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nonlinear@mii.vu.lt
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Antanas, Laurin[~] **cikas; Darius, Šiauciunas; Monika, Tekore**

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II


Laurinćikas Antanas antanas.laurincikas@mif.vu.lt

Vilnius University, Lithuania

 <https://orcid.org/0000-0002-7671-0282>


Šiauciunas Darius darius.siauciunas@sa.vu.lt

Vilnius University, Lithuania

 <https://orcid.org/0000-0002-9248-8917>

Tekore Monika monika.tekore@mif.stud.vu.lt

Vilnius University, Lithuania

 <https://orcid.org/0000-0002-3968-3377>

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Abstract: In the paper, a joint discrete universality theorem for periodic zeta-functions with multiplicative coefficients on the approximation of analytic functions by shifts involving the sequence $\{\gamma_k\}$ of imaginary parts of nontrivial zeros of the Riemann zeta-function is obtained. For its proof, a weak form of the Montgomery pair correlation conjecture is used. The paper is a continuation of [A. Laurinćikas, M. Tekore, Joint universality of periodic zeta-functions with multiplicative coefficients, *Nonlinear Anal. Model. Control*, 25(5):860–883, 2020] using nonlinear shifts for approximation of analytic functions.

Keywords: joint universality, nontrivial zeros of the Riemann zeta-function, periodic zeta-function, space of analytic functions, weak convergence.

1 Introduction

It is well known that some zeta- and L -functions, and even some classes of Dirichlet series, for example, the Selberg-Steuding class, see [29, 32], are universal in the Voronin sense, i.e., a wide class of analytic functions can be approximated by one and the same zeta-function. For example, in the case of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, analytic nonvanishing functions on the strip $D = \{s \in \mathbb{C}: 1/2 < \sigma < 1\}$ are approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$ (continuous case), or shifts $\zeta(s + ikh)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $h > 0$ (discrete case); see [1, 6, 13, 24, 32].

The above shifts are very simple, T and kh occur in them linearly. It turned out that the approximation remains valid also with more general shifts. A significant progress in this direction was made by Pan'kowski [31] using the shifts $\zeta(s + i\varphi(\tau))$ and $\zeta(s + i\varphi(k))$ with $\varphi(\tau) = \tau^\alpha \log^\beta \tau$ and a wide class of reals a and B . The papers [22] and [35] are also devoted to approximation of analytic functions by generalized shifts of zeta-functions. In [5], the shifts $\zeta(s + ih\gamma_k)$ were applied, where $\{\gamma_k: k \in \mathbb{N}\} = \{\gamma_k: 0 < \gamma_1 < \dots \leq \gamma_k \leq \gamma_{k+1} \leq \dots\}$ is

the sequence of imaginary parts of nontrivial zeros of the Riemann zeta-function.

Universality in the Voronin sense also has its joint version. In the joint case, a collection of analytic functions is approximated simultaneously by a collection of shifts of zeta- or L -functions. The first joint universality theorem belongs to Voronin who proved [36] the joint universality of Dirichlet L -functions $L(s, \chi_j)$, $j = 1, \dots, r$. Obviously, in joint universality theorems, the approximating shifts must be in some sense independent. Voronin required [36] for this the pairwise nonequivalence of Dirichlet characters, i.e., in fact, he considered joint universality of different Dirichlet L -functions. On the other hand, as it was observed by Pan'kowski [31], the independence of approximating shifts of Dirichlet L -functions can be ensured by different functions $s\varphi_j(\tau)$ in shifts $L(s + i\varphi_j(\tau), \chi_j)$ or $L(s + i\varphi_j(k), \chi_j)$ even with the same characters χ_j . This observation extends significantly classes of jointly universal functions. For example, the joint universality with generalized shifts was obtained in [16] and [20].

