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Articles

Finite-time ruin probability of a perturbed risk model with dependent main and delayed claims.

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Abstract: This paper considers a delayed claim risk model with stochastic return and Brownian perturbation in which each main claim may be accompanied with a delayed claim occurring after a stochastic period of time, and the price process of the investment portfolio is described as a geometric Lévy process. By means of the asymptotic results for randomly weighted sum of dependent subexponential random variables we obtain some asymptotics for finite-time ruin probability. A simulation study is also performed to check the accuracy of the obtained theoretical result via the crude Monte Carlo method.
Keywords: finite-time ruin probability, main and delayed claims, stochastic return, subexponential distribution, dependence.

1 Introduction

Consider a renewal risk model with main and delayed claims in which, for each positive integer i, an insurer's ith main claim X_i occurs at time τ_i accompanied with a delayed claim Y_i occurring at time $\tau_i + D_i$, where D_i denotes an uncertain delay time. Let $\{(X_i, Y_i), i \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) non-negative random vectors with generic random vector (X, Y) and marginal distributions F and G, respectively. The accident arrival times $\{\tau_i, i \geqslant 1\}$ constitute a renewal counting process

$$N(t) = \sup\{n \geqslant 0\colon \tau_n \leqslant t\}, \quad t \geqslant 0,$$

with a finite mean function $\lambda(t) = \mathbf{E}[N(t)]$, and denote the inter-arrival times by $\theta_i = \tau_i - \tau_{i-1}, i \geqslant 1$, with $\tau_0 = 0$, which are i.i.d. nonnegative random variables (r.v.s). The delay times $\{D_i, i \geqslant 1\}$ are a sequence of identically distributed and nonnegative (possibly degenerate at 0) r.v.s with common distribution H. The insurer is allowed to invest its surplus into a risk-free market. The price process of the investment portfolio is described by a geometric Lévy process $\{e^{R_t}, t \geqslant 0\}$. Here $\{R_t, t \geqslant 0\}$ is a nonnegative Lévy process, also representing the stochastic accumulated return rate process, which starts from zero and has independent and stationary increments.

For more discussions on Lévy processes, see [1,4,22]. Then the discounted value of the surplus process with stochastic return on investment can be defined as

$$U(t) = x + \int_0^t c(s)\mathrm{e}^{-R_s}\,\mathrm{d}s - \sum_{i=1}^{N(t)} X_i \mathrm{e}^{-R_{\tau_i}} - \sum_{i=1}^{\infty} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant t\}} + \delta \int_0^t \mathrm{e}^{-\tilde{R}_s} B(\mathrm{d}s), \quad t \geqslant 0,$$

where I_A denotes the indicator function of a set A, $x \geqslant 0$ is the initial risk reserve of the insurer, $c(t) \geqslant 0$ is the density function of premium income at time t, $\delta \geqslant 0$ is the volatility factor, $\tilde{R}_t$, $t \geqslant 0$, is another nonnegative Lévy process representing the stochastic interest process, and $B(t)$, $t \geqslant 0$, is the diffusion perturbation, which is a standard Brownian motion. As usual, assume that $\{(X_i, Y_i), i \geqslant 1\}$, $\{\theta_i, i \geqslant 1\}$, $\{D_i, i \geqslant 1\}$, $\{R_t, t \geqslant 0\}$, $\{\tilde{R}_t, t \geqslant 0\}$, and $B(t)$, $t \geqslant 0$ are mutually independent, but some certain dependence may exist within each pair (X_i, Y_i). Additionally, assume that the premium density function $c(t)$ is bounded, i.e., $0 \leqslant c(t) \leqslant c_0$ for some $c_0 > 0$ and all $t \geqslant 0$. For any fixed time $t \geqslant 0$, the finite-time ruin probability of risk model (1) can be defined as

$$\psi(x;t) = \mathbf{P}\Big(\inf_{0 \leqslant s \leqslant t} U(s) < 0 \;\Big|\; U(0) = x\Big),$$

and the corresponding infinite-time ruin probability is $\lim_{t\to\infty} \psi(x;t) = \psi(x)$.

Such a risk model (1) has been playing an important role in insurance practice since a severe accident may trigger more than one claim. First is the main claim caused immediately, while all the others are accumulated as another type, called as the delayed claim, occurring after a stochastic period of time. For example, a traffic accident may cause an immediate payoff for vehicle damage, as well as some medical claims for injures of both drivers and passengers in the subsequent periods. In this paper, we study the asymptotic expression for the finite-time ruin probability, which has immediate implications under modern insurance regulatory frameworks such as solvency capital requirement and insurance risk management.

The ruin probabilities of risk model (1) were initially studied by [28],who considered a Poisson accident-number process $N(t)$ and some light-tailed claims and established an exact formula for $\psi(x)$ without perturbation and investment (i.e., $\delta = 0$ and $R_t = 0$) in (1) by a martingale approach. In the presence of heavy-tailed claims, [15] studied the model with two deterministic linear functions for the premium income process and the stochastic accumulated return rate process (i.e., $c(t) = c$ and $R_t = rt$ for premium rate $c > 0$ and interest rate $r > 0$); [16] considered the case of $c(t) = c$ and $R_t = 0$; and both of these two literatures derived the asymptotic relation for $\psi(x)$. A few extensions with dependence structures and stochastic returns can be found in [9–11,27], who investigated the asymptotic behavior for both $\psi(x;t)$ and $\psi(x)$. Some related results in bidimensional risk models can be found in [2,3,26], among others. Remark that all the above works are restricted to some extremely heavy-tailed claims such as the ones with regularly varying tails or consistently varying tails. Recently, by using the asymptotics for the tailprobability of randomly weighted sum of i.i.d.subexponentialr.v.s [25] established an asymptotic formula for the finite-time ruin probability $\psi(x;t)$ under the independent model (1) with some moderately heavy-tailed (subexponential) claims, but $c(t) = c$, $R_t = rt$, and $\delta = 0$.

In this paper, we continue to seek the asymptotic behavior for the finite-time ruin probability in a more general risk model (1) with subexponential claims, risk-free investment, and diffusion perturbation, where each pair of the main and delayed claims may be interdependent to some extent. Our obtained results also confirm the intuition that the asymptotic finite-time ruin probability of risk model (1) with subexponential claims is insensitive to the Brownian perturbation, which coincide with the results of the models without delayed claims in [17] and [24]. Our adopting method is the tail asymptotics for the randomly weighted sum of dependent subexponential r.v.s, which may be interesting on its own right.

The rest of this paper consists of four sections. Section 2 states the main result after introducing some necessary preliminaries, and Section 3 performs a simulation study to check the accuracy of the theoretical result. Section 4 establishes some asymptotic formulas for the tail probability of finite randomly weighted sum generated by dependent subexponential r.v.s. The proof of the main result is postponed to Section 5.

