



Nonlinear Analysis: Modelling and Control
ISSN: 1392-5113
ISSN: 2335-8963
nonlinear@mii.vu.lt
Vilniaus Universitetas
Lituania

Feng–Liu-type fixed point result in orbitalb-metricspaces and application to fractal integral equation

Hemant Kumar, Nashine; Lakshmi Kanta, Dey; Rabha, W. Ibrahim; Stojan, Radenović

Feng–Liu-type fixed point result in orbitalb-metricspaces and application to fractal integral equation

Nonlinear Analysis: Modelling and Control, vol. 26, núm. 3, 2021

Vilniaus Universitetas, Lituania

Disponible en: <https://www.redalyc.org/articulo.oa?id=694172973008>

DOI: <https://doi.org/10.15388/namc.2021.26.22497>




Esta obra está bajo una Licencia Creative Commons Atribución 4.0 Internacional.

Feng–Liu-type fixed point result in orbital b-metric spaces and application to fractal integral equation


Nashine Hemant Kumar hemantkumarnashine@tdtu.edu.vn

Ton Duc Thang University, Vietnam

 <https://orcid.org/0000-0002-0250-9172>


Dey Lakshmi Kanta lakshmikdey@yahoo.co.in

National Institute of Technology Durgapur, India

 <https://orcid.org/0000-0001-5389-6048>


W. Ibrahim Rabha rabhaibrahim@yahoo.com

Ajman University, Emiratos Árabes Unidos

 <https://orcid.org/0000-0001-9341-025X>

Radenović Stojan radens@beotel.net

University of Belgrade, Serbia

 <https://orcid.org/0000-0001-8254-6688>

Nonlinear Analysis: Modelling and Control, vol. 26, núm. 3, 2021

Vilniaus Universitetas, Lituania

Recepción: 07 Marzo 2020

Revisado: 13 Septiembre 2020

Publicación: 01 Mayo 2021

DOI: <https://doi.org/10.15388/namc.2021.26.22497>

Redalyc: <https://www.redalyc.org/articulo.oa?id=694172973008>

Abstract: In this manuscript, we establish two Wardowski–Feng–Liu-type fixed point theorems for orbitally lower semicontinuous functions defined in orbitally complete b -metric spaces. The obtained results generalize and improve several existing theorems in the literature. Moreover, the findings are justified by suitable nontrivial examples. Further, we also discuss ordered version of the obtained results. Finally, an application is presented by using the concept of fractal involving a certain kind of fractal integral equations. An illustrative example is presented to substantiate the applicability of the obtained result in reducing the energy of an antenna.

Keywords: b -metric space, orbitally lower semicontinuous, F -contraction, fixed point of a multivalued mapping.

1 Introduction

Through the whole of the last century, mathematicians engrossed themselves with touching up the underlying metric framework of the acclaimed Banach contraction principle. In their research articles, Bakhtin [3] and afterwards Czerwik [6–8] put forward another natural and impressive setting as an extension of metric spaces, b -metric spaces, to work with. Since then, plenty of fixed point, common fixed point and related results involving various classes of single-valued and multi-valued operators defined in such kind of spaces or in allied structures are published (see [1, 2, 5, 10, 11, 13–18, 20]). Here we recall some basic definitions, notations and essential results, which will play crucial roles in this manuscript. Throughout this article, \mathbb{R}^+ and $\mathbb{R}^+ / 0$ denote the set of all positive, respectively, nonnegative real numbers, and \mathbb{N} stands for the

set of positive integers. First of all, we recollect the definition of b -metric spaces.

Definition 1.

A b -metric on a nonempty set X is a function $d_b : X \times X \rightarrow \mathbb{R}_0^+$ such that for a constant $b \geq 1$ and all $x, y, z \in X$, the following conditions hold:

(M1) $d_b(x, y) = 0$ if and only if $x = y$;

(M2) $d_b(x, y) = d_b(y, x)$;

(M3) $d_b(x, y) \leq b(d_b(x, z) + d_b(z, y))$.

Then (X, d_b) is called a b -metric space.

One may note that each metric space is a b -metric space considering $b = 1$. However, the converse does not hold. Furthermore, the topology on a b -metric space, the concept of Cauchy sequences, convergent sequences and the completeness of the setting are analogous to that of standard metric spaces. However, in general, a b -metric is not a continuous mapping in both variables (see, e.g., [13]).

On the other hand, Cosentino et al. in [5] introduced the set of functions TF in the line of Wardowski's [19] approach to b -metric space as follows:

Definition 2.

