{1-\alpha}{1-\beta}\right)\right]^{1-\beta}.$$

It follows from Lemma 9 that the solution $x(t)$ is bounded for all $t \geq 0$, and the bound- edness is denoted as M . The proof is completed.

Now we can give the following theorem of stability for the equilibrium solution of system

Theorem 4.

If the function f satisfies the condition in Lemma . and also there exist two positive constants δ. and M such that

$$\int_0^\infty s^{(\alpha-\beta)/\beta} \left(M(s,\delta)\right)^{1/\beta} \mathrm{d}s \leqslant \overline{M}$$

for all $0 \leqslant \delta \leqslant \delta_0$, *, then the equilibrium solution* $x \quad (t) = 0$ *of system* (1) *is stable in mean sense.*

Proof. Let $\varepsilon > 0$ and $x(t)$ be the solution of system (1). We choose . such that

$$\delta(\varepsilon) = \min\left\{\frac{\varepsilon}{2}, \delta_0, \frac{\varepsilon}{2\widetilde{M}}\right\},$$

where M is the constant boundedness in Lemma 10. Then, for *#x₀#* < *δ(ε),* we have

$$\mathbf{E}\big\|x(t)\big\| \leqslant \mathbf{E}\big\{\|x_0\| + \|x - x_0\|\big\} \leqslant \frac{\varepsilon}{2} + \|x_0\|\widetilde{M} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that the equilibrium solution $x(t) = 0$ of system (1) is stable in mean sense. The proof is completed.

Finally, we will discuss the Mittag-Leffler stability of system (1) using integral in- equality.

Theorem 5.

Assume that system (1) can be decomposed into

$$({}^{C}D^{\alpha}_{0+}x)(t) = Ax(t) + g(t,\omega(t),x(t)), \quad t \geqslant 0, \qquad x(0) = x_0. \tag{16}$$

Also, assume that

(i) There exist two constants λ, K > 0 such that $\|e^{At}\| \leqslant Ke^{-\lambda t}, t \geqslant 0;$

(ii) For any $t \geq 0$, it has $g(t, \omega(t), 0) = 0$, and there exists a constant $L > 0$ with $L < \lambda$ such that for $x, y \in \mathbb{R}^n$, it has

$$\mathbf{E}\big\|g(t,\omega(t),x) - g(t,\omega(t),y)\big\| \leqslant L\mathbf{E}\|x-y\|. \, , \tag{17}$$

Then the equilibrium solution $x(t) = 0$ of system (1) is Mittag-Leffler stable in mean sense.

Proof. Combining the fact $g(t, \omega(t), 0) \# 0$ and inequality (17), we have

$$\mathbf{E}\big\|g(t,\omega(t),x)\big\| \leqslant L\mathbf{E}\|x\|.$$

Let $x(t)$ be the solution of system (16). Then by Lemma 1 we have

$$x(t) = E_\alpha(At^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-\tau)^\alpha\big)g\big(\tau,\omega(\tau),x(\tau)\big)\,\mathrm{d}\tau. \tag{18}$$

According to equality (2), we have the estimate

$$\|E_\alpha(At^\alpha)\| \leqslant \int_0^\infty \|\mathrm{e}^{At^\alpha s}\| M_\alpha(s)\,\mathrm{d}s \leqslant \int_0^\infty K\mathrm{e}^{-\lambda t^\alpha s} M_\alpha(s)\,\mathrm{d}s$$
$$= K\sum_{n=0}^\infty \frac{(-\lambda)^n t^{n\alpha}}{n!}\int_0^\infty s^n M_\alpha(s)\,\mathrm{d}s$$
$$= K\sum_{n=0}^\infty \frac{(-\lambda)^n t^{n\alpha}}{n!}\frac{\Gamma(n+1)}{\Gamma(n\alpha+1)} = KE_\alpha(-\lambda t^\alpha). \tag{19}$$

Using the similar arguments to (19), we can deduce that

$$\|E_{\alpha,\alpha}(At^\alpha)\| \leqslant KE_{\alpha,\alpha}(-\lambda t^\alpha).$$

Then, taking the expectation on the both sides of inequality (19), we get

$$\mathbf{E}\|x(t)\| \leqslant KE_\alpha(-\lambda t^\alpha)\|x_0\| + LK\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)\mathbf{E}\|x(\tau)\|\,\mathrm{d}\tau. \tag{20}$$

Applying Lemma 6 to inequality (20), we have

$$\mathbf{E}\|x(t)\| \leqslant K^2\sum_{k=0}^\infty L^k t^{k\alpha}E^{k+1}_{\alpha,k\alpha+1}(-\lambda t^\alpha)\|x_0\|.$$

In fact, it has

$$\sum_{k=0}^\infty L^k t^{k\alpha}E^{k+1}_{\alpha,k\alpha+1}(-\lambda t^\alpha) = \sum_{k=0}^\infty\sum_{n=0}^\infty L^k t^{k\alpha}\frac{(k+1)_n(-\lambda)^n t^{n\alpha}}{\Gamma(n\alpha+k\alpha+1)n!} = \sum_{k=0}^\infty\sum_{m=k}^\infty \frac{L^k(-\lambda)^{m-k}t^{m\alpha}m!}{\Gamma(m\alpha+1)k!(m-k)!}$$
$$= \sum_{m=0}^\infty \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\sum_{k=0}^m \frac{m!L^k(-\lambda)^{m-k}}{k!(m-k)!} = \sum_{m=0}^\infty \frac{(L-\lambda)^m t^{m\alpha}}{\Gamma(m\alpha+1)} = E_\alpha((L-\lambda)t^\alpha).$$

It follows that *E#x(t)# ≤ K2Eα((L − λ)tα)#x0#* . The proof is completed.