In general, joint universality of zeta-functions was widely studied, and many results are known; see, for example, general results obtained in [7–11, 14, 26, 30] and other papers by authors of the mentioned works. In this note, we focus on joint universality of so-called periodic zeta-functions with generalized shifts involving the sequence $\{\gamma_k; k \in \mathbb{N}\}$. N of imaginary parts of nontrivial zeros of the function $\zeta(s)$. We will mention some joint universality results involving the latter sequence. Note that the behaviour of the sequence $\{\gamma_k\}$, as of nontrivial zeros of $\zeta(s)$, is very complicated, and at the moment, its known properties are not sufficient for the proof of universality. Therefore, in [5], the conjecture that, for $c > 0$,

$$\sum_{\substack{\gamma_k, \gamma_l \leq T \\ |\gamma_k - \gamma_l| < c/\log T}} 1 \ll T \log T \quad (1)$$

was introduced. This conjecture is inspired by the Montgomery pair correlation conjecture [28] that

$$\sum_{\substack{\gamma_k, \gamma_l \leq T \\ 2\pi\alpha_1/\log T \leq \gamma_k - \gamma_l \leq 2\pi\alpha_2/\log T}} 1 \sim \left(\int_{\alpha_1}^{\alpha_2} \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) du + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T,$$

where $\alpha_1 < \alpha_2$ are arbitrary real numbers, and

$$\delta(\alpha_1, \alpha_2) = \begin{cases} 1 & \text{if } 0 \in [\alpha_1, \alpha_2], \\ 0 & \text{otherwise.} \end{cases}$$

Now we will state a joint universality theorem for Dirichlet L -functions involving the sequence $\{\gamma_k\}$ obtained in [18]. Denote by the class of compact subsets of the strip $\sigma_1 < \sigma < \sigma_2$ with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous nonvanishing functions on $\sigma_1 < \sigma < \sigma_2$ that are analytic in the interior of K .

Theorem 1.

Suppose that χ_1, \dots, χ_r are pairwise nonequivalent Dirichlet characters, and estimate (1) is true. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$ and $h > 0$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \#\left\{1 \leq k \leq N: \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + ih\gamma_k, \chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover “lim inf” can be replaced by “lim” for all but at most countably many $\varepsilon > 0$.

Here $\#A$ denotes the cardinality of the set A , and k runs over the set \mathbb{N} .

Now we recall the definition of the periodic zeta-function, which is an object of investigation of the present note. Let $\mathbf{a} = \{a_m; m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$. Then the periodic zeta-function $\zeta(s; \mathbf{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathbf{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

and has an analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\frac{1}{q} \sum_{l=1}^q a_l.$$

The sequence \mathbf{a} is called multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprimes $m, n \in \mathbb{N}$. If $0 < \alpha \leq 1$ is a fixed number, then the function

$$\zeta(s, \alpha; \mathbf{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}, \quad \sigma > 1,$$

and its meromorphic continuation are called the periodic Hurwitz zeta-function. In [15] and [3], under hypothesis (1), joint universality theorems involving sequence γ for the pair consisting from the Riemann and Hurwitz zeta-functions and their periodic analogues, respectively, were obtained, while in [23], such theorems were proved for Hurwitz zeta-functions.

For $j = 1, \dots, r$, let $\mathbf{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequences of complex numbers with minimal period $q_j \in \mathbb{N}$, and let $\zeta(s; \mathbf{a}_j)$ be the corresponding zeta-function. The main result of the paper is the following theorem.

Theorem 2.

Suppose that the sequences $\mathbf{a}_1, \dots, \mathbf{a}_r$ are multiplicative, h_1, \dots, h_r are positive algebraic numbers linearly independent over the field of rational numbers, and estimate (1) is true. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + ih_j \gamma_k; \mathbf{a}_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover “lim inf” can be replaced by “lim” for all but at most countably many $\varepsilon > 0$.

In [21], joint continuous universality theorems for periodic zeta-functions with shifts defined by means of certain differentiable functions were obtained.