2 Preliminaries and main results

Throughout the paper, all limit relationships hold as $x \to \infty$ unless stated otherwise. For two positive functions $f(\cdot)$ and $g(\cdot)$, write $f(x) \lesssim g(x)$ if lim sup $f(x)/g(x) \leqslant 1$, write $f(x) \sim g(x)$ if lim $f(x)/g(x) = 1$, write f $f(x) = o(g(x))$ if lim $f(x)/g(x) = 0$, write $f(x) = O(g(x))$ if lim sup $f(x)/g(x) < \infty$, and write $f(x) \asymp g(x)$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$. For two real-valued numbers *x* and *y*, denote by $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and denote the positive and negative parts of *x*, respectively, *by x* $x^{+} = x \vee$ *0 and* $x^{-} = -x \wedge$ *0.* The indicator function of a set A is denoted by 1_A . Furthermore, for two positive bivariate functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$, we write $f(x,t) \sim g(x,t)$ uniformly for all t in a nonempty set A if

$$\lim_{x\to\infty} \sup_{t\in A} \left| \frac{f(x,t)}{g(x,t)} - 1 \right| = 0;$$

and write $f(x,t) \lesssim g(x,t)$ uniformly for all $t \in A$ if

$$\limsup_{x\to\infty}\sup_{t\in A}\frac{f(x,t)}{g(x,t)}\leqslant 1.$$

Denote by F_ξ the distribution of a *r.v.* ξ letting the notation speak for itself. For two real- valued r.v.s ξ and η, we say that ξ *ξ* is stochastically not greater than *η*, denoted by $\xi\leqslant_{st}\eta$, if $\mathbf{P}(\xi>x)\leqslant\mathbf{P}(\eta>x)$ for all $x\in\mathbb{R}$.

2.1 Heavy-tailed distributions

It is well known that most insurance claims possess the heavy-tailed feature since many insurance data are characterised by right heavy-tailedness; see [5,6,21]. We will use heavy-tailed distributions to model the claims. A distribution *V on* $\mathbb{R}_+=[0,\infty)$ is said to be subexponential, denoted by $V\in\mathcal{S}$, if $\overline{V}(x)=1-V(x)>0$ for all $x\geqslant 0$ and $\overline{V^{n*}}(x)\sim n\overline{V}(x)$ holds for all (or, equivalently, for some) $n\geqslant 2$, where V^{n*} is then-fold convolution of *V*. More generally, a distribution V on $\mathbb{R}$ is still said to be subexponential if the distribution $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ is subexponential. The class *S* contains a lot of important distributions such as Pareto, lognormal, and heavy-tailed Weibull distributions. Clearly, by Lemma 1.3.5(a) of [7], if a distribution *V* on $\mathbb{R}$ is subexponential, then it holds that

$$\overline{V}(x+y)\sim\overline{V}(x)$$

for all $y\in\mathbb{R}$, which defines the class of long-tailed distributions denoted by $\mathcal{L}$. Automatically, relation (3) holds uniformly on every compact set of *y*. Hence, it is easy to see that there exists some positive function $h(\cdot)$, with $h(x)=o(x)$ and $h(x)\uparrow\infty$, such that relation (3) holds uniformly for all | $|y|\leqslant h(x)$. An important subclass of *S* is that of regularly varying tailed distributions. A distribution V on $\mathbb{R}$ is said to be regularly varying tailed with index $\alpha>0$, denoted by $V\in\mathcal{R}_{-\alpha}$, if $\overline{V}(xy)\sim y^{-\alpha}\overline{V}(x)$ for all $y>0$. A typical example is the Pareto distribution

$$V(x)=1-(x+\mu)^{-\alpha},\quad x\geqslant 1-\mu,$$

with parameters $\alpha>0$ and $\mu>0$. The reader is referred to monographs [7] and [8] for reviews of some related heavy-tailed distributions.

2.2 Main results

Our main result is established under the following assumptions, which describe some weak dependence among variables.

Assumption 1. Suppose that real-valued r.v.s $\xi_1,\dots,\xi_n$ satisfy the relation

$$\lim_{x_i \wedge x_j \to \infty} \mathbf{P}\left(|\xi_i| > x_i \mid \xi_j > x_j\right) = 0$$

for all $1 \leqslant i \neq j \leqslant n$.

This concept is related to what is called asymptotic independence, see, e.g., [18], and indicates that neither too positively nor too negatively can ξ_i and ξ_j be dependent.

Assumption 2. Suppose that for real-valued r.v.s $\xi_1, \ldots, \xi_n$, there exist two positive constants x_0 and M such that

$$\mathbf{P}\left(|\xi_i| > x_i \mid \xi_j = x_j \text{ with } j \in J\right) \leqslant M\mathbf{P}(\xi_i > x_i)$$

holds for all $i = 1, \ldots, n, \emptyset \neq J \subset \{1, \ldots, n\} \setminus \{i\}$, and $x_i \wedge x_j \geqslant x_0 \text{ with } j \in J$.

When x_j is not a possible value of ξ_j, i.e., $\mathbf{P}(\xi_j \in \Delta) = 0$ for some open set Δ containing x_j, the conditional probability in Assumption 2 is understood as 0. This dependence structure was introduced by [12] and is related to the so-called negative (or positive) regression dependence proposed by [14]. As pointed by [12], if r.v.s $\xi_1, \ldots, \xi_n$ follow a joint n-dimensional Farlie–Gumbel–Morgenstern (FGM) distribution of the form

$$F(x_1, \ldots, x_n) = \prod_{k=1}^{n} V_k(x_k)\left(1 + \sum_{1 \leqslant i < j \leqslant n} \gamma_{ij} \overline{V_i}(x_i)\overline{V_j}(x_j)\right),$$

where γ_{ij} are real-valued numbers such that $F(x_1, \ldots, x_n)$ is a proper *n*-dimensional distribution, $V_1, \ldots, V_n$ are absolutely continuous marginal distributions satisfying $V_k(-x) = o(\overline{V_k}(x)),\ k = 1, \ldots, n,$ then Assumptions 1 and 2 are both satisfied. In addition, Assumption 2 implies Assumption 1.

Now we are ready to state our main result in which Assumption 2 is satisfied for $\xi_1 = X$ and $\xi_2 = Y$ with $n = 2$ and remark that the delay times $D_i, i \geqslant 1$, can be arbitrarily dependent.

Theorem 1. *Consider the risk model (1) in which the generic random vector* (X, Y) *satisfies Assumption 2 with* $F \in \mathcal{S}$. *Let* $T > 0$ *be any fixed time such that* $\mathbf{P}(\tau_1 \leqslant T) > 0$.

(i) *If* $G \in \mathcal{S}$ *and* $\overline{G}(x) \asymp \overline{F}(x)$, *then*

$$\psi(x;T) \sim \int_{0-}^{T} \mathbf{P}\left(X\mathrm{e}^{-R_s} > x\right) \lambda(\mathrm{d}s) + \int_{0-}^{T} \mathbf{P}\left(Y\mathrm{e}^{-R_s} > x\right) (\lambda * H)(\mathrm{d}s),$$

Where $(\lambda * H)(s) = \int_{0-}^{s} H(s-t)\, \lambda(\mathrm{d}t)$.

