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Articles

Existence of a unique solution for a third-order boundary value problem with nonlocal conditions of integral type.

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Abstract: The existence of a unique solution for a third-order boundary value problem with integral condition is proved in several ways. The main tools in the proofs are the Banach fixed point theorem and the Rus's fixed point theorem. To compare the applicability of the obtained results, some examples are considered.
Keywords: third-order nonlinear boundary value problems, integral boundary conditions, existence and uniqueness of solutions, Green's function, Banach fixed point theorem, Rus's fixed point theorem.

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1 Introduction

We study boundary value problem consisting of the nonlinear third-order differential equation

$$x''' + f(t,x) = 0, \quad t \in [a,b],$$

and the integral-type boundary conditions

$$x(a) = 0, \quad x(b) = 0, \qquad \int_a^b x(\xi)\,\mathrm{d}\xi = 0,$$

where $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and $f(t,0) \neq 0$ for all $t \in [a,b]$.

By a solution of (1), (2) we mean $C^3[a,b]$ function that satisfies the problem. The assumption $f(t,0) \neq 0$ excludes the possibility of the trivial solution.

The purpose of the paper is to give and compare results on the existence of a unique solution to (1), (2) by applying fixed point theorems. Our main results state that if the function f satisfies the Lipschitz condition, the length of the interval $[a,b]$ is not large, then the problem has a unique nontrivial solution.

To obtain these results, we first rewrite problem (1), (2) as an equivalent integral equation by constructing the corresponding Green's function. Then, we apply the Banach fixed point theorem on an infinite strip. Next, in order for the result to be applicable to a wider class of functions, we apply the Banach fixed point theorem within a closed and bounded set. Finally, we apply the Rus's fixed point theorem [16] to increase the length of the interval where the result is valid. To compare the obtained results, we consider examples.

Investigation of the existence of solutions for boundary value problems is often related to the construction of corresponding Green's functions. Thus, Green's functions play an important role in the theory of boundary value problems. A survey of results on the Green's functions for stationary problems with nonlocal boundary conditions is presented in [17]. Green's functions for third-order boundary value problems with different additional conditions were studied in [15]. Green's matrix for a system of first-order ordinary differential equations with nonlocal conditions was considered in [14].

Fixed point theorems are very useful and powerful tools to obtain the existence or uniqueness of solutions to nonlinear boundary value problems. There is a vast literature on this subject because boundary value problems appear in almost all branches of physics, engineering, and technology [4].

Many authors studied the existence of solutions for nonlinear boundary value prob- lems using different fixed point theorems, for instance, Schauder theorem [19], Krasnosel- skii theorem [12], Leggett–Williams theorem [13], Guo–Krasnoselskii theorem [11], etc. Let us mention some recent results. The existence of solutions to a third-order three- point boundary value problem using the Krasnoselskii and Leggett and Williams fixed point theorems was studied in [2]. In [10], the authors applied the Guo–Krasnoselskii fixed point theorem in the study of existence of positive solutions for a second-order three-point boundary value problem. Generalized Krasnoselskii fixed point theorem was used in [5] to establish an existence result for a second-order two-point boundary value problem. In [8], the author has proved the existence of at least three symmetric positive solutions to a second-order two-point boundary value problem using a generalization of the Leggett–Williams fixed point theorem. Applying the upper and lower solution method and the Schauder fixed point theorem, the existence of solutions for a third- order three-point boundary value problem was proved in [6]. The Krasnoselkii fixed point theorem together with two fixed point results of Leggett–Williams type was used in [7] to prove the existence of one or multiple solutions to an nth order two-point boundary value problem. The existence of at least one solution for a fourth-order three- point boundary value problem using the Leray–Schauder nonlinear alternative was studied in [3]. The existence of a unique solution to a third-order three-point boundary value problem applying the Banach fixed point theorem and fixed point theorem of Maia type given by Rus [16] was investigated in [1].

Despite this, third-order boundary value problems with integral type boundary conditions are not sufficiently investigated. Note that nonlocal boundary conditions in particu- lar integral-type boundary conditions often give more precise models. Let us mention some results for such types of problems. Sufficient conditions for the existence and nonexistence of positive solutions were obtained in [9] applying the Guo–Krasnoselskii fixed point theorem. The existence of solutions applying the method of upper and lower functions and Leray–Schauder degree theory was obtained in [18]. The existence, non- existence, and the multiplicity of positive solutions by means of fixed point principle in a cone was studied in [20].