Let $b \geq 1$ be a real number. TF denotes the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ with the following properties:

(F1) F is strictly increasing;

(F2) For each sequence $\{a_n\}$ of positive numbers, $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(a_n) = -\infty$;

(F3) For each sequence $\{a_n\}$ of positive numbers with $\lim_{n \rightarrow \infty} a_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} a_n^k F(a_n) = 0$;

(F4) There exists $\tau \in \mathbb{R}^+$ such that for each sequence $\{a_n\}$ of positive numbers, if $\tau + F(ba_n) \leq F(a_{n-1})$ for all $n \in \mathbb{N}$.

Definition 3.

Let $b \geq 1$ be a real number. \mathfrak{TF}_b denotes the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ having the properties (F1)–(F4) and the following property:

(F5) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

It is easy to see that the function $F(X) = \ln x$ or $F(x) = x + \ln x$ satisfies properties (F1)–(F5) for $x > 0$.

We recall the following result from [5]. In this result, \mathcal{H}_b denotes the b -Hausdorff–Pompeiu metric.

Theorem 1.

(See [5, Thm. 3.4].) Let (X, d_b) be a complete b -metric space, and let $T : X \rightarrow P_{cb}(X)$. Assume that there exists a continuous from the right function $F \in TF$ and $\tau \in \mathbb{R}^+$ such that

$$2\tau + F(b\mathcal{H}_b(Tx, Ty)) \leq F(d_b(x, y))$$

for all $x, y \in X, Tx \neq Ty$. Then has a fixed point.

Moreover, Feng and Liu obtained the subsequent result (recall that a function $f : X \rightarrow \mathbb{R}$ is said to be lower semicontinuous if for all sequences $\{x_n\}$ in X with $\lim_{n \rightarrow \infty} x_n = x \in X$, it satisfies $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$).

Theorem 2.

Let (X, d) be a metric space, $T : X \rightarrow CL(X)$, and let the function $f : X \rightarrow \mathbb{R}$, $f(x) = d(x, Tx)$ be lower semicontinuous. If there exist $b, c \in (0, 1)$ with $b < c$ such that for any $x \in X$, there is $y \in Tx$ satisfying

$$cd(x, y) \leq f(x) \quad \text{and} \quad f(y) \leq bd(x, y),$$

then T has a fixed point.

Against this background, we obtain fixed point results for multivalued mappings satisfying Wardowski–Feng–Liu-type conditions for orbitally lower semicontinuous functions in orbitally complete b -metric spaces. These results generalize, complement and unify the findings proposed in [2, 12]. Besides, we illustrate a couple of examples to validate our obtained results, and also, we consider the ordered version of the attained results. Finally, we dish out an interesting application concerning our derived theorem and employing the notion of fractals to a certain type of fractal integral equations. An illustrative example is presented to show the applicability of the obtained result in reducing the energy of an antenna.

2 Main results

Let $CL(\cdot)$ denotes the family of nonempty closed subsets of X . Let $T : X \rightarrow CL(X)$ be a multivalued map, $F \in \mathbb{R}^+$ and $\eta : (0, \infty) \rightarrow (0, \infty)$. For $x \in X$ with $d_b(x, Tx) > 0$, define a set $F_\eta^x \subseteq X$ as

$$F_\eta^x = \{y \in Tx : F(d_b(x, y)) \leq F(d_b(x, Tx)) + \eta(d_b(x, y))\}.$$

Let $T : X \rightarrow X$, and for some $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$ be the orbit of x_0 . A function $f : X \rightarrow \mathbb{R}$ is called T -orbitally lower semicontinuous if $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ for all sequences $\{x_n\} \subset \mathcal{O}(x_0)$ with $\lim_{n \rightarrow \infty} x_n = x \in X$.

Cirić defined the orbit and orbital completeness in case of multivalued mappings in his paper [4]. In a similar fashion, we define these concepts in case of b -metric spaces. For a set X , we use the notation 2^X to denote the set of all subsets of X .

Definition 4.

Let $T : X \rightarrow 2^X$ be a multivalued mapping on a b -metric space (X, d_b) . An orbit for T at a point $x_0 \in X$ is denoted by $O(x_0)$ and is defined as a sequence

$$\{x_n : x_n \in Tx_{n-1}\}.$$

Definition 5.

Let $T : X \rightarrow 2^X$ be a multivalued mapping on a b -metric space (X, d_b) . A b -metric space X is said to be T -orbitally complete if every Cauchy sequence of the form $\{x_n : x_n \in Tx_{n-1}\}$ converges in X .

It is obvious that an orbitally complete b -metric space may not be complete. Now we present one of our main result.

Theorem 3.

Let (X, d_b) be an orbitally complete b -metric space with $b \geq 1$, $T : X \rightarrow 2^X$, $CL(X)$ and $F \in \mathfrak{F}_b^*$. Assume that the following conditions hold:

- (i) The mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous;
- (ii) There exist functions $\tau, \eta : (0, \infty) \rightarrow (0, \infty)$ such that for all $t \geq 0$,

$$\tau(t) > \eta(t), \quad \liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s);$$

- (iii) For any $x \in X$ with $d_b(x, Tx) > 0$, there exists $y \in F_\eta^x$ satisfying

$$\tau(d_b(x, y)) + F(bd_b(y, Ty)) \leq F(d_b(x, y)).$$

Then T has a fixed point.