2 The sequence $\{\gamma_k\}$

From the

functional equation for the Riemann zeta-function

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

it follows that $\zeta(-2m) = 0$ for all $m \in \mathbb{N}$, and the zeros $s = -2m$ of $\zeta(s)$ are called trivial. Moreover, it is known that $\zeta(s)$ has infinitely many of so-called complex nontrivial zeros $\rho_k = \beta_k + i\gamma_k$ lying in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. The famous Riemann hypothesis, one of seven Millennium problems, asserts that $\beta_k = 1/2$, i.e., all nontrivial zeros lie on the critical line $\sigma = 1/2$. There exists a conjecture that all nontrivial zeros of $\zeta(s)$ are simple.

We recall some properties of the sequence

$$\{\gamma_k : k \in \mathbb{N}\} = \{\gamma_k : 0 < \gamma_1 < \dots \leq \gamma_k \leq \gamma_{k+1} \leq \dots\}.$$

By the definition, a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1, if, for every subinterval $(a, b] \subset (0, 1]$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n I_{(a,b]}(\{x_k\}) = b - a,$$

where $I_{(a,b]}$ is the indicator function of $(a, b]$, and $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$. Though the sequence $\{\gamma_k\}$ is distributed irregularly, the following statement is true for it.

Lemma 1

The sequence $\{\gamma_k a; k \in \mathbb{N}\} \subset \mathbb{N}$ with every $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1

Proof. Proof of the lemma is given in [33], and in the above form, was applied in [5].

For convenience, we recall the Weyl criterion on the uniform distribution modulo 1; see, for example, [12].

Lemma 2.

A sequence $\{x_k; k \in \mathbb{N}\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{2\pi i m x_k} = 0.$$

Obviously, the uniform distribution modulo 1 of the sequence shows its nonlinear character.

The following statement is well known; see, for example, [34].

Lemma 3.

For $k \rightarrow \infty$,

$$\gamma_k = \frac{2\pi k}{\log k} (1 + o(1)).$$

3 Limit theorems

Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta. We will derive Theorem 2 from a limit theorem on the weak convergence of probability measures in the space

$$H^r(D) = \underbrace{H(D) \times \cdots \times H(D)}_r.$$

Therefore, we start with a certain probability model.

Let $\mathcal{B}(X)$ be the Borel σ -field of the space X , and \mathcal{P} denote the set of all prime numbers. Define

$$\Omega = \prod_{p \in \mathbb{P}} \mathbb{X}_p,$$

where $\mathbb{X}_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. Then Ω is a compact topological Abelian group. Moreover, let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then again Ω^r is a compact topological Abelian group.

Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r can be defined. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. Denote by $w(p)$ the p -th component, $p \in \mathbb{P}$, of an element $\omega_j \in \Omega_j$, $j = 1, \dots, r$. For brevity, let $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$, $\omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r$, $\underline{a} = (a_1, \dots, a_r)$, and on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$, define the $H^r(D)$ -valued random element

$$\underline{\zeta}(s, \omega; \underline{a}) = (\zeta(s, \omega_1; a_1), \dots, \zeta(s, \omega_r; a_r)),$$

Where

$$\zeta(s, \omega_j; a_j) = \prod_{p \in \mathbb{P}} \left(1 + \sum_{l=1}^{\infty} \frac{a_j p^l \omega_j^l(p)}{p^{ls}} \right), \quad j = 1, \dots, r.$$

Note that the latter products, for almost all ω , are uniformly convergent on compact subsets of the strip D . Since the periodic sequences a_j , $j = 1, \dots, r$, are bounded, the proofs of the above assertions completely coincides with those of Lemma 5.1.6 and Theorem 5.1.7 from [13]. More general results are given in [1]. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \omega; \underline{a})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H^r \{ \omega \in \Omega^r : \underline{\zeta}(s, \omega; \underline{a}) \in A \}, \quad A \in \mathcal{B}(H^r(D)).$$

Put $\underline{h} = (h_1, \dots, h_r)$, and, for $A \in \mathcal{B}(H^r(D))$, define

$$P_N(A) = \frac{1}{N} \#(1 \leq k \leq N : \underline{\zeta}(s + ih\gamma_k; \underline{a}) \in A),$$

where

$$\underline{\zeta}(s; \underline{a}) = (\zeta(s; a_1), \dots, \zeta(s; a_r)).$$

In this section, we will prove the following limit theorem.