Since our main tools in this paper are the Banach fixed point theorem and the Rus's fixed point theorem, let us state here these theorems for the reader's convenience.

Theorem 1. *(See [19].) Let X be a nonempty set, and let d be a metric on X such that* (X, d) *forms a complete metric space. If the mapping* $T: X \to X$ *satisfies*

$$d(Tx, Ty) \leqslant \alpha d(x, y) \quad \textit{for some } \alpha \in (0, 1) \textit{ and all } x, y \in X;$$

then there is a unique $x_0 \in X$ such that $Tx_0 = x_0$.

Here we state the Rus's fixed point theorem given in [16].

Theorem 2. *(See [16].) Let X be a nonempty set, and let d and ρ be two metrics on X such that (X, d) forms a complete metric space. If the mapping* $T: X \to X$ *is continuous with respect to d on X and*

(i) *there exists* $c > 0$ *such that*

$$d(Tx, Ty) \leqslant c\rho(x, y) \quad \textit{for all } x, y \in X;$$

(ii) *there exists* $\alpha \in (0, 1)$ *such that*

$$\rho(Tx, Ty) \leqslant \alpha\rho(x, y) \quad \textit{for all } x, y \in X;$$

then there is a unique $x_0 \in X$ such that $Tx_0 = x_0$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function employing the variation of parameters formula. Section 3 is devoted to the estimation of an integral that involves the Green's function. In Section 4, we prove our main theorems on the existence and uniqueness of a solution to the problem. Also, to illustrate and compare the results, we consider examples.

2 Construction of the Green's function

The goal of this section is to rewrite problem (1), (2) as an equivalent integral equation.

So, let us consider the linear equation

$$x''' + h(t) = 0, \quad t \in [a, b],$$

together with boundary conditions

Proposition 1. *If* $h:[a,b]\to\mathbb{R}$ *is continuous function, then boundary value problem (3), (2) has a unique solution.*

$$x(t) = \int_a^t \frac{(a-s)^2(b-t)(-(b-a)^2+(t-a)(2(b-s)+(b-a)))}{2(b-a)^3} h(s)\,\mathrm{d}s + \int_t^b \frac{(b-s)^2(t-a)((b-a)^2-(b-t)(2(s-a)+(b-a)))}{2(b-a)^3} h(s)\,\mathrm{d}s$$

that we can rewrite as

$$x(t) = \int_a^b G(t,s)h(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{(a-s)^2(b-t)(-(b-a)^2+(t-a)(2(b-s)+(b-a)))}{2(b-a)^3}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{(b-s)^2(t-a)((b-a)^2-(b-t)(2(s-a)+(b-a)))}{2(b-a)^3}, & a \leqslant t \leqslant s \leqslant b. \end{cases}$$

Proof. To prove the proposition, we use the variation of parameters formula

$$x(t) = c_1 + c_2 t + c_3 t^2 - \frac{1}{2}\int_a^t (s-t)^2 h(s)\,\mathrm{d}s.$$

Using boundary conditions (2), we can obtain

$$c_1 = \frac{a(a+2b)}{2(b-a)^2}\int_a^b (s-b)^2 h(s)\,\mathrm{d}s + \frac{ab}{(b-a)^3}\int_a^b (s-b)^3 h(s)\,\mathrm{d}s,$$

$$c_2 = -\frac{2a+b}{(b-a)^2}\int_a^b (s-b)^2 h(s)\,\mathrm{d}s - \frac{b+a}{(b-a)^3}\int_a^b (s-b)^3 h(s)\,\mathrm{d}s,$$

$$c_3 = \frac{3}{2(b-a)^2}\int_a^b (s-b)^2 h(s)\,\mathrm{d}s + \frac{1}{(b-a)^3}\int_a^b (s-b)^3 h(s)\,\mathrm{d}s.$$