Proof. Suppose that X has no fixed points. Then for all $x \in X$, we have $d_b(x, Tx) > 0$. Since $Tx \in CL(X)$ for every $x \in X$ and $F \in \mathfrak{F}_b^*$, then it is easy to prove that the set F_η^x is nonempty for every $x \in X$ (proof will follow in the line of [12]). If $x_0 \in X$ is any initial point, then there exists $x_1 \in F_\eta^{x_0}$ such that

$$\tau(d_b(x_0, x_1)) + F(bd_b(x_1, Tx_1)) \leq F(d_b(x_0, x_1))$$

and for $x_1 \in X$, there exists $x_2 \in F_\eta^{x_1}$ satisfying

$$\tau(d_b(x_1, x_2)) + F(bd_b(x_2, Tx_2)) \leq F(d_b(x_1, x_2)).$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in F_\eta^{x_n}$ and

$$\tau(d_b(x_n, x_{n+1})) + F(bd_b(x_{n+1}, Tx_{n+1})) \leq F(d_b(x_n, x_{n+1})) \quad (1)$$

for all $n \in \mathbb{N} \cup \{0\}$. It follows by (1) and property (F4) that

$$\tau(d_b(x_n, x_{n+1})) + F(b^{n+1}d_b(x_{n+1}, Tx_{n+1})) \leq F(b^n d_b(x_n, x_{n+1})) \quad (2)$$

for all $n \in \mathbb{N}$. We verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in F_\eta^{x_n}$, then by the definition of F_η^x we have

$$F(d_b(x_n, x_{n+1})) \leq F(d_b(x_n, Tx_n)) + \eta(d_b(x_n, x_{n+1})),$$

which implies that

$$F(b^n d_b(x_n, x_{n+1})) \leq F(b^n d_b(x_n, Tx_n)) + \eta(d_b(x_n, x_{n+1})). \quad (3)$$

From (2) and (3) we have

$$F(b^{n+1}d_b(x_{n+1}, Tx_{n+1})) \leq F(b^n d_b(x_n, Tx_n)) + \eta(d_b(x_n, x_{n+1})) - \tau(d_b(x_n, x_{n+1})), \quad (4)$$

i.e.,

$$F(b^{n+1}d_b(x_{n+1}, x_{n+2})) \leq F(b^n d_b(x_n, x_{n+1})) + \eta(d_b(x_n, x_{n+1})) - \tau(d_b(x_n, x_{n+1})). \quad (5)$$

Let $\varrho_n = d_b(x_n, x_{n+1})$ for $n \in \mathbb{N}$; then $\varrho_n > 0$, and from (5) $\{\varrho_n\}$ is decreasing. Therefore, there exists $\delta > 0$ such that $\lim_{n \rightarrow \infty} \varrho_n = \delta$. Now let $\delta > 0$. Let $B(t) = \liminf_{t \rightarrow s^+} \tau(t) - \liminf_{t \rightarrow s^+} \eta(t) > 0$. Then using (5), the following holds:

$$\begin{aligned} F(b^{n+1}\varrho_{n+1}) &\leq F(b^n\varrho_n) - \beta(\varrho_n) \\ &\leq F(b^{n-1}\varrho_{n-1}) - \beta(\varrho_n) - \beta(\varrho_{n-1}) \\ &\dots \\ &\leq F(\varrho_0) - \beta(\varrho_n) - \beta(\varrho_{n-1}) - \dots - \beta(\varrho_0). \end{aligned}$$

Let p_n be the greatest number in $\{0, 1, \dots, n-1\}$ such that

$$\beta(\varrho_{p_n}) = \min\{\beta(\varrho_0), \beta(\varrho_1), \dots, \beta(\varrho_n)\}$$

for all $n \in \mathbb{N}$. In this case, $\{\varrho_n\}$ is a nondecreasing sequence. From (6) we get

$$F(b^n\varrho_n) \leq F(\varrho_0) - n\beta(\varrho_{p_n}). \quad (7)$$

In a similar way, from (4) we can obtain

$$F(b^{n+1}d_b(x_{n+1}, Tx_{n+1})) \leq F(d_b(x_0, x_1)) - n\beta(\varrho_{p_n}).$$

Now consider the sequence $\{\beta(\varrho_{p_n})\}$. We distinguish two cases.

Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\beta(\varrho_{p_m}) > \beta(\varrho_{p_n})$. Then we obtain a subsequence $\{\varrho_{p_{n_k}}\}$ of $\{\varrho_{p_n}\}$ with $\beta(\varrho_{p_{n_k}}) > \beta(\varrho_{p_{n_{k+1}}})$ for all k . Since $\varrho_{p_{n_k}} \rightarrow \delta^+$, we deduce that

$$\liminf_{t \rightarrow s^+} \beta(\varrho_{p_{n_k}}) > 0.$$

Hence,

$$F(b^{n_k}\varrho_{n_k}) \leq F(\varrho_0) - n^k\beta(\varrho_{p_{n_k}})$$

for all k . Consequently, $\lim_{k \rightarrow \infty} F(b^{n_k}\varrho_{n_k}) = -\infty$, and by (F2) $\lim_{k \rightarrow \infty} b^{n_k}\varrho_{n_k} = 0$, which contradicts the fact that $\lim_{k \rightarrow \infty} \varrho_{n_k} > 0$ as $b > 1$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\beta(\varrho_{p_n}) > \beta(\varrho_{p_m})$ for all $m > n_0$. Then $F(\varrho_m) \leq F(\varrho_0) - m\beta(\varrho_{p_{n_0}})$ for all $m > n_0$. Hence, $\lim_{m \rightarrow \infty} F(\varrho_m) = -\infty$, and by (F2) $\lim_{m \rightarrow \infty} \varrho_m = 0$, which contradicts the fact that $\lim_{m \rightarrow \infty} \varrho_m > 0$. Thus, $\lim_{m \rightarrow \infty} \varrho_m = 0$. From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (b^n\varrho_n)^k F(b^n\varrho_n) = -\infty.$$

By (7) the following holds for all $n \in \mathbb{N}$:

$$\begin{aligned}
 & (b^n \varrho_n)^k F(b^n \varrho_n) - (b^n \varrho_n)^k F(\varrho_0) \\
 & \leq (b^n \varrho_n)^k (F(\varrho_0) - n\beta(\varrho_{p_n})) - (\varrho_n)^k F(\varrho_0) \\
 & = -n(b^n \varrho_n)^k \beta(\varrho_{p_n}) \leq 0.
 \end{aligned}
 \tag{9}$$

Passing to limit as $n \rightarrow \infty$ in (9), we obtain

$$\lim_{n \rightarrow \infty} n(b^n \varrho_n)^k \beta(\varrho_{p_n}) = 0.$$

Since $\zeta := \liminf_{n \rightarrow \infty} \beta(\varrho_{p_n}) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(\varrho_{p_n}) > \zeta/2$ for all $n \geq n_0$. Thus,

$$n(b^n \varrho_n)^k \frac{\zeta}{2} < n(b^n \varrho_n)^k \beta(\varrho_{p_n})$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$ in (10), we have $0 \leq \lim_{n \rightarrow \infty} n(b^n \varrho_n)^k \zeta/2 < \lim_{n \rightarrow \infty} n(b^n \varrho_n)^k \beta(\varrho_{p_n}) = 0$, that is,

$$\lim_{n \rightarrow \infty} n(b^n \varrho_n)^k = 0.$$

From (11) there exists $n_1 \in \mathbb{N}$ such that $(b^n \varrho_n)^k \leq 1$ for all $n \geq n_1$. So, we have $\varrho_n \leq 1/(b^n n^{1/k})$ for all $n \geq n_1$. Now, the last limit implies that the series $\sum_{n=1}^{+\infty} b^n \varrho_n$ is convergent, and hence, $\{x_n\}$ is a Cauchy sequence in X . Since X is a orbitally complete b -metric space, there exists $z \in \mathcal{O}(x_0)$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. On the other hand, from (8) and (F2) we have $\lim_{n \rightarrow \infty} d_b(x_n, Tx_n) = 0$. Since $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous, we have

$$\begin{aligned}
 0 & \leq d_b(z, Tz) \leq \liminf_{n \rightarrow \infty} d_b(x_n, Tx_n) \leq \lim_{n \rightarrow \infty} d_b(x_n, x_{n+1}) \\
 & \leq \lim_{n \rightarrow \infty} b[d_b(x_n, x) + d_b(x, x_{n+1})] = 0.
 \end{aligned}$$

Therefore, $z \in Tz$. Hence, T has a fixed point.

Next, we present an example in order to substantiate the above result.

Example 1. Let us consider the set $X = [1/9, \infty)$ and define a mapping $d_b : X \times X \rightarrow \mathbb{R}$ by $d_b(x, y) = |x - y|^2$. Then d_b is a b -metric on X with $b = 2$. Next, we define a mapping $T : X \rightarrow CL(X)$ by

$$Tx = \left\{ \frac{x+1}{4}, x + 2^x \right\}$$

for all $x \in X$. Let us take $F(\alpha) = \ln \alpha$ for all $\alpha \in \mathbb{R}^+$ and $\tau(t) = 1/4$, $\eta(t) = 1/6$ for all $t \in (0, \infty)$. Then, clearly, $F \in \mathfrak{F}F_b$ and $\tau(t) > \eta(t)$, $\liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s)$ for all $t \geq 0$.