Theorem 3.

Suppose that the sequences a_1, \dots, a_r are multiplicative, b_1, \dots, b_r are positive algebraic numbers linearly independent over \mathbb{Q} , and estimate (1) is valid. Then P_N converges weakly to $\mu_{\mathbb{Q}}^r$ as $N \rightarrow \infty$.

We start the proof of Theorem 3, as usual, with a limit lemma in the space Ω^r . In this lemma, the uniform distribution modulo 1 of the sequence $\{\gamma_k a\}$, $a \in \mathbb{R} \setminus \{0\}$, and the property of the numbers b_1, \dots, b_r essentially are applied.

For $A \in B(\Omega^r)$, define

$$Q_N(A) = \frac{1}{N} \# \{1 \leq k \leq N: ((p^{-i h_1 \gamma_k}: p \in \mathbb{P}), \dots, (p^{-i h_r \gamma_k}: p \in \mathbb{P})) \in A\}.$$

Before the statement of a limit theorem for Q_N , we recall one result of Diophantine type.

Lemma 4.

Suppose that $\lambda_1, \dots, \lambda_r$ are algebraic numbers such that the logarithms $\log \lambda_1, \dots, \log \lambda_r$ are linearly independent over \mathbb{Q} . Then, for any algebraic numbers β_0, \dots, β_r , not all zero, we have

$$|\beta_0 + \beta_1 \log \lambda_1 + \dots + \beta_r \log \lambda_r| > H^{-C},$$

where H is the maximum of the heights of $\beta_0, \beta_1, \dots, \beta_r$ and C is an effectively computable number depending on r and the maximum of the degrees of $\beta_0, \beta_1, \dots, \beta_r$.

The lemma is the well-known Baker theorem on logarithm forms; see, for example [2].

Lemma 5.

Suppose that b_1, \dots, b_r are real algebraic numbers linearly independent over \mathbb{Q} . Then Q_N converges weakly to the Haar measure $\mu_{\mathbb{Q}}^r$ as $N \rightarrow \infty$.

Proof. As usual, we apply the Fourier transform method. The characters of the group Ω^r are of the form

$$\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_{jp}}(p),$$

where the star “*” shows that only a finite number of integers k_{jp} are distinct from zero. Therefore, the Fourier transform of Q_N is

$$g_N(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^r \prod_{p \in \mathbb{P}}^* \omega_j^{k_{jp}}(p) \right) dQ_N,$$

where $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P}), j = 1, \dots, r$. Thus, by the definition of Q_N ,

$$\begin{aligned} g_N(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{N} \sum_{k=1}^N \prod_{j=1}^r \prod_{p \in \mathbb{P}}^* p^{-i h_j k_{jp} \gamma_k} \\ &= \frac{1}{N} \sum_{k=1}^N \exp \left\{ -i \gamma_k \sum_{j=1}^r h_j \sum_{p \in \mathbb{P}}^* k_{jp} \log p \right\}. \end{aligned} \quad (2)$$

Obviously,

$$g_N(\underline{0}, \dots, \underline{0}) = 1. \quad (3)$$

Now, suppose that $\underline{k} \neq (\underline{0}, \dots, \underline{0})$. Then there exists $j \in \{1, \dots, r\}$ such that $k_{jp} \neq 0$. Thus, there exists a prime number p such that $k_{jp} \neq 0$. Define

$$a_p = \sum_{j=1}^r h_j k_{jp}.$$

Then, in view of a property of the numbers h_1, \dots, h_r , we have $a_p \neq 0$. The numbers a_p are algebraic, and the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over \mathbb{Q} . Therefore, by Lemma 4,

$$a_{\underline{k}_1, \dots, \underline{k}_r} \stackrel{\text{def}}{=} \sum_{j=1}^r h_j \sum_{p \in \mathbb{P}}^* k_{jp} \log p = \sum_{p \in \mathbb{P}}^* a_p \log p \neq 0.$$