Thus, we get

$$x(t) = \int_a^b \frac{(a-t)(a+2b-3t)(s-b)^2}{2(b-a)^2} h(s)\,\mathrm{d}s$$

$$- \int_a^b \frac{(a-t)(t-b)(s-b)^3}{(b-a)^3} h(s)\,\mathrm{d}s - \frac{1}{2}\int_a^t (s-t)^2 h(s)\,\mathrm{d}s$$

$$= \int_a^t \frac{(a-s)^2(b-t)(-(b-a)^2 + (t-a)(2(b-s)+(b-a)))}{2(b-a)^3} h(s)\,\mathrm{d}s$$

$$+ \int_t^b \frac{(b-s)^2(t-a)((b-a)^2 - (b-t)(2(s-a)+(b-a)))}{2(b-a)^3} h(s)\,\mathrm{d}s.$$

To prove the uniqueness, assume that $y(t)$ is also a solution of (3), (2), that is,

$$y'''(t) + h(t) = 0, \quad t \in [a,b],$$

$$y(a) = 0, \quad y(b) = 0, \qquad \int_a^b y(\xi)\,\mathrm{d}\xi = 0.$$

Let us consider $z(t) = y(t) - x(t),\ t \in [a,b]$. So, we have

$$z'''(t) = y'''(t) - x'''(t) = h(t) - h(t) = 0, \quad t \in [a,b].$$

Hence $z(t) = c_1t^2 + c_2t + c_3$, where c_1, c_2, and c_3 are constants that we will determine. We get $z(a) = y(a) - x(a) = 0,\ z(b) = y(b) - x(b) = 0$ or $c_1a^2 + c_2a + c_3 = 0$, $c_1b^2 + c_2b + c_3 = 0$.Further

$$\int_a^b z(\xi)\,\mathrm{d}\xi = \int_a^b \big(y(\xi) - x(\xi)\big)\,\mathrm{d}\xi = \int_a^b y(\xi)\,\mathrm{d}\xi - \int_a^b x(\xi)\,\mathrm{d}\xi = 0,$$

or

$$\int_a^b \left(c_1\xi^2 + c_2\xi + c_3\right)\mathrm{d}\xi = \frac{c_1}{3}\left(b^3 - a^3\right) + \frac{c_2}{2}\left(b^2 - a^2\right) + c_3(b-a) = 0.$$

We obtain homogeneous system

$$\begin{aligned} &a^2c_1 + ac_2 + c_3 = 0,\\ &b^2c_1 + bc_2 + c_3 = 0,\\ &\frac{1}{3}\left(b^3 - a^3\right)c_1 + \frac{1}{2}\left(b^2 - a^2\right)c_2 + (b-a)c_3 = 0 \end{aligned}$$

with determinant

$$\begin{vmatrix} a^2 & a & 1\\ b^2 & b & 1\\ \frac{1}{3}\left(b^3 - a^3\right) & \frac{1}{2}\left(b^2 - a^2\right) & (b-a) \end{vmatrix} = \frac{(b-a)^4}{6} \neq 0.$$

Thus, the homogeneous system has only the trivial solution, and hence $z(t) \equiv 0,\ t \in [a,b]$, or $x(t) \equiv y(t),\ t \in [a,b]$. The proof is complete.

Therefore, boundary value problem (1), (2) can be rewritten as an equivalent integral equation

$$x(t) = \int_a^b G(t,s) f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in [a,b],$$

where the Green's function $G(t,s)$ is defined by (4). To show that a continuous solution x of (5) is actually a classical $C^3[a,b]$ solution of (1), (2), one can differentiate thrice equation (5) and verify the continuity.

3 Estimation of the Green's function

In this section, we prove a useful inequality for integral that involves the Green's function.

Proposition 2. *The Green's function $G(t,s)$ in (4) satisfies*

$$\int_a^b |G(t,s)|\,\mathrm{d}s \leqslant \frac{5}{96}(b-a)^3 \quad \textit{for all } t \in [a,b].$$