(ii) *If* $\overline{G}(x) = o(\overline{F}(x))$, *then*

$$\psi(x;T) \sim \int_{0^-}^{T} \mathbf{P}\big(X e^{-R_s} > x\big)\,\lambda(\mathrm{d}s).$$

We remark that in Theorem 1, the condition of (X,Y) satisfying Assumption 2 can be reduced to

$$\mathbf{P}\big(|X| > x \mid Y = y\big) \leqslant M\overline{F}(x), \qquad \mathbf{P}\big(|Y| > x \mid X = y\big) \leqslant M\overline{G}(x)$$

when $x \wedge y \geqslant x_0$ for some $M > 0$ and $x_0 > 0$. The following corollary is a simplified version of Theorem 1.

Corollary 1. *Under the conditions of Theorem 1,assume that* $F \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0$ *, the process* $\{N(t), t \geqslant 0\}$ *is a homogeneous Poisson process with intensity* $\lambda > 0$ *, and the distribution H of the delay time is exponentially distributed with intensity* $\lambda_H > 0$ *. Let* $T > 0$ *be any fixed time such that* $\mathbf{P}(\tau_1 \leqslant T) > 0$ *.*

(i) If $G \in \mathcal{R}_{-\alpha}$ *and* $\overline{G}(x) \asymp \overline{F}(x)$ *, then*

$$\psi(x;T) \sim \lambda\left(\frac{e^{T\phi(\alpha)} - 1}{\phi(\alpha)}\overline{F}(x) + \frac{e^{T(\phi(\alpha)-\lambda_H)} - 1}{\lambda_H - \phi(\alpha)}\overline{G}(x)\right)$$

Where $\phi(z) = \log \mathbf{E}[e^{-zR_1}]$.

(ii) *If* $\overline{G}(x) = o(\overline{F}(x))$ *, then*

$$\psi(x;T) \sim \frac{\lambda(e^{T\phi(\alpha)} - 1)}{\phi(\alpha)}\overline{F}(x).$$

3 A simulation study

In this section, we use some numerical simulations to verify the accuracy of the asymptotic result for $\psi(x;T)$ in Corollary 1. To this end, via the crude Monte Carlo (CMC) method we compare the simulated ruin probability $\psi(x;T)$ in (2) with the asymptotic one on the right-hand side of (6).

Throughout this section, model specifications for the numerical studies are listed below:

- The main and delayed claims X and Y are modelled by a bivariate FGM distribution of (5), which can be reduced to $\mathbf{P}(X \leqslant x, Y \leqslant y) = F(x)G(y)(1 + \gamma\overline{F}(x)\overline{G}(y))$ with parameter $\gamma \in [-1,1]$; and their marginal distributions are identical to a Pareto distribution (4) with parameters $\alpha > 0$ and $\mu > 0$. Clearly, $F = G \in \mathcal{R}_{-\alpha}$.
- The accident arrival counting process $\{N(t), t \geqslant 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$. That is, the accident inter-

arrival times $\{\theta_i, i \geqslant 1\}$ are i.i.d. nonnegative r.v.s with a common exponential distribution having parameter $\lambda > 0$.

- The delay times $\{D_i, i \geqslant 1\}$ are i.i.d. nonnegative r.v.s with a common exponential distribution having parameter $\lambda_H > 0$.
- The stochastic accumulated return rate process $\{R_t, t \geqslant 0\}$ is specialised to

$$R_t = r_0 t + \sum_{i=1}^{M(t)} Z_i,$$

where $r_0 \geqslant 0$ is a constant, $\{M(t), t \geqslant 0\}$ is a homogeneous Poisson process with intensity $\rho > 0$, and $\{Z_i, i \geqslant 1\}$ are i.i.d. nonnegative r.v.s; see a similar discussion in [13]. Clearly, such an R_t constitutes a nonnegative Lévy process [4, Prop. 3.10], it can be calculated that in Corollary 1,

$$\phi(z) = \log \mathbf{E}\left[\mathrm{e}^{-zR_1}\right] = -r_0 z + \left(\mathbf{E}\left[\mathrm{e}^{-zZ_1}\right] - 1\right)\rho.$$

Further, assume that Z_1 is uniformly distributed on [0,1].

- The stochastic interest process $\{\tilde{R}_t, t \geqslant 0\}$ reduces to $\tilde{R}_t = rt$ with constant interest rate $r > 0$).

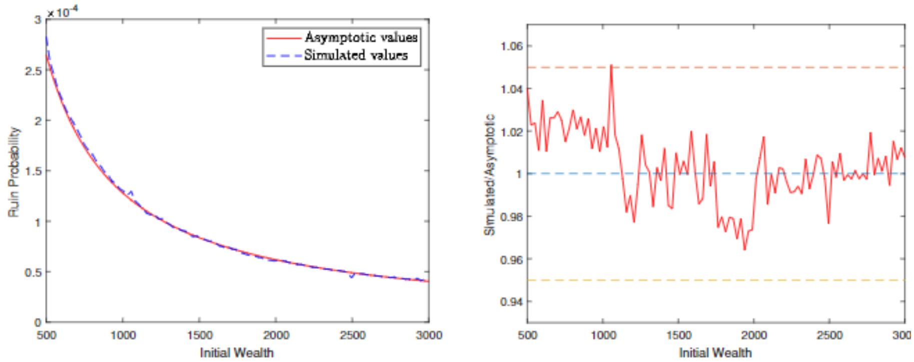
The various parameters are set to:

$$T = 10, \quad c(s) = 1, \quad \delta = 1, \quad r = 1,$$
$$\gamma = 0.5, \quad \alpha = 1.2, \quad \mu = 1,$$
$$\lambda = 1, \quad \lambda_H = 0.5, \quad r_0 = 1, \quad \rho = 1.$$

For the simulated estimation $\hat{\psi}(x;T)$, we first divide the time interval [*0,T*] into *n* parts, and for the given $t_k = kT/n, k = 1, \ldots, n$, we generate m samples $N^{(j)}(t_k)$, $j = 1, \ldots, m$. Then, for each $j = 1, \ldots, m$, generate $N^{(j)}(t_k)$ pairs of $(X_i^{(j)}, Y_i^{(j)})$, the accident inter-arrival times $\theta_i^{(j)}$, and the delay times $D_i^{(j)}, i = 1, \ldots, N^{(j)}(t_k)$For each $j = 1, \ldots, m$, generate $R_{t_k}^{(j)}$ according to (7). Thus, the discounted value of the surplus process $U^{(j)}(t_k)$ can be calculated according to (1). In this way, the ruin probability $\hat{\psi}(x;T)$ can be estimated by

$$\hat{\psi}(x;T) = \frac{1}{m} \sum_{j=1}^{m} \mathbf{1}_{\{\bigwedge_{k=1}^{n} U^{(j)}(t_k) < 0\}}.$$

In Fig. 1, we compare the CMC estimate $\hat{\psi}(x;T)$ in (8) with the asymptotic value given by (6) on the left and show their ratio on the right.