Example 2. We consider the stability of system (10) in Example 1. Obviously, it has *L(t, u) = #A(t)#u. If #A(t)#* satisfies the condition

$$\int_0^\infty s^{\alpha/\beta-1}\|A(s)\|^{1/\beta}\,\mathrm{d}s \leqslant \overline{M},$$

where M is a positive constant, then by Theorem 4 system (10) is stable in mean sense.

For the particular case when *#A(t)# ≤ e−λt (λ > 0)*, it has

$$\int_0^\infty s^{\alpha/\beta-1}\|A(s)\|^{1/\beta}\,\mathrm{d}s \leqslant \Gamma\left(\frac{\alpha}{\beta}\right)\left(\frac{\lambda}{\beta}\right)^{\alpha/\beta}.$$

Then by Theorem 5 system (10) is Mittag-Leffler stable in mean sense.

For example, we take $\alpha = 0.5$, and

$$A(t) = \begin{bmatrix} \mathrm{e}^{-5t} & -\mathrm{e}^{-5t} & -\mathrm{e}^{-3t} \\ -\mathrm{e}^{-3t} & \mathrm{e}^{-3t} & -\mathrm{e}^{-t} \\ \mathrm{e}^{-2t} & -\mathrm{e}^{-t} & -\mathrm{e}^{-3t} \end{bmatrix}.$$

Then, according to Theorem 5, the equilibrium solution $x(t) = 0$ is stable. Figure 2 is the numerical result. From Fig. 2 one sees that the numerical results agree with the theory analysis.

Example 3. Consider the following fractional stochastic differential equation:

$$\mathrm{d}\mathcal{I}_{0+}^{1-\alpha}\big(x(t) - x(0)\big) = A(t)x(t)\,\mathrm{d}t + \mu x(t)\,\mathrm{d}B(t), \quad t > 0,$$
$$x(0) = x_0, \tag{21}$$

where $B(t)$ is the standard Brownian motion.

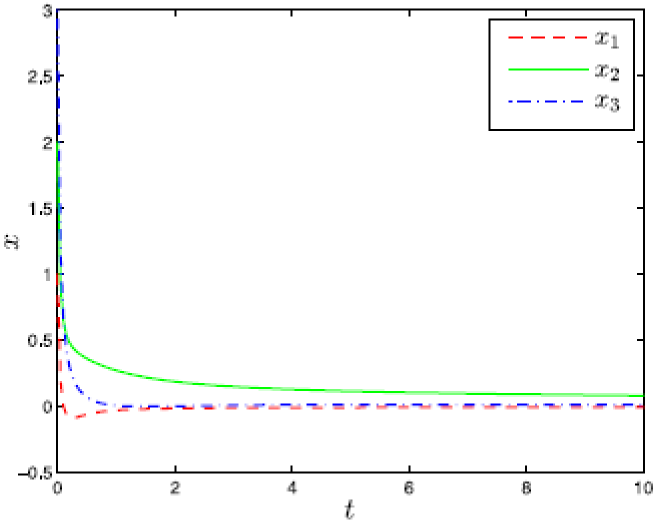

Figure 2.
Transient behavior of system (10) in the case $\alpha = 0.5$.

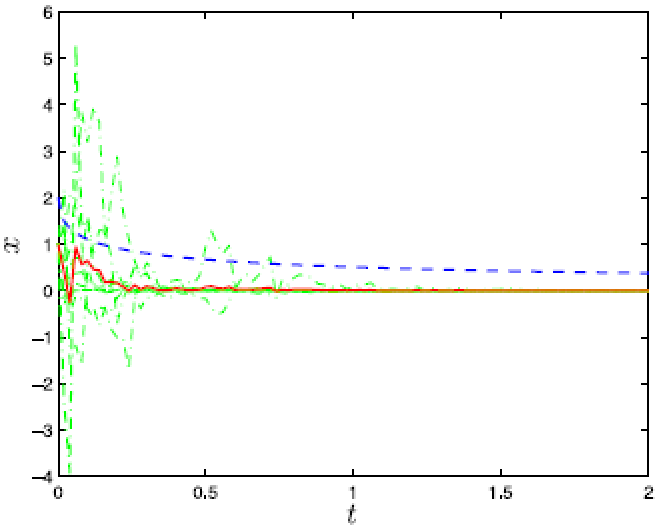

Figure 3.
Transient behavior of system (21) in the case $\mu = 1$, where the green lines (- ·) denote ten sample paths x(t), the red line (–) denotes the mean value of x, and the blue line (- -) denotes the Mittag-Leffer function $x0E\alpha(-\lambda t\alpha)$.

Take $\alpha = 0.5, A(t) = 5t,$ and $x0 = 1$. Then it has $L = 0$ and $\lambda = 1$. Therefore, according to Theorem 5, the equilibrium solution $x(t) = 0$ is Mittag-Leffler stable in mean sense. Figure 3 is the numerical result, and from it one sees that the numerical result agrees with the theory analysis.

4 Conclusions

In this paper, we provided two approaches to assess the stability of fractional-order sys- tems with randomly time-varying parameters. The first approach is based on the gener- alized Lyapunov functionals. We can construct suitable Lyapunov functionals satisfying some conditions to discuss the stability of such systems. This approach has important theoretical significance, although the finding of suitable Lyapunov functional is difficult in applications. The second approach is based on integral inequalities and ingenious mathematical reduction. Several examples show that the derived results are effective and reliable to check the stability. Compared with the first approach, the second approach is a convenient way to handle the stability of the fractional-order systems with randomly time-varying parameters. Besides that, the derived criteria improve the existing related results. We believe these results with weak conditions are useful for the analysis of the stability of fractional differential equations in the future.

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