For any $x \in X$, we have $d_b(x, Tx) = ((3x - 1)/4)^2$. So the mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous. Again, for any $x \in X$, we have $y = (x + 1)/4 \in F_\eta^x$, and for this y , we have

$$\begin{aligned} \tau(d_b(x, y)) + F(bd_b(y, Ty)) \\ = \frac{1}{4} + \ln\left(2\left(\frac{3x-1}{16}\right)^2\right) &\leq \ln\left(3\left(\frac{3x-1}{16}\right)^2\right) \leq \ln\left(\left(\frac{3x-1}{4}\right)^2\right) \\ &= F(d_b(x, y)). \end{aligned}$$

Thus, we see that all the conditions of Theorem 3 hold true. So by that theorem, T has a fixed point, and note that $z = 1/3$ is a fixed point of T .

Our second result is related to multivalued mappings T on the b -metric space X , where Tx is compact for all $x \in X$. By taking into account Case 1, we can take $n \geq 0$. Therefore, the proof of the following theorem is obvious.

Let us denote $k(X)$ as the set of all nonempty compact subsets of X .

Theorem 4.

Let (X, d_b) be an orbitally complete b -metric space, $b \geq 1$, $T : X \rightarrow K(X)$ and $F \in \mathfrak{F}F_b^*$. Assume that the following conditions hold:

- (i) The mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous;
- (ii) There exist functions $\tau : (0, \infty) \rightarrow (0, \infty)$ and $\eta : (0, \infty) \rightarrow [0, \infty)$ such that, for all $t \geq 0$,

$$\tau(t) > \eta(t), \quad \liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s);$$

- (iii) For any $x \in X$ with $d_b(x, Tx) > 0$, there exists $y \in F_\eta^x$ satisfying

$$\tau(d_b(x, y)) + F(bd_b(y, Ty)) \leq F(d_b(x, y)).$$

Then T has a fixed point.

Now we have the ordered version of the above results. To do this, we recall the definition of ordered b -metric space. (X, d_b, \preccurlyeq) is called an ordered b -metric space if d_b is a b -metric space on X and (X, \preccurlyeq) is a partially ordered set. Further, if (X, \preccurlyeq) is a partially ordered set, then $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds.

For $x \in X$ with $d_b(x, Tx) > 0$, define a set $F_{\eta, \preccurlyeq}^x \subseteq X$ as

$$\begin{aligned} F_{\eta, \preccurlyeq}^x = \{y \in Tx : F(d_b(x, y)) &\leq F(d_b(x, Tx)) + \eta(d_b(x, y)), \\ F(kd_b(x, y)) &\leq F(kd_b(x, Tx)) + \eta(d_b(x, y)), x \preccurlyeq y, k > 1\}. \end{aligned}$$

Theorem 5.

Let (X, d_b, \preccurlyeq) be an ordered orbitally complete b -metric space with $b \geq 1$, $T : X \rightarrow CL(X)$ and $F \in \mathfrak{F}F_b$. Assume that the following conditions hold:

- (i) The mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous;
- (ii) There exist functions $\tau, \eta : (0, \infty) \rightarrow (0, \infty)$ such that, for all $t \geq 0$,

$$\tau(t) > \eta(t), \quad \liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s);$$

- (iii) For any $x \in X$ with $d_b(x, Tx) > 0$, there exists $y \in F_{\eta, \tau}^x$ satisfying

$$\tau(d_b(x, y)) + F(bd_b(y, Ty)) \leq F(d_b(x, y)).$$

If the condition

- (C) If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \rightarrow z$ in X , then $x_n \preceq z$ for all n

holds, then T has a fixed point.

Proof. Following the line of proof of Theorem 3 and definition of $F_{\eta, \tau}^x \subseteq X$, we can show that $\{x_n\}$ is a Cauchy sequence in (X, d_b, \preceq) with $x_{n-1} \preceq x_n$ for $n \in \mathbb{N}$. From the orbital completeness of X there exists $z \in \mathcal{O}(x_0)$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. By assumption (C) $x_n \preceq z$ for all n . Rest is followed from the proof of Theorem 3.

Theorem 6.

Let (X, d, \preceq) be an ordered orbitally complete b -metric space with $b \geq 1$, $T : X \rightarrow K(X)$ and $F \in \mathfrak{F}_b^+$. Assume that the following conditions hold:

- (i) The mapping $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous;
- (ii) There exist functions $\tau : (0, \infty) \rightarrow (0, \infty)$ and $\eta : (0, \infty) \rightarrow [0, \infty)$ such that, for all $t \geq 0$,

$$\tau(t) > \eta(t), \quad \liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s);$$

- (iii) For any $x \in X$ with $d_b(x, Tx) > 0$, there exists $y \in F_{\eta, \tau}^x$ satisfying

$$\tau(d_b(x, y)) + F(bd_b(y, Ty)) \leq F(d_b(x, y)).$$

Then T has a fixed point provided (C) holds.