(a) (b)

Figure 1
Comparison between the simulated and asymptotic values of finite-time ruin probability (a) and their ratio (b).

The CMC simulation is conducted with the sample size $m = 7 \times 10^6$, the time step size $T/n = 10^{-3}$ with $n = 10^4$,and the initial wealth x from 500 to 3000. From Fig.1 it can be seen that with the increase of the initial wealth x, both estimates decrease gradually and the two lines get closer. In addition, the ratio of the simulated and asymptotic values for the finite-time ruin probability are close to 1. The fluctuation is due to the poor performance of the CMC method, which requires a sufficiently large sample size to meet the demands of high accuracy.

4 Tail behavior of randomly weighted sum

In this section, we investigate the asymptotic tail probability of finite randomly weighted sum generated by dependent subexponential r.v.s, which plays an important role in proving our main result and may be interesting on its own right. Before giving the results, we firstly introduce a series of lemmas. In the sequel, let $(\xi_1^*, \dots, \xi_n^*)$ be a random vector with independent components and the same marginal distributions as those of $(\xi_1, \dots, \xi_n)$, which is independent of all the other random sources.

Lemma1.

(i) *If* $V_1 \in \mathcal{S}$, $V_2 \in \mathcal{L}$, *and* $\overline{V}_1(x) \asymp \overline{V}_2(x)$, *then* $V_1 * V_2 \in \mathcal{S}$.

(ii) *If* $V_1 \in \mathcal{S}$ *and* $\overline{V_2}(x) = o(\overline{V_1}(x))$, *then* $V_1 * V_2 \in \mathcal{S}$ *and* $\overline{V_1 * V_2}(x) \sim \overline{V_1}(x)$.

(iii) *Let* (ξ_1, ξ_2) *be a real-valued random vector with marginal distributions* V_1 *and* V_2*,respectively, and satisfying Assumption 2 with* $n = 2$ *. If* $V_1 \in \mathcal{S}$ *and* $\overline{V_2}(x) = o(\overline{V_1}(x))$, *then* $F_{\xi_1+\xi_2} \in \mathcal{S}$ *and*

$$\mathbf{P}(\xi_1 + \xi_2 > x) \sim \overline{V_1}(x).$$

Proof. The proofs of parts (i) and (ii) are referred to Theorem 3.11 (or Corollary 3.16) and Corollary 3.18 of [8], respectively.

(iii) It is easy to see that $F_{\xi_1+\xi_2} \in \mathcal{S}$ if (9) holds. By $V_1 \in \mathcal{S} \subset \mathcal{L}$ there exists a function $h(x) \uparrow \infty$ such that $h(x) = o(x)$, and

$$\overline{V_1}\big(x + yh(x)\big) \sim \overline{V_1}(x)$$

holds for any fixed $y \in \mathbb{R}$.

On the one hand, for sufficiently large *x*, according to ξ_2 belonging to $(-\infty, h(x)]$, $(h(x), x - h(x)]$, and $(x - h(x), \infty)$, we divide the tail probability $\mathbf{P}(\xi_1 + \xi_2 > x)$ into three parts denoted by $I_1, I_2,$ and I_3. By (10) and $\overline{V_2}(x) = o(\overline{V_1}(x))$ we have

$$I_1 \leqslant \overline{V_1}\big(x - h(x)\big) \sim \overline{V_1}(x)$$

and

$$I_3 \leqslant \overline{V_2}\big(x - h(x)\big) = o\big(\overline{V_1}(x)\big).$$

As for I_2, by Assumption 2, for sufficiently large *x*,

$$\begin{aligned} I_2 &= \int_{h(x)}^{x-h(x)} \mathbf{P}\big(\xi_1 > x - u \mid \xi_2 = u\big)\, V_2(\mathrm{d}u) \\ &\leqslant M\mathbf{P}\big(\xi_1^* + \xi_2^* > x,\ h(x) < \xi_2^* \leqslant x - h(x)\big) \\ &= M\big(\mathbf{P}(\xi_1^* + \xi_2^* > x) - \mathbf{P}\big(\xi_1^* + \xi_2^* > x,\ \xi_2^* \leqslant h(x)\big) \\ &\quad - \mathbf{P}\big(\xi_1^* + \xi_2^* > x,\ \xi_2^* > x - h(x)\big)\big) \\ &=: M(I_{21} - I_{22} - I_{23}). \end{aligned}$$

Clearly, part (ii) gives $I_{21} \sim \overline{V_1}(x)$, and similarly to (12), $I_{23} = o(\overline{V_1}(x))$. According to the dominated convergence theorem and $V_1 \in \mathcal{S} \subset \mathcal{L}$, we have $I_{22} \sim \overline{V_1}(x)$. Thus,

$$I_2 = o\big(\overline{V_1}(x)\big).$$

Combining (11)–(13), we obtain the upper bound

$$\mathbf{P}(\xi_1 + \xi_2 > x) \lesssim \overline{V_1}(x).$$

On the other hand,

$$\begin{aligned}\mathbf{P}(\xi_1+\xi_2>x) &\geqslant \mathbf{P}\big(\xi_1+\xi_2>x,\ \xi_2\geqslant -h(x)\big)\\ &\geqslant \mathbf{P}\big(\xi_1>x+h(x),\ \xi_2\geqslant -h(x)\big)\\ &= \mathbf{P}\big(\xi_1>x+h(x)\big)-\mathbf{P}\big(\xi_1>x+h(x),\ \xi_2<-h(x)\big)\\ &=: J_1-J_2.\end{aligned}$$

It is easy to see that $J_1\sim\overline{V_1}(x)$, and further, by Assumption 2 and (10),

$$\begin{aligned}J_2 &= \int_{x+h(x)}^{\infty}\mathbf{P}\big(\xi_2<-h(x)\mid \xi_1=u\big)\,V_1(\mathrm{d}u)\\ &\leqslant M\mathbf{P}\big(\xi_2>h(x)\big)\overline{V_1}\big(x+h(x)\big)\\ &= o\big(\overline{V_1}(x)\big).\end{aligned}$$

Therefore, the lower bound

$$\mathbf{P}(\xi_1+\xi_2>x)\gtrsim\overline{V_1}(x)$$

is derived.