Hence, in virtue of Lemma 1, the sequence

$$\left\{ \frac{1}{2\pi} \gamma_k a_{\underline{k}_1, \dots, \underline{k}_r} : k \in \mathbb{N} \right\}$$

is uniformly distributed modulo 1. This, together with (2) and Lemma 2, shows that, in the case $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$,

$$\lim_{N \rightarrow \infty} g_N(\underline{k}_1, \dots, \underline{k}_r) = 0.$$

Thus, in view of (3),

$$\lim_{N \rightarrow \infty} g_N(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}), \end{cases}$$

and the lemma is proved because the right-hand side of the latter equality is the Fourier transform of the Haar measure $m_{\mathbb{H}^r}^*$.

Lemma 5 implies a limit lemma in the space $H^r(D)$ for absolutely convergent Dirichlet series. Let, for a fixed $\vartheta > 1.2$,

$$v_n(m) = \exp \left\{ - \left(\frac{m}{n} \right)^\theta \right\}, \quad m, n \in \mathbb{N},$$

and

$$\zeta_n(s; \mathbf{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} v_n(m)}{m^s}, \quad j = 1, \dots, r.$$

Then the latter series are absolutely convergent for $\sigma > 1.2$. Actually, since $v_n(m) \ll m^{-L/n^\theta}$ with every $L > 0$, the latter series are absolutely convergent even in the whole

complex plane. For $B(H^r(D))$, define

$$V_{N,n}(A) = \frac{1}{N} \# \{ 1 \leq k \leq N : \underline{\zeta}_n(s + i h \gamma_k; \underline{\mathbf{a}}) \in A \},$$

where

$$\underline{\zeta}_n(s; \underline{\mathbf{a}}) = (\zeta_n(s; \mathbf{a}_1), \dots, \zeta_n(s; \mathbf{a}_r)).$$

Moreover, let

$$\zeta_n(s, \omega_j; \mathbf{a}_j) = \sum_{m=1}^{\infty} \frac{a_{jm} \omega_j(m) v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

$$\underline{\zeta}_n(s, \omega; \underline{\mathbf{a}}) = (\zeta_n(s, \omega_1; \mathbf{a}_1), \dots, \zeta_n(s, \omega_r; \mathbf{a}_r)),$$

and let $u_n : \Omega^r \rightarrow H^r(D)$ be given by the formula

$$u_n(\omega) = \underline{\zeta}_n(s, \omega; \underline{\mathbf{a}}).$$

Lemma 6.

Suppose that h_1, \dots, h_r are real algebraic numbers linearly independent over \mathbb{Q} . Then $V_{N,n}$ as $N \rightarrow \infty$, converges weakly to a measure $V_n = \text{def } m_H^r u_n^{-1}$, where

$$m_H^r u_n^{-1}(A) = m_H^r(u_n^{-1}A), \quad A \in \mathcal{B}(H^r(D)).$$

Proof. Since the series for $\zeta_n(s, \omega_j, a_j)$ are absolutely convergent for $\sigma > 1.2$, the function u_n is continuous, hence $(B(\Omega^r), B(H^r(D)))$ -measurable. Therefore, the measure V_n is defined correctly. The definitions of Q_N , $V_{N,n}$ and u_n imply the equality $V_{N,n} = Q_N u_n^{-1}$. Therefore, the lemma follows from Lemma 5 and a preservation of weak convergence under continuous mappings; see [4, Thm. 5.1].

The limit measure V_n in Lemma 6 is independent on h and $\{\gamma_k\}$ and has a good convergence property, which is the next lemma.

Lemma 7.

Suppose that the sequences a_1, \dots, a_r are multiplicative. Then V_n converges weakly to $P_{\underline{a}}$ as $n \rightarrow \infty$.