Now we present an example to authenticate Theorem 5.

Example 2. Let us take $X = \mathbb{N} \cup \{0\}$ and consider a relation $<$ on X by defining $x < y$ if and only if y divides x . Then it is easy to verify that $(X, <)$ is a partially ordered set. Now we define $d_b : X \times X \rightarrow \mathbb{R}$ by

$$d_b(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 3\left(\frac{1}{n} + \frac{1}{m}\right) & \text{if } x = n, y = m \text{ and } n \neq m; \\ \frac{1}{n} & \text{if } x = n, y = 0 \text{ or } x = 0, y = n. \end{cases}$$

Then (X, d_b) is a partially ordered b -metric space with $b = 3$.

Now we define a mapping $T : X \rightarrow CL(X)$ by

$$Tx = \{2x, 3x, 4x\} \quad \text{for all } x \in X.$$

Let us choose $F(a) = \ln a$ for all $a \in \mathbb{R}^+$ and $T(t) = 1/8$, $n(t) = 1.10$ for all $t \in (0, \infty)$. Then it is obvious that $F \in \mathfrak{F}_b$ and $T(t) > \eta(t)$, $\liminf_{s \rightarrow t^+} \tau(s) > \liminf_{s \rightarrow t^+} \eta(s)$ for all $t \geq 0$. Further, for $x \in X$, we have

$$d_b(x, Tx) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{15}{4x} & \text{if } x \neq 0. \end{cases}$$

Therefore, $x \mapsto d_b(x, Tx)$ is orbitally lower semicontinuous. For $x \in X$, we have $y = 4x \in F_{\eta, \varphi}^x$, and for this y , we have

$$d_b(x, Tx) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{15}{4x} & \text{if } x \neq 0. \end{cases}$$

Thus, we see that all conditions of Theorem 5 hold. So by the same theorem T has a fixed point. Indeed $z = 0$ is a fixed point of T .

3 Application

Fredholm equations stand up obviously in the scheme of signal processing, for instance, as the well-known spectral concentration issue. The operators convoluted are the similar as linear filters. They besides normally stand up in linear forward forming and inverse problems. In physics, the result of such integral equations permits for investigation spectra to be correlated to several fundamental disseminations, for example, the mass supply of polymers in a polymeric melts, or the supply of reduction times in the scheme. Moreover, Fredholm integral equations indicate in fluid mechanics issues connecting hydrodynamic connections near finite-sized elastic borders (see [9] for recent work). In antenna manufacturing, side lobes are the lobes (local maxima with local minimum energy) of the distant field radiation design of an antenna or other radioactivity foundation, which are not the chief lobe. The problem statement is that: is there a solution for which the side lobe energy is minimum? The answer of this question is to find a solution for Fredholm integral equation. In our discussion, we use a generalize fractal Fredholm integral equation based on the fractal integral [21] as follows:

$$\chi(t) = \frac{1}{\Gamma(\varphi + 1)} \int_{\alpha}^{\beta} A(t, \varsigma, \chi(\varsigma)) (d\varsigma)^{\varphi} + \sigma(t), \quad t, \varsigma \in J := [\alpha, \beta],$$

where $A: J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ is defined and continuous on a fractal set of fractal dimension $\varphi \in (0, 1)$, and $\sigma: J \rightarrow \mathbb{R}$ is continuous function. Our aim is to show that (12) has a solution by using Theorem 3.

Theorem 7.

Let $X = C[J, \mathbb{R}]$, and let \mathcal{Q} the operator defined by

$$(\mathcal{Q}\chi)(t) = \frac{1}{\Gamma(\wp + 1)} \int_{\alpha}^{\beta} \Lambda(t, \varsigma, \chi(\varsigma)) (d\varsigma)^{\wp} + \sigma(t), \quad t, \varsigma \in J = [\alpha, \beta], \quad \wp \in (0, 1),$$

where $\Lambda : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ is defined and continuous on a fractal set of fractal dimension $\wp \in (0, 1)$, and $\sigma : J \rightarrow \mathbb{R}$, is continuous function. Moreover, assume the following conditions:

(i) There occurs a continuous function $\gamma : \mathbb{X} \rightarrow [0, \infty)$ satisfying

$$|\Lambda(t, \varsigma, \chi(\varsigma)) - \Lambda(t, \varsigma, (\mathcal{Q}\chi)(\varsigma))| \leq \gamma(\varsigma) |\chi(\varsigma) - (\mathcal{Q}\chi)(\varsigma)|;$$

(ii) $\int_{\alpha}^{\beta} \gamma(\varsigma) (d\varsigma)^{\wp} \leq \sqrt{e^{-\mathfrak{T}}}$, $\mathfrak{T} := \tau(d(\chi, \mathcal{Q}\chi)) > 0$.