Lemma 2. *Let* $(\xi_1,\dots,\xi_n)$ *be n real-valued r.v.s with distributions* $V_1,\dots,V_n$ *, respectively. If Assumption 1 is satisfied, then for every set* $\emptyset\neq I\subsetneq\{1,\dots,n\}$ *, every* $j\in\{1,\dots,n\}\setminus I$ *, and any* $b>0$ *,*

$$\lim_{x\wedge y\to\infty}\ \sup_{c_i\in[0,b],\ i\in I}\mathbf{P}\bigg(\bigg|\sum_{i\in I}c_i\xi_i\bigg|>x\ \bigg|\ \xi_j>y\bigg)=0.$$

Proof. Clearly.

$$\sup_{c_i\in[0,b],\,i\in I}\mathbf{P}\bigg(\bigg|\sum_{i\in I}c_i\xi_i\bigg|>x\ \bigg|\ \xi_j>y\bigg)\leqslant\sum_{i\in I}\mathbf{P}\bigg(|\xi_i|>\frac{x}{bn}\ \bigg|\ \xi_j>y\bigg),$$

which tends to 0 as $x\wedge y\to\infty$ by Assumption 1.

The next lemma can be derived by using Lemma 2 and the arguments in the proof of Lemma 4.3 of [12]. We omit its detailed proof.

Lemma 3. *Let* $(\xi_1,\dots,\xi_n)$ *be . real-valued r.v.s with distributions* $V_1,\dots,V_n$ *, respectively. If Assumption 1 issatisfied and* $V_i\in\mathcal{L},\ i=1,\dots,n$ *, then for any* $0<a\leqslant b<\infty$ *and uniformly for all* $c_i\in[a,b],\ i=1,\dots,n$ *,*

$$\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \gtrsim \sum_{i=1}^{n} \mathbf{P}(c_i\xi_i > x).$$

Along the similar line of the proofs of Lemmas 5.1 and 5.2 in [12], we can obtain the following two lemmas in which the uniformity for all $c_i \in [a,b],\ i=1,\dots,n$, holds by addressing the uniform convergence for the weighted sum with independent subexponential summands and nonrandom weights; see, e.g., Lemma 1 of [23].

Lemma 4. *Let* $(\xi_1,\dots,\xi_n)$ *be n real-valued r.v.s with distributions* $V_1,\dots,V_n$, *respectively. If Assumption 2 is satisfied,* $V_i \in \mathcal{L}$ *and* $\overline{V_i}(x) \asymp \overline{V}(x)$ *for some distribution* $V \in \mathcal{S},\ i=1,\dots,n$, *then there exist two positive numbers* x_0 *and* d_n *such that for any* $0 < a \leqslant b < \infty$ *and each* $k=1,\dots,n$,

$$\mathbf{P}\left(\sum_{i=1, i\neq k}^{n} c_i\xi_i > x \,\middle|\, \xi_k = x_k\right) \leqslant d_n \mathbf{P}\left(\sum_{i=1, i\neq k}^{n} c_i\xi_i^* > x\right)$$

holds for all $x \wedge x_k \geqslant x_0$ and $c_i \in [a,b],\ i=1,\dots,n$.

Lemma 5. *Let* $(\xi_1,\dots,\xi_n)$ *be n independent and nonnegative r.v.s with distributions* $V_1,\dots,V_n$, *respectively. If* $V_i \in \mathcal{L}$ *and* $\overline{V_i}(x) \asymp \overline{V}(x)$ *for some distribution* $V \in \mathcal{S}$, $i=1,\dots,n$, *then, for any function* $h(x)$ *satisfying* $h(x) < x$ *and* $h(x) \uparrow \infty$, *any* $k=1,\dots,n$, *and* $0 < a \leqslant b < \infty$, *it holds that uniformly for all* $c_i \in [a,b],\ i=1,\dots,n$,

$$\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i^* > x,\ h(x) < c_k\xi_k^* \leqslant x\right) = o(1)\sum_{i=1}^{n} \mathbf{P}(c_i\xi_i^* > x).$$

Lemma 6. *Under the conditions of Lemma 4, for* $0 < a \leqslant b < \infty$, *it holds that uniformly for all* $c_i \in [a,b],\ i=1,\dots,n$,

$$\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \lesssim \sum_{i=1}^{n} \mathbf{P}(c_i\xi_i > x).$$

Proof. By noting $\mathbf{P}(\sum_{i=1}^{n} c_i\xi_i > x) \leqslant \mathbf{P}(\sum_{i=1}^{n} c_i\xi_i^+ > x)$ we only prove (14) for nonnegative $(\xi_1,\dots,\xi_n)$. Recalling $h(\cdot)$ in (10), it holds that

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x\right) &\leqslant \sum_{i=1}^{n}\mathbf{P}\big(c_i\xi_i > x - h(x)\big)\\ &\quad + \mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x,\ h(x) < \bigvee_{k=1}^{n} c_k\xi_k \leqslant x - h(x)\right)\\ &=: I_1 + I_2.\end{aligned}$$

By (10), $I_1 \sim \sum_{i=1}^{n}\mathbf{P}(c_i\xi_i > x)$ holds uniformly for all $c_i \in [a,b],\ i = 1,\dots,n$. As for I2, by Lemmas 4 and 5, it holds that for sufficiently large *x* and uniformly for all $c_i \in [a,b],\ i = 1,\dots,n$,

$$\begin{aligned}I_2 &\leqslant \sum_{k=1}^{n}\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x,\ h(x) < c_k\xi_k \leqslant x - h(x)\right)\\ &= \sum_{k=1}^{n}\int_{h(x)/c_k}^{(x-h(x))/c_k}\mathbf{P}\left(\sum_{i=1,\, i\neq k}^{n} c_i\xi_i > x - c_k y \,\middle|\, \xi_k = y\right)V_k(\mathrm{d}y)\\ &\leqslant d_n\sum_{k=1}^{n}\int_{h(x)/c_k}^{(x-h(x))/c_k}\mathbf{P}\left(\sum_{i=1,\, i\neq k}^{n} c_i\xi_i^* > x - c_k y\right)V_k(\mathrm{d}y)\\ &= d_n\sum_{k=1}^{n}\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i^* > x,\ h(x) < c_k\xi_k^* \leqslant x - h(x)\right)\\ &= o(1)\sum_{i=1}^{n}\mathbf{P}(c_i\xi_i > x).\end{aligned}$$

Therefore, the desired relation (14) follows.

Combining Lemmas 3 and 6 gives the first result on the uniform asymptotics for the tail probability of the weighted sum with nonrandom weights.

Proposition 1. *Under the conditions of Lemma 4, for* $0 < a \leqslant b < \infty$*, it holds that uniformly for all* $c_i \in [a,b],\ i = 1,\dots,n$*,*

$$\mathbf{P}\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \sim \sum_{i=1}^{n}\mathbf{P}(c_i\xi_i > x).$$

Let $(\xi_1,\dots,\xi_n)$ be n nonnegative r.v.s satisfying Assumption 2, and $\theta_1,\dots,\theta_n$ be *n* arbitrarily dependent, nondegenerate at 0,andnonnegative r.v.s independent of $(\xi_1,\dots,\xi_n)$.