Proof. In [17], the weak convergence for

$$\hat{P}_T(A) = \frac{1}{T} \text{meas} \{ \tau \in [0, T]: \zeta(s + i\tau; \underline{a}) \in A \}, \quad A \in \mathcal{B}(H^r(D)),$$

was considered, and it was obtained its weak convergence to $P_{\underline{a}}$ as $T \rightarrow \infty$, and that V_n also converges weakly to $P_{\underline{a}}$ as $n \rightarrow \infty$. In other words, V_n and \hat{P}_T have the same limit measure $P_{\underline{a}}$.

In view of Lemma 7, to prove Theorem 3, it suffices to show that P_N , as $N \rightarrow \infty$, and V_n , as $n \rightarrow \infty$, have a common limit measure. For this, a certain closeness of $\zeta(s; \underline{a})$ and $\zeta_n(s; \underline{a})$ is needed.

There exists a sequence $\{K_l: l \in \mathbb{N}\} \subset D$ of compact subsets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, for all $l \in \mathbb{N}$, and if $K \# D$ is a compact set, then $K \# K_l$ for some l . Then, putting for $g_1, g_2 \# H(D)$,

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

we have a metric in $H(D)$ inducing its topology of uniform convergence on compacta.

Hence,

$$\rho(\underline{g}_1, \underline{g}_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j}),$$

$$\underline{g}_1 = (g_{11}, \dots, g_{1r}), \underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D),$$

is a metric in $H^r(D)$ inducing its product topology. Note that, in the proof of the next lemma, the multiplicativity of the sequences $a_j, j = 1, \dots, r$, is not used.

Lemma 8.

Suppose that estimate (1) is true. Then, for every positive h_1, \dots, h_r and a_1, \dots, a_r ,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \rho(\zeta(s + ih_k \gamma_k; \mathbf{a}), \zeta_n(s + ih_k \gamma_k; \mathbf{a})) = 0. \quad (4)$$

Proof. By the definitions of the metrics p and p , it is sufficient to show that, for every compact set $K \# D$,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |\zeta(s + ih_j \gamma_k; \mathbf{a}_j) - \zeta_n(s + ih_j \gamma_k; \mathbf{a}_j)| = 0, \quad (5)$$

$j = 1, \dots, r$. The equality of type (5) was already used in [3], therefore, only for fullness, we give remarks on its proof.

Thus, let $h > 0$ and \cdot be arbitrary. We consider $\zeta(s + ih \gamma_k; \mathbf{a})$ and $\zeta_n(s + ih \gamma_k; \mathbf{a})$.

Let 0 be as in the definition of $v_n(m)$. Then the representation

$$\zeta_n(s; \mathbf{a}) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z; \mathbf{a}) l_n(z) \frac{dz}{z},$$

where

$$l_n(z) = \frac{z}{\theta} \Gamma\left(\frac{z}{\theta}\right) n^z,$$

is valid. Hence, for $\theta_1 < 0$,

$$\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a}) = \frac{1}{2\pi i} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \zeta(s+z; \mathfrak{a}) l_n(z) \frac{dz}{z} + R_n(s; \mathfrak{a}), \quad (6)$$

where

$$R_n(s; \mathfrak{a}) = \frac{al_n(1-s)}{1-s},$$

and a is the residue of $\zeta(s; \mathfrak{a})$ at the point $s = 1$. Let $K \# D$ be an arbitrary compact set, and $\varepsilon > 0$ be such that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ for $s \in K$. Then, in view of (6), for $s = \sigma + iv \in K$,

$$|\zeta_n(s; \mathfrak{a}) - \zeta(s; \mathfrak{a})| \ll \int_{-\infty}^{\infty} |\zeta(s - \theta_1 + it; \mathfrak{a})| \frac{|l_n(-\theta_1 + it)|}{|-\theta_1 + it|} dt + |R_n(s; \mathfrak{a})|.$$