Proof. For all $t \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |G(t,s)|\,\mathrm{d}s \\
&= \int_a^t |G(t,s)|\,\mathrm{d}s + \int_t^b |G(t,s)|\,\mathrm{d}s \\
&= \int_a^t \left|\frac{(a-s)^2(b-t)(-(b-a)^2+(t-a)(2(b-s)+(b-a)))}{2(b-a)^3}\right|\mathrm{d}s \\
&\quad + \int_t^b \left|\frac{(b-s)^2(t-a)((b-a)^2-(b-t)(2(s-a)+(b-a)))}{2(b-a)^3}\right|\mathrm{d}s \\
&\leqslant \int_a^t \frac{(a-s)^2(b-t)((b-a)^2+(t-a)(2(b-s)+(b-a)))}{2(b-a)^3}\,\mathrm{d}s \\
&\quad + \int_t^b \frac{(b-s)^2(t-a)((b-a)^2+(b-t)(2(s-a)+(b-a)))}{2(b-a)^3}\,\mathrm{d}s \\
&= \frac{(t-a)^3(b-t)(5a^2-10ab+2b^2+6bt-3t^2)}{12(b-a)^3} \\
&\quad + \frac{(t-a)(b-t)^3(2a^2-10ab+5b^2+6at-3t^2)}{12(b-a)^3}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{(t-a)^3(b-t)}{12(b-a)^3}\max_{a\leqslant t\leqslant b}\left(5a^2-10ab+2b^2+6bt-3t^2\right) \\
&\quad + \frac{(t-a)(b-t)^3}{12(b-a)^3}\max_{a\leqslant t\leqslant b}\left(2a^2-10ab+5b^2+6at-3t^2\right) \\
&= \frac{(t-a)^3(b-t)}{12(b-a)^3}5(b-a)^2 + \frac{(t-a)(b-t)^3}{12(b-a)^3}5(b-a)^2 \\
&= \frac{5}{12}\cdot\frac{1}{b-a}\left((t-a)^3(b-t)+(t-a)(b-t)^3\right) \\
&\leqslant \frac{5}{12}\cdot\frac{1}{b-a}\max_{a\leqslant t\leqslant b}\left((t-a)^3(b-t)+(t-a)(b-t)^3\right) = \frac{5}{96}(b-a)^3
\end{aligned}$$

4 Existence of a unique solution

In this section, we prove our main results on the existence of a unique solution for problem (1), (2) applying fixed point theorems. Then, we compare the obtained results.

So, let X be the set of continuous functions on $[a,b]$, and consider two metrics

$$d(x,y) = \max_{a\leqslant t\leqslant b} \left|x(t)-y(t)\right|, \quad x,y\in X,$$

and

$$\rho(x,y) = \left(\int_a^b \left|x(t)-y(t)\right|^2 \mathrm{d}t\right)^{1/2}, \quad x,y\in X.$$

The pair (X,d) is a complete metric space and (X,ρ) is a metric space (but not complete).

Application of the Banach fixed point theorem on an infinite strip

Theorem 3. *Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be continuous and $f(t,0)\neq 0$ for all $t\in[a,b]$. Suppose also that f satisfies a uniform Lipschitz condition with respect to x on $[a,b]\times\mathbb{R}$, namely, there is a constant $L>0$ such that, for every $(t,x),(t,y)\in[a,b]\times\mathbb{R}$,*

$$\left|f(t,x)-f(t,y)\right| \leqslant L|x-y|.$$

If $(b-a)$ satisfies the inequality

$$\frac{5}{96}(b-a)^3 < \frac{1}{L},$$

then there exists a unique (nontrivial) solution of (1), (2).

Proof. Since boundary value problem (1), (2) is equivalent to integral equation (5), we need to prove that the mapping $T:X\to X$ defined by

$$(Tx)(t) = \int_a^b G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s, \quad t\in[a,b],$$

has a unique fixed point.

To establish the existence of a unique fixed point for T, we show that the conditions of Theorem 1 hold. Let $x,y\in X$ and consider

$$\begin{aligned}\left|(Tx)(t)-(Ty)(t)\right| &\leqslant \int_a^b \left|G(t,s)\right|\left|f\left(s,x(s)\right)-f\left(s,y(s)\right)\right|\mathrm{d}s\\ &\leqslant \int_a^b \left|G(t,s)\right|L\left|x(s)-y(s)\right|\mathrm{d}s\\ &\leqslant L\,d(x,y)\int_a^b \left|G(t,s)\right|\mathrm{d}s\\ &\leqslant L\,\frac{5}{96}\,(b-a)^3 d(x,y)\quad \text{for } t\in[a,b].\end{aligned}$$

Taking the maximum of both sides of inequality (7) over $[a,b]$, we get

$$d(Tx,Ty)\leqslant L\,\frac{5}{96}\,(b-a)^3 d(x,y)\quad \text{for all } x,y\in X.$$

In view of (6), the mapping T satisfies all of the conditions of Theorem 1 and hence has a unique fixed point, which yields a unique solution to (1), (2).