If $\theta_1,\dots,\theta_n$ are bounded from above, then for each $1 \leqslant i \neq j \leqslant n$,

$$\begin{aligned}\mathbf{P}(\theta_i\xi_i > x,\ \theta_j\xi_j > x) &\leqslant \mathbf{P}\left(\theta_i\xi_i > x,\ \xi_j > \frac{x}{b}\right)\\ &= \int_0^b \int_{x/b}^{\infty} \mathbf{P}\left(\xi_i > \frac{x}{u} \,\middle|\, \xi_j = y\right) \mathbf{P}(\xi_j \in \mathrm{d}y)\, \mathbf{P}(\theta_i \in \mathrm{d}u)\\ &\leqslant M \int_0^b \int_{x/b}^{\infty} \mathbf{P}\left(\xi_i > \frac{x}{u}\right) \mathbf{P}(\xi_j \in \mathrm{d}y)\, \mathbf{P}(\theta_i \in \mathrm{d}u)\\ &= M\mathbf{P}\left(\xi_j > \frac{x}{b}\right)\mathbf{P}(\theta_i\xi_i > x)\\ &= o(1)\mathbf{P}(\theta_i\xi_i > x),\end{aligned}$$

where $b > 0$ is the common upper bound of $\theta_1, \ldots, \theta_n$.

By using (15) and Proposition 1 we can mimic the proof of Theorem 1 of [23] to establish the asymptotic formula for the randomly weighted sum of dependent subexponential and nonnegative summands.

Proposition 2. *Let* $(\xi_1, \ldots, \xi_n)$ *be n nonnegative r.v.s with distributions* $V_1, \ldots, V_n$ *, respectively, and satisfying Assumption 2; and let* $\theta_1, \ldots, \theta_n$ *be n nonnegative r.v.s, which are arbitrarily dependent, bounded from above, nondegenerate at 0, and independent of* $(\xi_1, \ldots, \xi_n)$ *. If* $V_i \in \mathcal{L}$ *and* $\overline{V_i}(x) \asymp \overline{V}(x)$ *for some distribution* $V \in \mathcal{S}, i = 1, \ldots, n$ *, then*

$$\mathbf{P}\left(\sum_{i=1}^{n} \theta_i\xi_i > x\right) \sim \sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x).$$

Now we state the last result, which is an extension of Proposition 2.

Proposition 3. *Assume that all conditions of Proposition 2 are satisfied. Let η be a real- valued r.v. independent of all other sources.* *If* $\mathbf{P}(\eta > x) = o(\overline{V}(x/c))$ *for all* $c > 0,$ *, then*

$$\mathbf{P}\left(\sum_{i=1}^{n} \theta_i\xi_i + \eta > x\right) \sim \sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x).$$

Proof. On the one hand, since θ_i is nondegenerate at *0*, there exists some small $\varepsilon_0 > 0$ such that $\mathbf{P}(\theta_1 \geqslant \varepsilon_0) > 0$. For such ε_0 , by the condition $\mathbf{P}(\eta > x) = o(\overline{V}(x/\varepsilon))$ we have that for any $\varepsilon > 0$, there exists some large x_0 such that $\mathbf{P}(\eta > x) = o(\overline{V}(x/\varepsilon))$ for all $x \geqslant x_0$. Construct a new nonnegative r.*v.* ξ , independent of all other sources, with tail distribution

$$\mathbf{P}(\zeta > x) = \begin{cases} \varepsilon\overline{V}\left(\frac{x}{\varepsilon_0}\right), & x \geqslant x_0, \\ 1, & x < x_0. \end{cases}$$

Clearly, $F_\zeta \in \mathcal{S}$ and $\eta \leqslant_{\mathrm{s.t.}} \zeta$. Then, by Proposition 2,

$$\begin{aligned} \mathbf{P}\left(\sum_{i=1}^{n} \theta_i\xi_i + \eta > x\right) &\leqslant \mathbf{P}\left(\sum_{i=1}^{n} \theta_i\xi_i + \zeta > x\right) \\ &\sim \sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x) + \varepsilon\overline{V}\left(\frac{x}{\varepsilon_0}\right) \\ &\sim \sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x), \end{aligned}$$

by letting $x \to \infty$ then $\varepsilon \downarrow 0$, where in the last step, we used $\overline{V_i}(x) \asymp \overline{V}(x)$ and the fact $\mathbf{P}(\theta_1\xi_1 > x) \geqslant \mathbf{P}(\theta_1\xi_1 > x, \theta_1 \geqslant \varepsilon_0) \geqslant \mathbf{P}(\theta_1 \geqslant \varepsilon_0)\overline{V}(x/\varepsilon_0)$.

On the other hand, according to Fatou's lemma and Proposition 2,

$$\begin{aligned} &\liminf_{x\to\infty} \frac{\mathbf{P}(\sum_{i=1}^{n} \theta_i\xi_i + \eta > x)}{\sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x)} \\ &\geqslant \liminf_{x\to\infty} \frac{\mathbf{P}(\sum_{i=1}^{n} \theta_i\xi_i - \eta^- > x)}{\sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x)} \\ &\geqslant \int_{0-}^{\infty} \liminf_{x\to\infty} \frac{\mathbf{P}(\sum_{i=1}^{n} \theta_i\xi_i > x + u)}{\sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x)} \mathbf{P}(\eta^- \in \mathrm{d}u) \\ &= \int_{0-}^{\infty} \liminf_{x\to\infty} \frac{\sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x + u)}{\sum_{i=1}^{n} \mathbf{P}(\theta_i\xi_i > x)} \mathbf{P}(\eta^- \in \mathrm{d}u) \\ &\geqslant \int_{0-}^{\infty} \liminf_{x\to\infty} \bigwedge_{i=1}^{n} \frac{\mathbf{P}(\theta_i\xi_i > x + u)}{\mathbf{P}(\theta_i\xi_i > x)} \mathbf{P}(\eta^- \in \mathrm{d}u) \\ &= 1, \end{aligned}$$

where in the last step, we used $F_{\theta_i\xi_i} \in \mathcal{L}$ due to Lemma 2 of [23].

Remark 1. Propositions 2 and 3 still hold if not all weights are degenerate at 0.

5 Proofs of main results

Write the last stochastic integral on the right-hand side of (1) as

$$p(t) = \int_0^t \mathrm{e}^{-\tilde{R}_s} B(\mathrm{d}s)$$

and its supremum and infimum as

$$p^*(t) = \sup_{0\leqslant s\leqslant t} p(s) \geqslant 0 \quad \text{and} \quad p_*(t) = \inf_{0\leqslant s\leqslant t} p(s) \leqslant 0$$

for any $t \geqslant 0$. Before proving the main result, we firstly establish two lemmas. The first one gives the distribution of the above supremum and infimum of the stochastic integral, which may be interesting in its own right.