Hence, taking t in place of $t + v$ and $\theta_1 = \sigma - \varepsilon - 1/2$, we have

$$\frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |\zeta(s + ih\gamma_k; \mathfrak{a}) - \zeta_n(s + ih\gamma_k; \mathfrak{a})| \ll I + Z, \quad (7)$$

where

$$I = \int_{-\infty}^{\infty} \left(\frac{1}{N} \sum_{k=1}^N \left| \zeta\left(\frac{1}{2} + \varepsilon + ih\gamma_k + it; \mathfrak{a}\right) \right| \right) \sup_{s \in K} \left| \frac{l_n(1/2 + \varepsilon - s + it)}{1/2 + \varepsilon - s + it} \right| dt$$

and

$$Z = \frac{1}{N} \sum_{k=1}^N \sup_{s \in K} |R_n(s + ih\gamma_k; \mathfrak{a})|.$$

Estimate (1) is applied for estimation of the first factor of the integrated function in the integral I . It is well known that, for $T \in \mathbb{R}$,

$$\int_0^T \left| \zeta\left(\frac{1}{2} + \varepsilon + i\tau + it; \mathfrak{a}\right) \right|^2 dt \ll_{\varepsilon} T(1 + |\tau|). \quad (8)$$

The same estimate is also true for the derivative of $\zeta(s; \mathfrak{a})$. Let $\delta = ch(\log \gamma_N)^{-1}$ and

$$N_\delta(h\gamma_k) = \sum_{\substack{\gamma_k, \gamma_l \leq \gamma_N \\ |\gamma_l - \gamma_k| < \delta}} 1.$$

Then, in view of (1) and Lemma 3,

$$\sum_{k=1}^N N_\delta(h\gamma_k) = \sum_{\substack{\gamma_k, \gamma_l \leq \gamma_N \\ |\gamma_k - \gamma_l| < c(\log \gamma_N)^{-1}}} 1 \ll \gamma_N \log \gamma_N \ll N.$$

This, (6) and an application of the Gallagher lemma connecting discrete and continuous mean squares for some function, see Lemma 1.4 of [27], give

$$\begin{aligned} & \sum_{k=1}^N \left| \zeta \left(\frac{1}{2} + \varepsilon + ih\gamma_k + it; \mathfrak{a} \right) \right| \\ & \leq \left(\sum_{k=1}^N N_\delta(h\gamma_k) \sum_{k=1}^N N_\delta^{-1}(h\gamma_k) \left| \zeta \left(\frac{1}{2} + \varepsilon + ih\gamma_k + it; \mathfrak{a} \right) \right|^2 \right)^{1/2} \\ & \ll N^{1/2} \left(\frac{1}{\delta} \int_{h\gamma_1}^{h\gamma_N} \left| \zeta \left(\frac{1}{2} + \varepsilon + i\tau + it; \mathfrak{a} \right) \right|^2 d\tau \right. \\ & \quad \left. + \left(\int_{h\gamma_1}^{h\gamma_N} \left| \zeta \left(\frac{1}{2} + \varepsilon + i\tau + it; \mathfrak{a} \right) \right|^2 d\tau \int_{h\gamma_1}^{h\gamma_N} \left| \zeta' \left(\frac{1}{2} + \varepsilon + i\tau + it; \mathfrak{a} \right) \right|^2 d\tau \right)^{1/2} \right) \\ & \ll_{\varepsilon, h} N(1 + |t|). \end{aligned}$$

Therefore, the classical estimate for the gamma-function and the definition of $l_n(s)$ show that

$$I \ll_{\varepsilon, h, K} n^{-\varepsilon} \quad \text{and} \quad Z \ll_{h, K} n^{1/2-2\varepsilon} \frac{\log N}{N}.$$

This, together with (7), proves (5), thus (4).