Then the integral equation (12) admits a solution.

Proof. We have to prove that the operator \mathcal{Q} achieves all the assumptions of Theorem 3 in the single-valued type. Let $X \# X$, then we obtain

$$\begin{aligned} & |(\mathcal{Q}\chi)(t) - \mathcal{Q}(\mathcal{Q}\chi(t))|^2 \\ & \leq \frac{1}{\Gamma^2(\wp + 1)} \left(\int_{\alpha}^{\beta} |\Lambda(t, \varsigma, \chi(\varsigma)) - \Lambda(t, \varsigma, \mathcal{Q}\chi(\varsigma))| (d\varsigma)^{\wp} \right)^2 \\ & \leq \frac{1}{\Gamma^2(\wp + 1)} \left(\int_{\alpha}^{\beta} \gamma(\varsigma) |\chi(\varsigma) - \mathcal{Q}\chi(\varsigma)| (d\varsigma)^{\wp} \right)^2 \\ & = \frac{1}{\Gamma^2(\wp + 1)} d(\chi, \mathcal{Q}\chi) \left(\int_{\alpha}^{\beta} \gamma(\varsigma) (d\varsigma)^{\wp} \right)^2 \\ & \leq \frac{1}{\Gamma^2(\wp + 1)} e^{-\mathfrak{T}} d(\chi, \mathcal{Q}\chi). \end{aligned}$$

Thus, we get the following inequality:

$$\Gamma^2(\wp + 1) |(\mathcal{Q}\chi)(t) - \mathcal{Q}(\mathcal{Q}\chi(t))|^2 \leq e^{-\mathfrak{T}} d(\chi, \mathcal{Q}\chi).$$

Since the natural logarithm indicates to be in ${}^{\mathfrak{N}}F_b$, employing it on above inequality, we conclude that

$$\mathfrak{T} + \ln[\Gamma^2(\wp + 1) d(\mathcal{Q}\chi, \mathcal{Q}^2\chi)] \leq \ln[d(\chi, \mathcal{Q}\chi)].$$

Consequently, we obtain

$$\tau(d(\chi, Q\chi)) + \ln[\Gamma^2(\wp + 1)d(Q\chi, Q^2\chi)] \leq \ln[d(\chi, Q\chi)].$$

Now let $b := \Gamma^2(\wp + 1)$, thus, we indicate the following inequality:

$$\tau(d(\chi, Q\chi)) + \ln[b, d(Q\chi, Q^2\chi)] \leq \ln[d(\chi, Q\chi)].$$

Hence, in view of Theorem 3 (single-valued) with $F(\psi) = \ln \psi$, the operator admits at least one fixed point, which is corresponding to the solution of Eq. (12). This completes the proof.

Example 3. Consider the following data: $J = [0, 1]$, $A(t, \varsigma, \chi) = \varsigma\chi(\varsigma)/4$, $\varsigma \in J$. From Fig. 1 we can say that the relation between \cdot and \cdot certainly minimize the energy of the antenna. Obviously, the value of $\wp = 0.4616$ minimized the energy. It is clear that $\int_0^1 \gamma(\varsigma)(d_\varsigma)^\wp = 0.25 < \sqrt{e^{-\mathfrak{T}}} = 0.36$, $\mathfrak{T} = 1 = \tau(1) > 0$. Hence, in view of Theorem 7, Eq. (12) has a solution, which minimize the energy.

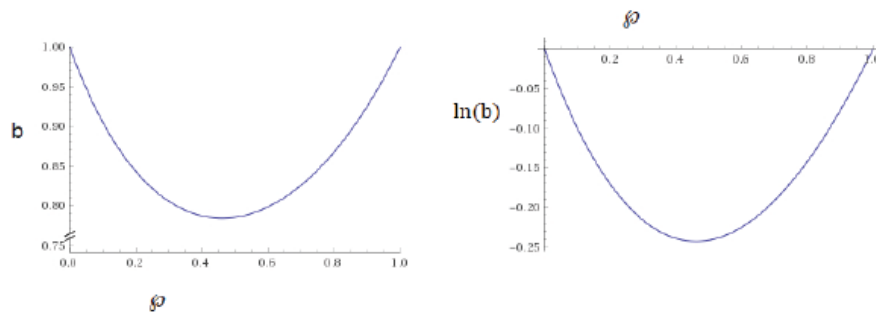


Figure 1.