Lemma 7. *Let* $\{\tilde{R}_t, t \geqslant 0\}$ *be a Lévy process, and* $\{B(t), t \geqslant 0\}$ *be a Brownian motion independent of* $\{\tilde{R}_t, t \geqslant 0\}$ *Then, for any fixed* $T > 0$ *and any* $x > 0$*,*

$$\mathbf{P}\left(p^*(T) > x\right) = \mathbf{P}\left(p_*(T) < -x\right) = 2\mathbf{P}\left(Z\sqrt{\xi(T)} > x\right),$$

where Z is a standard normal r.v. independent of $\xi(t) = \int_0^t \mathrm{e}^{-2\tilde{R}_s}\,\mathrm{d}s, t \geqslant 0.$

We remark that if $\tilde{R}_t = rt$ for some $r > 0$), then Lemma 7 reduces to Theorem D.3 (ii) of [20].

Proof of Lemma 7. Since $\{\tilde{R}_t, t \geqslant 0\}$ is independent of $\{B(t), t \geqslant 0\}$, according to Proposition C.2 of [19], the stochastic integral $p(t)$ in (16) is a continuous Ocone martingale; that is, it can be expressed as a time-changed Brownian motion

$$p(t) = W\left([p,p]_t\right) = W\left(\xi(t)\right), \quad t \geqslant 0,$$

for some Brownian motion $\{W(t), t \geqslant 0\}$ independent of $\{\xi(t), t \geqslant 0\}$, where $[p,p]_t$ is the quadratic variation of $p(t)$. It follows from (18) that

$$p^*(T) \overset{\mathrm{d}}{=} -p_*(T) \overset{\mathrm{d}}{=} |Z|\sqrt{\xi(T)},$$

which coincides with (17), where $\overset{\mathrm{d}}{=}$ represents equality in distribution.

Further, if $\{\tilde{R}_t, t \geqslant 0\}$ is a nonnegative Lévy process, then $\xi(T) \leqslant T$ for any fixed $T > 0$. Hence, by Lemma 7 we have that for any $x > 0$,

$$\mathbf{P}\big(\delta p^*(T) > x\big) = \mathbf{P}\big(\delta p_*(T) < -x\big) \leqslant 2\overline{\Phi}\left(\frac{x}{\delta\sqrt{T}}\right),$$

where Φ is the standard Gaussian distribution function. Therefore, by mimicking the proof of Theorem 1.1 of [24] we derive that

Lemma 8. *Let* $\{\xi_i,\ i \geqslant 1\}$ *be a sequence of i.i.d. nonnegative r.v.s with common distribution belonging to S, let* $\{N(t),\ t \geqslant 0\}$ *be a renewal counting process with arrival times* $\tau_1, \tau_2, \dots$ *and mean function* $\lambda(t)$ *, and let* $\{R_t,\ t \geqslant 0\}$ *and* $\{\tilde{R}_t,\ t \geqslant 0\}$ *be two nonnegative Lévy processes. Assume that {* $\{\xi_i,\ i \geqslant 1\}, \{N(t),\ t \geqslant 0\}, \{R_t,\ t \geqslant 0\},$*, are mutually independent. Then, for any* $T > 0$ *such that* $\mathbf{P}(\tau_1 \leqslant T) > 0$*, it holds that*

$$\lim_{m\to\infty} \sup_{x\geqslant 0} \frac{\mathbf{P}(\sum_{i=1}^{N(T)} \xi_i \mathrm{e}^{-R_{\tau_i}} + p^*(T) > x,\ N(T) > m)}{\int_{0-}^{T} \mathbf{P}(\xi_1 \mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)}$$
$$= \lim_{m\to\infty} \sup_{x\geqslant 0} \frac{\sum_{n=m+1}^{\infty} \sum_{i=1}^{n} \mathbf{P}\big(\xi_i \mathrm{e}^{-R_{\tau_i}} > x,\ N(T) = n\big)}{\int_{0-}^{T} \mathbf{P}(\xi_1 \mathrm{e}^{-R_s} > x)\,\lambda(\mathrm{d}s)} = 0.$$

5.1 Proof of Theorem 1

We firstly prove part (i). We deal with the upper bound for $\psi(x;t)$. Choosing some large *m*, we have that for all $x > 0$,

$$\psi(x;T) \leqslant \mathbf{P}\left(\sum_{i=1}^{N(T)} X_i \mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{N(T)} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i\leqslant T\}} + p^*(T) > x\right)$$
$$= \left(\sum_{n=1}^{m} + \sum_{n=m+1}^{\infty}\right) \mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{n} Y_i \mathrm{e}^{-R_{\tau_i+D_i}}\right.$$
$$\left. \times \mathbf{1}_{\{\tau_i+D_i\leqslant T\}} + p^*(T) > x,\ N(T) = n\right)$$
$$=: I_1 + I_2.$$

For I_2, by Proposition *1,* $\mathbf{P}(X+Y>x) \sim \overline{F}(x) + \overline{G}(x) \sim \mathbf{P}(X^* + Y^* > x)$, which implies $F_{X+Y} \in \mathcal{S}$ due to Lemma 1(i), and $\mathbf{P}(X+Y > x) = O(\overline{F}(x))$ due to $\overline{F}(x) \asymp \overline{G}(x)$. Then

$$\int_{0-}^{T} \mathbf{P}\left((X+Y)\mathrm{e}^{-R_s} > x\right) \lambda(\mathrm{d}s)$$
$$= \int_{0-}^{T}\int_{0}^{\infty} \frac{\mathbf{P}(X+Y > x\mathrm{e}^{u})}{\overline{F}(x\mathrm{e}^{u})} \cdot \overline{F}\left(x\mathrm{e}^{u}\right) \mathbf{P}(R_s \in \mathrm{d}u)\, \lambda(\mathrm{d}s)$$
$$= O(1) \int_{0-}^{T} \mathbf{P}\left(X\mathrm{e}^{-R_s} > x\right) \lambda(\mathrm{d}s).$$

This, together with Lemma 8, leads to

$$\lim_{m\to\infty} \limsup_{x\to\infty} \frac{I_2}{\int_{0-}^{T} \mathbf{P}(X\mathrm{e}^{-R_s} > x)\, \lambda(\mathrm{d}s)}$$
$$\leqslant \lim_{m\to\infty} \limsup_{x\to\infty} \frac{\sum_{n=m+1}^{\infty} \mathbf{P}(\sum_{i=1}^{n}(X_i+Y_i)\mathrm{e}^{-R_{\tau_i}} + p^*(T) > x,\ N(T)=n)}{\int_{0-}^{T} \mathbf{P}(X\mathrm{e}^{-R_s} > x)\, \lambda(\mathrm{d}s)}$$
$$= 0.$$

Now we turn to I_1. For each $1 \leqslant n \leqslant m$,

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{n} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T\}} + p^*(T) > x,\ N(T)=n\right)$$
$$\leqslant \mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} + \sum_{i=1}^{n} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T,\, N(T)=n\}} + p^*(T) > x\right).$$