Example 1. Consider the problem

$$x''' + 1 + t^2 + \frac{x^2}{x^2+1} = 0,$$

$$x(0)=0,\quad x(1)=0,\qquad \int_0^1 x(\xi)\,\mathrm{d}\xi = 0.$$

Function $f(t,x)=1+t^2+x^2/(x^2+1)$ is continuous in $[0,1]\times\mathbb{R}$, and $f(t,0)=1+t^2\neq 0$ for all $t\in[0,1]$. Further, for every $(t,x),(t,y)\in[0,1]\times\mathbb{R}$, consider

$$\begin{aligned}\left|f(t,x)-f(t,y)\right| &= \left|\frac{x^2}{x^2+1}-\frac{y^2}{y^2+1}\right| = \left|\frac{(x-y)(x+y)}{(x^2+1)(y^2+1)}\right|\\ &= |x-y|\cdot\left|\frac{x}{(x^2+1)(y^2+1)}+\frac{y}{(x^2+1)(y^2+1)}\right|\\ &\leqslant |x-y|\cdot\left|\frac{x}{x^2+1}+\frac{y}{y^2+1}\right|\leqslant |x-y|.\end{aligned}$$

Thus, f satisfies the Lipschitz condition with respect to x on $[0,1]\times\mathbb{R}$ with constant $L = 1$. Moreover, inequality (6) holds since $5/96<1$. Therefore, by Theorem 3, problem (8), (9) has a unique nontrivial solution $x(t)$, which together with its antiderivative is depicted

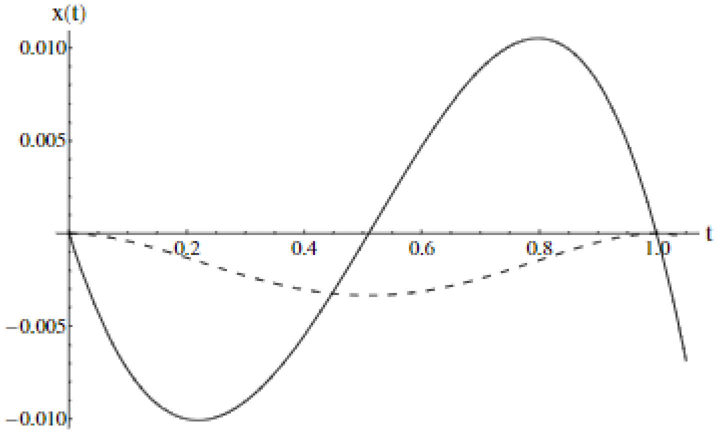

Figure 1.
The unique solution of (8), (9) (solid) with antiderivative (dashed).

in Fig. 1. The initial conditions for this solution are $x(0) = 0,\ x'(0) \approx -0.1,\ x''(0) \approx\ 0.5666$.

Example 2. Consider the same equation from Example 1, but let us change the length of the interval in the boundary conditions. So, consider the problem for equation (8) with boundary conditions

$$x(0) = 0, \quad x(5) = 0, \qquad \int_0^5 x(\xi)\, \mathrm{d}\xi = 0.$$

Theorem 3 is not applicable in this case because inequality (6) does not hold.

Example 3. Now let us change the function in the equation from Example 1, but consider the same length of the interval in the boundary conditions. So, consider the equation

$$x''' - 5 + t^2x^3 = 0$$

with boundary conditions (9). We also cannot use Theorem 3 in this case because the function $f(t,x) = -5 + t^2x^3$ does not satisfy the Lipschitz condition with respect to *x* on $[0,1]\times\mathbb{R}$.

In view of the above examples, let us improve Theorem 3 such that the results will be applicable to a wider range of problems.