The relation between $b = \Gamma^2(\# + 1)$ and $\#$ in the left, while the right graph is the relation between $\ln(b) = F$ with respect to $\#$. The minimum value of b is at $\# = 0.4616$.

Acknowledgments

The authors are very grateful to the referees for their constructive comments and suggestions, which substantially improve the paper. The authors express their gratitude to Hiranmoy Garai for his valuable suggestions for the improvement of this paper.

References

1. I. Altun, G. Minak, M. Olgun, Fixed points of multivalued nonlinear \cdot -contractions on complete metric spaces, *Nonlinear Anal. Model. Control*, **21**(2):201–210, 2016, <https://doi.org/10.15388/NA.2016.2.4>.
2. I. Altun, M. Olgun, G. Minak, On a new class of multivalued weakly Picard operators on complete metric spaces, *Taiwanese J. Math.*, **19**(3):659–672, 2015, <https://doi.org/10.11650/tjm.19.2015.4752>.
3. I.A. Bakhtin, The contraction mapping principle in almost quasispaces, *Funkts. Anal.*, **30**:26–37, 1989 (in Russian).
4. Lj.B. Ćirić, Fixed point for generalized multivalued contractions, *Mat. Vesn.*, **26**:265–272, 1972.

5. M. Cosentino, M. Jleli, B. Samet, C. Vetro, Solvability of integrodifferential problems via fixed point theory in \mathcal{M} -metric spaces, *Fixed Point Theory Appl.*, **2015**:70, 2015, <https://doi.org/10.1186/s13663-015-0317-2>.
6. S. Czerwik, Contraction mappings in \mathcal{M} -metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **5**: 5–11, 1993.
7. S. Czerwik, Nonlinear set-valued contraction mappings in \mathcal{M} -metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, **46**(2):263–276, 1998.
8. S. Czerwik, K. Dłutek, S.L. Sing, Round-off stability of iteration procedures for set-valued operators in \mathcal{M} -metric spaces, *J. Nat. Phys. Sci.*, **11**:87–94, 2007.
9. A. Daddi-Moussa-Ider, B. Kaoui, H. Löwen, Axisymmetric flow due to a stokeslet near a finite- sized elastic membrane, *J. Phys. Soc. Japan*, **88**(5):054401, 2019, <https://doi.org/10.7566/JPSJ.88.054401>.
10. Y. Feng, S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, *J. Math. Anal. Appl.*, **317**(1):103–112, 2006, <https://doi.org/10.1016/j.jmaa.2005.12.004>.
11. Z. Kadelburg, S. Radenovic', Notes on some recent papers concerning \mathcal{M} -contractions in \mathcal{M} -metric spaces, *Constr. Math. Anal.*, **2**(2):108–112, 2018, <https://doi.org/10.33205/cma.468813>.
12. G. Minak, M. Olgun, I. Altun, A new approach to fixed point theorems for multivalued contractive maps, *Carpathian J. Math.*, **31**(2):241–248, 2015.
13. H.K. Nashine, Z. Kadelburg, Cyclic generalized \mathcal{M} -contractions in \mathcal{M} -metric spaces and an application to integral equations, *Filomat*, **28**(10):2047–2057, 2014, <https://doi.org/10.2298/FIL1410047N>.
14. A. Nicolae, Fixed point theorems for multi-valued mappings of Feng–Liu type, *Fixed Point Theory*, **12**(1):145–154, 2011.
15. T. Senapati, L.K. Dey, D. Dolic'anin-Dekic', Extensions of Ćirić' and Wardowski type fixed point theorems in \mathcal{M} -generalized metric spaces, *Fixed Point Theory Appl.*, **2016**:33, 2016, <https://doi.org/10.1186/s13663-016-0522-7>.
16. C. Vetro, D. Wardowski, Krasnosel'skii–Schaefer type method in the existence problems, *Topol. Methods Nonlinear Anal.*, **54**(1):131–139, 2019, <https://doi.org/10.12775/TMNA.2019.028>.
17. V. Vetro, Fixed point for \mathcal{M} -contractions and first-order periodic differential problem, *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM*, **113**(3):1823–1837, 2019, <https://doi.org/10.1007/s13398-018-0586-9>.
18. V. Vetro, C. Vetro, On an idea of Bakhtin and Czerwik for solving a first-order periodic problem, *J. Nonlinear Convex Anal.*, **18**(12):2123–2134, 2017.
19. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**:94, 2012, <https://doi.org/10.1186/1687-1812-2012-94>.
20. D. Wardowski, N. Van Dung, Fixed points of \mathcal{M} -weak contractions on complete metric spaces, *Demonstr. Math.*, **47**(1):146–155, 2014, <https://doi.org/10.2478/dema-2014-0012>.

21. X.J. Yang, D. Baleanu, H.M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, 2015, <https://doi.org/10.1016/C2014-0-04768-5>.