If $\mathbf{P}(N(T)=n) > 0$, then the first n weights are nondegenerate at 0. Note that by (19) and $F \in \mathcal{S}$ we have that for all $c > 0$,

$$\mathbf{P}\left(p^*(T) > x\right) = o\left(\overline{F}\left(\frac{x}{c}\right)\right).$$

Then, according to Proposition 3 and Remark 1,

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{n} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T\}} + p^*(T) > x,\ N(T)=n\right)$$
$$\lesssim \sum_{i=1}^{n} \mathbf{P}\left(X_i \mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} > x\right)$$
$$+ \sum_{i=1}^{n} \mathbf{P}\left(Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T,\ N(T)=n\}} > x\right).$$

Trivially, if $\mathbf{P}(N(T)=n)>0$, then both sides of (21) are 0,and notation $\lesssim$ is understood as =. Thus,

$$I_1 \lesssim \left(\sum_{n=1}^{\infty} - \sum_{n=m+1}^{\infty}\right) \sum_{i=1}^{n} \Big(\mathbf{P}\big(X\mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} > x\big) \\ + \mathbf{P}\big(Y\mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T,\ N(T)=n\}} > x\big)\Big) \\ =: I_{11} - I_{12}.$$

Interchanging the order of the sum in I_{11} yields

$$I_{11} = \int_{0^-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\, \lambda(\mathrm{d}s) + \int_{0^-}^{T} \mathbf{P}\big(Y\mathrm{e}^{-R_s} > x\big)\, (\lambda * H)(\mathrm{d}s).$$

As for I_{12},

$$I_{12} \leqslant \sum_{n=m+1}^{\infty} \sum_{i=1}^{n} \big(\mathbf{P}\big(X\mathrm{e}^{-R_{\tau_i}} > x,\ N(T)=n\big) + \mathbf{P}\big(Y\mathrm{e}^{-R_{\tau_i}} > x,\ N(T)=n\big)\big).$$

Applying $\overline{F}(x) \asymp \overline{G}(x)$ and Lemma 8 gives that

$$\lim_{m\to\infty} \limsup_{x\to 0} \frac{I_{12}}{\int_{0^-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\, \lambda(\mathrm{d}s)} = 0.$$

Combining (20) and (22)–(24), we obtain the upper bound for $\hat{\psi}(x;T)$. Now we estimate the lower bound for $\hat{\psi}(x;T)$. Since $p^*(T)>0$, we have that for any fixed integer *m* and all $x>0$,

$$\psi(x;T) \geqslant \mathbf{P}\left(\sum_{i=1}^{N(T)} X_i\mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{N(T)} Y_i\mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T\}} - p^*(T) - c_0T > x\right) \\ \geqslant \sum_{n=1}^{m} \mathbf{P}\left(\sum_{i=1}^{n} X_i\mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} + \sum_{i=1}^{n} Y_i\mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T,\, N(T)=n\}} \\ - p^*(T) - c_0T > x\right).$$

Since $-p^*(T)-c_0T$ is nonpositive, we have that for all $c>0$ and *any* $x>0$,

$$\mathbf{P}\big(-p^*(T) - c_0T > x\big) = 0 = o\left(\overline{F}\left(\frac{x}{c}\right)\right).$$

As done in dealing with I_1, we can derive the lower bound for $\hat{\psi}(x;T)$.

The proof of part (ii) is much similar to that of (i), and we only show the difference. For the upper bound, it is easy to see that (20) still holds by using Lemma 1(iii) and the fact

$$\int_{0-}^{T} \mathbf{P}\big((X+Y)\mathrm{e}^{-R_s} > x\big)\,\lambda(\mathrm{d}s) \sim \int_{0-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\,\lambda(\mathrm{d}s).$$

Note that for each $1 \leqslant n \leqslant m$,

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} + \sum_{i=1}^{n} Y_i \mathrm{e}^{-R_{\tau_i+D_i}} \mathbf{1}_{\{\tau_i+D_i \leqslant T\}} + p^*(T) > x,\ N(T) = n\right)$$
$$\leqslant \mathbf{P}\left(\sum_{i=1}^{n} (X_i + Y_i)\mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} + p^*(T) > x\right).$$

Then, as done in (22)–(24), by using Lemma 1(iii), Proposition 3, Lemma 8, and (26) we can obtain

$$I_1 \lesssim \int_{0-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\,\lambda(\mathrm{d}s),$$

which, together with (20), leads to

$$\psi(x;T) \lesssim \int_{0-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\,\lambda(\mathrm{d}s).$$

For the lower bound, by (25), for any fixed integer m and all $x > 0$,

$$\psi(x;T) \geqslant \sum_{n=1}^{m} \mathbf{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-R_{\tau_i}} \mathbf{1}_{\{N(T)=n\}} - p^*(T) - c_0 T > x\right)$$

As done in (22)–(24), we can derive

$$\psi(x;T) \gtrsim \int_{0-}^{T} \mathbf{P}\big(X\mathrm{e}^{-R_s} > x\big)\,\lambda(\mathrm{d}s).$$

5.2 Proof of Corollary 1

We only prove part (i) and omit the similar proof of (ii). By Proposition 3.14 of [4] we have that for any $t \geqslant 0$ *and* $z \in \mathbb{R}$,

$$\mathbf{E}\left[e^{-zR_t}\right] = e^{t\phi(z)}.$$

Since $N(t)$ is a homogeneous Poisson process with intensity λ and the distribution H is exponential with parameter $\lambda_H > 0$, it can be calculated that $(\lambda * H)(ds) = \lambda(1 - e^{-\lambda_H s})$ ds. Then by $G \in \mathcal{R}_{-\alpha}$ and (27) we have

$$\begin{aligned}
&\int_{0^-}^{T} \mathbf{P}\left(Y e^{-R_s} > x\right)(\lambda * H)(ds) \\
&= \lambda \int_0^T \left(1 - e^{-\lambda_H s}\right) \int_0^1 \overline{G}\left(\frac{x}{y}\right) \mathbf{P}\left(e^{-R_s} \in dy\right) ds \\
&\sim \lambda \overline{G}(x) \int_0^T \left(1 - e^{-\lambda_H s}\right) \mathbf{E}\left[e^{-\alpha R_s}\right] ds \\
&= \lambda \left(\frac{e^{T\phi(\alpha)} - 1}{\phi(\alpha)} - \frac{e^{T(\phi(\alpha) - \lambda_H)} - 1}{\phi(\alpha) - \lambda_H}\right) \overline{G}(x),
\end{aligned}$$

where in the second step, we used the dominated convergence theorem. Similarly,

$$\int_{0^-}^{T} \mathbf{P}\left(X e^{-R_s} > x\right) \lambda(ds) \sim \frac{\lambda\left(e^{T\phi(\alpha)} - 1\right)}{\phi(\alpha)} \overline{F}(x).$$

Therefore, the desired relation follows from Theorem 1.

References

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