Application of the Banach fixed point theorem within a closed and bounded set

Theorem 4. *Let* $f:[a,b]\times[-N,N]\to\mathbb{R}$ *be continuous and* $f(t,0)\neq 0$ *for all* $t\in[a,b]$ *. Suppose also that there exists a constant* $L>0$ *such that, for every* $(t,x),(t,y)\in[a,b]\times\ [-N,N]$,

$$\left|f(t,x)-f(t,y)\right| \leqslant L|x-y|.$$

If $(5/96)(b-a)^3 < 1/L$ and $(5/96)(b-a)^3 \leqslant N/M$, where $M = \max|f(t,x)|$ for $t \in [a,b]$, $t \in [a,b]$, $|x| \leqslant N$, then there exists a unique (nontrivial) solution of (1), (2) such that $|x(t)| \leqslant N$ for all $t \in [a,b]$.

Proof. Consider the ball

$$B_N = \{x \in X\colon d(x,0) \leqslant N\}.$$

Since B_N is a closed subspace of X, the pair (B_N,d) forms a complete metric space. Consider the mapping $T: B_N \to X$ defined by

$$(Tx)(t) = \int_a^b G(t,s) f\big(s,x(s)\big)\,\mathrm{d}s, \quad t \in [a,b].$$

Let us prove that $T: B_N \to B_N$. For $x \in B_N$ and $t \in [a,b]$, consider

$$\begin{aligned}\left|(Tx)(t)\right| &\leqslant \int_a^b \left|G(t,s)\right| \cdot \left|f\big(s,x(s)\big)\right|\mathrm{d}s \\ &\leqslant M \cdot \frac{5}{96}(b-a)^3 \leqslant N.\end{aligned}$$

Thus, $d(Tx,0) \leqslant N$ or for all $x \in B_N$ we have $Tx \in B_N$ Therefore, $T: B_N \to B_N$.

To prove that the mapping $T: B_N \to B_N$ has a unique fixed point, we use similar arguments to that of the proof of Theorem 3.

Remark 1. Note that Theorem 4 does not exclude the existence of other solutions to the problem for which the inequality $|x(t)| \leqslant N$ does not hold for every $t \in [a,b]$. We illustrate this idea in the next example.

Example 4. Consider problem (11), (9). Choose $N = 2$. Function $f(t,x) = -5 + t^2x^3$ is continuous on $\Omega = [0,1] \times [-2,2]$, and $f(t,0) = -5 \neq 0$ for all $t \in [0,1]$. Next, for every $(t,x),(t,y) \in \Omega$ consider

$$\begin{aligned}\left|f(t,x)-f(t,y)\right| &= \left|t^2x^3 - t^2y^3\right| = \left|t^2\right| \cdot \left|x^3-y^3\right| \\ &\leqslant \max_{0\leqslant t\leqslant 1}\left|t^2\right| \cdot \left|x^3-y^3\right| = |x-y| \cdot \left|x^2+xy+y^2\right| \\ &\leqslant \max_{\Omega}\left|x^2+xy+y^2\right| \cdot |x-y| = 12 \cdot |x-y|.\end{aligned}$$

So, f satisfies the Lipschitz condition with respect to x on Ω with constant $L = 12$. Moreover, $5/96 < 1/L$ and $5/96 \leqslant N/M$, where $M = \max_\Omega |f(t,x)| = \max_\Omega |-5 + t^2x^3| = 13$. Therefore, by Theorem 4, problem (11), (9) has a unique nontrivial solution $x(t)$ such that $|x(t)| \leqslant 2$ for all $t \in [0,1]$. This solution together with its antiderivative is depicted in Fig. 2. The initial conditions for this solution are $x(0) = 0,\ x'(0) \approx 0.4166$, $x''(0) \approx -2.5$

There is another solution to the problem with initial conditions $x(0) = 0,\ x'(0) \approx$ $109.914,\ x''(0) \approx -240.8$ for which the inequality $|x(t)| \leqslant 2$ does not hold for every $t \in [0,1]$ (see Fig. 3).