Proof of Theorem 3. We will use the random element language. Denote by $\underline{X}_n = \underline{X}_n(s)$ the $H^r(D)$ -valued random element having the distribution V_n , where V_n is the limit measure in Lemma 6. Then, by Lemma 7,

$$\underline{X}_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P_{\underline{\zeta}}, \quad (9)$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. Now, let the random variable η_N be defined on a certain probability space with a measure μ , and

$$\mu\{\eta_N = \gamma_k\} = \frac{1}{N}, \quad k = 1, \dots, N.$$

Define the $H^r(D)$ -valued random element

$$\underline{X}_{N,n} = \underline{X}_{N,n}(s) = \underline{\zeta}_n(s + i\hbar\eta_N; \underline{a}).$$

Then, in virtue of Lemma 7,

$$\underline{X}_{N,n} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \underline{X}_n.$$

Let

$$\underline{Y}_N = \underline{Y}_N(s) = \underline{\zeta}(s + i\hbar\eta_N; \underline{a}).$$

Then Lemma 8 implies that, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \mu\{\rho(\underline{Y}_n(s), \underline{X}_{N,n}(s)) \geq \varepsilon\} \\ & \leq \lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N\varepsilon} \sum_{k=1}^N \rho(\underline{\zeta}(s + i\hbar\gamma_k; \underline{a}), \underline{\zeta}_n(s + i\hbar\gamma_k; \underline{a})) = 0. \end{aligned}$$

Therefore, this, (9), (10) and Theorem 4.2 of [4] show that $\underline{Y}_N \xrightarrow[N \rightarrow \infty]{P_N} P_\zeta$ and the theorem is proved.

4 Proof of Theorem 2

We start with the explicit form of the support of the measure $P_{\underline{\zeta}}$. Recall that the support of a probability measure P is a minimal closed set S such that $P(S) = 1$.

$$\text{Let } S = \{g \in H(D): g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Lemma 9.

The support of the measure P_ζ is the set S^r .

Proof. The space $H^r(D)$ is separable. Therefore [4],

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_r.$$

From this it follows that it suffices to consider the measure P_ζ on the rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in \mathcal{B}(H(D)).$$

Denote by m_{jH} the Haar measure on Ω_j , $j = 1, \dots, r$. Then the Haar measure m_H is the product of the measures m_{1H}, \dots, m_{rH} . These remarks imply the equality

$$\begin{aligned} P_{\underline{\zeta}}(A) &= m_H^r \{ \omega \in \Omega^r : \underline{\zeta}(s, \omega; \underline{a}) \in A \} \\ &= m_{1H} \{ \omega_1 \in \Omega_1 : \zeta(s, \omega_1; \mathbf{a}_1) \in A_1 \} \\ &\quad \cdots m_{rH} \{ \omega_r \in \Omega_r : \zeta(s, \omega_r; \mathbf{a}_r) \in A_r \}. \end{aligned} \quad (11)$$

It is known [19] that the support of

$$P_{\underline{\zeta}_j}(A_j) = m_{jH} \{ \omega_j \in \Omega_j : \zeta(s, \omega_j; \mathbf{a}_j) \in A_j \}, \quad j = 1, \dots, r,$$

is the set S . Therefore, (11) and the minimality of the support prove the lemma.

Proof of Theorem 2. The theorem is corollary of Theorem 3, the Mergelyan theorem on the approximation of analytic functions by polynomials [25], and Lemma 9, and it is standard. By the Mergelyan theorem, there exist polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}. \quad (12)$$

In view of Lemma 9, the set

$$G_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \right\}$$

is an open neighbourhood of an element of the support of the measure $P_{\underline{\zeta}}$. Hence,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \quad (13)$$

Therefore, by Theorem 3 and the equivalent of weak convergence of probability measures in terms of open sets,

$$\liminf_{N \rightarrow \infty} P_N(G_\varepsilon) \geq P_{\underline{\zeta}}(G_\varepsilon) > 0.$$

This, the definitions of P and G , together with inequality (12), prove the first part of the theorem.

For the proof of the second part of the theorem, we define one more set

$$\hat{G}_\varepsilon = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then \hat{G}_ε is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$, moreover, in view of (12), the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$ is valid. Therefore, Theorem 3, the equivalent of weak convergence of probability measures in terms of continuity sets and (13) lead the inequality

$$\lim_{N \rightarrow \infty} P_N(\hat{G}_\varepsilon) = P_\zeta(\hat{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This, the definitions of P_N and \hat{G}_ε prove the second part of the theorem.

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