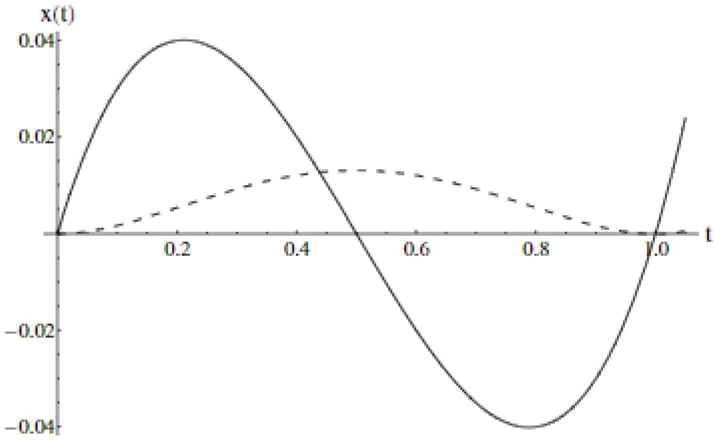

Figure 2.
The unique solution of (11), (9) (solid) with antiderivative (dashed) such that |*x(t)*| ≤ *2 for* all *t # [0, 1].*

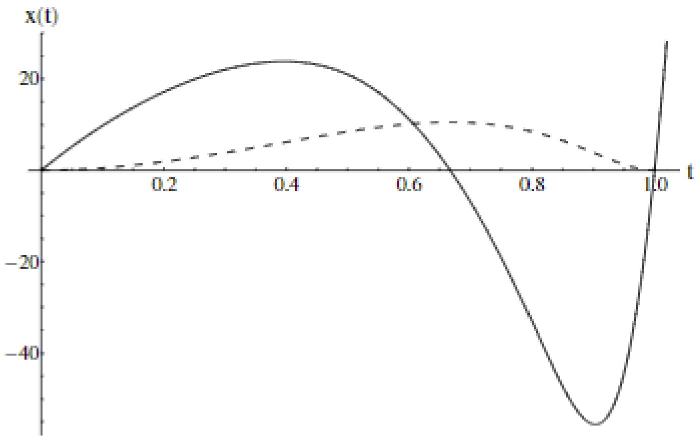

Figure 3.
Another solution of (11), (9) (solid) with antiderivative (dashed) for which the inequality |*x(t)*| ≤ *2* does not hold for every *t # [0, 1].*

Application of the Rus's fixed point theorem on an infinite strip

Theorem 5. *Let* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous and* $f(t,0) \neq 0$ *for all* $t \in [a,b]$ *. Suppose also that f satisfies a uniform Lipschitz condition with respect to x on* $[a,b] \times \mathbb{R}$ *, namely, there is a constant* $L > 0$ *such that, for every (* $(t,x),(t,y) \in [a,b] \times \mathbb{R}$ *,*

$$\left|f(t,x)-f(t,y)\right| \leqslant L|x-y|.$$

If $(b-a)$ satisfies the inequality

$$\frac{(b-a)^3}{30\sqrt{21}} < \frac{1}{L},$$

then there exists a unique (nontrivial) solution of (1), (2).

Proof. Here, we also need to prove that the mapping $T:X\to X$ has a unique fixed point. To establish the existence of a unique fixed point for *T*, we show that the conditions of Theorem 2 hold. Let $x,y\in X$ and consider

$$\begin{aligned}\left|(Tx)(t)-(Ty)(t)\right| &\leqslant \int_a^b \left|G(t,s)\right|\cdot\left|f\big(s,x(s)\big)-f\big(s,y(s)\big)\right|\mathrm{d}s\\ &\leqslant L\int_a^b \left|G(t,s)\right|\cdot\left|x(s)-y(s)\right|\mathrm{d}s\\ &\leqslant L\left(\int_a^b \left|G(t,s)\right|^2\mathrm{d}s\right)^{1/2}\left(\int_a^b \left|x(s)-y(s)\right|^2\mathrm{d}s\right)^{1/2}\\ &\leqslant L\max_{a\leqslant t\leqslant b}\left(\int_a^b \left|G(t,s)\right|^2\mathrm{d}s\right)^{1/2}\rho(x,y)\quad \text{for } t\in[a,b].\end{aligned}$$

So, defining

$$c = L\max_{a\leqslant t\leqslant b}\left(\int_a^b \left|G(t,s)\right|^2\mathrm{d}s\right)^{1/2},$$

we get $d(Tx,Ty)\leqslant c\rho(x,y)$ for all $x,y\in X$. Thus, (i) from Theorem 2 holds In view of.

$$\begin{aligned}\rho(x,y) &= \left(\int_a^b \left|x(t)-y(t)\right|^2\mathrm{d}t\right)^{1/2} \leqslant \max_{a\leqslant t\leqslant b}\left|x(t)-y(t)\right|\left(\int_a^b \mathrm{d}t\right)^{1/2}\\ &= (b-a)^{1/2}d(x,y),\quad x,y\in X,\end{aligned}$$

we obtain $d(Tx,Ty)\leqslant c\rho(x,y)\leqslant c(b-a)^{1/2}d(x,y)$ for all $x,y\in X$. Hence, given any $\varepsilon>0$, we can choose $\delta=\varepsilon/(c(b-a)^{1/2})$ such that d(Tx, Ty) < ε whenever $d(Tx,Ty)<\varepsilon$. Therefore, *T* is continuous with respect to *d* on *X*. From (13), for each $x,y\in X$, consider

$$\left(\int_a^b \left|(Tx)(t)-(Ty)(t)\right|^2\right)^{1/2} \leqslant L\rho(x,y)\left(\int_a^b\left(\int_a^b |G(t,s)|^2\,\mathrm{d}s\right)\mathrm{d}t\right)^{1/2}$$
$$= L\,\rho(x,y)\,\frac{(b-a)^3}{30\sqrt{21}}.$$

It follows that for all $x, y \in X$,

$$\rho(Tx,Ty) \leqslant \alpha\rho(x,y), \quad \alpha = L\frac{(b-a)^3}{30\sqrt{21}}.$$

In view of (12), $\alpha < 1$ and (ii) from Theorem 2 holds. Hence T has a unique fixed point, which yields a unique solution to (1), (2)

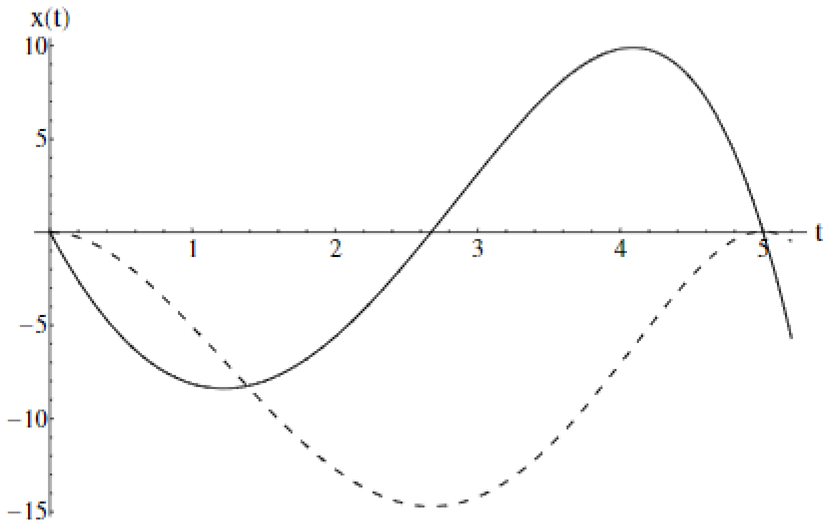

Figure 4.
The unique solution of (8), (10) (solid) with antiderivative (dashed).

Example 5. Consider problem (8), (10). As we see in Example 2, Theorem 3 cannot be used in this case due to inequality (6). But Theorem 5 is applicable here because inequality (12) holds since $5^3 < 30\sqrt{21}$. Therefore, problem (8), (10) has a unique (nontrivial) solution $x(t)$, which together with its antiderivative is depicted in Fig. 4. The initial conditions for this solution are $x(0) = 0, x'(0) \approx -14.3781, x''(0) \approx 13.068$.

5 Conclusions

First, we have proved the existence of a unique solution for a third-order boundary value problem with integral condition using the Banach fixed point theorem. Then the obtained result was improved in two directions. Applying the Banach fixed point theorem within a closed and bounded set, we have generalized the result to a wider class of functions. Applying the Rus's fixed point theorem, the length of the interval in which the result is valid was increased. The larger the length of the interval, the more applicable the result. As we have seen, the Rus's fixed point theorem, where space is endowed with two metrics, gives a much longer interval.

Note that our future research may be concerned with the study of the number of solutions for the problems of the type (11), (9).

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