Extension of Darbo's fixed point theorem via shifting distance functions and its application*

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Articles

Extension of Darbo's fixed point theorem via shifting distance functions and its application*

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Abstract: In this paper, we discuss solvability of infinite system of fractional integral equations (FIE) of mixed type. To achieve this goal, we first use shifting distance function to establish a new generalization of Darbo's fixed point theorem, and then apply it to the FIEs to establish the existence of solution on tempered sequence space. Finally, we verify our results by considering a suitable example.
Keywords: fractional integral equations, measure of noncompactness, fixed point theorem.

1 Introduction

Integral equations have multiple practical applications in modelling specific real world problems and different types of real-life situations, e.g., in laws of physics, in the theory of radioactive transmission, in the theory of statistical mechanics, and in the cytotoxic ac- tivity. The theory of infinite systems of fractional integral equations (FIEs) plays a pivotal role in different fields, which includes various implications in the scaling system theory, the theory of algorithms, etc. There are many real-life problems, which can be modelled by infinite systems of integral equations with fractional order in a very effective manner.

In recent times, the fixed point theory (FPT) has applications in various scientific fields. Also, FPT can be applied seeking solutions for FIE.

Recently, number of articles have been published in connection: with scalar linear im- pulsive Riemann–Liouville fractional differential equations with constant delay-explicit solutions, coupled systems of integral equations of Urysohn Volterra–Chandrasekhar mixed type, noninstantaneous impulsive fractional integro–differential equations, frac- tional differential equations, and proximity theory (the readers can consult the papers [14, 16–18, 22] and references therein).

In [14], Gabeleh and Künzi have established that the existence of best proximity points for cyclic nonexpansive mappings is equivalent to the

existence of best proximity pairs for noncyclic nonexpansive mappings in the setting of strictly convex Banach spaces by using the projection operator. They have also discussed the convergence of best proximity pairs for noncyclic contractions by applying the convergence of iterative sequences for cyclic contractions.

In [16], Harjani et al. established sufficient condition about the length of the interval for the existence and uniqueness of mild solutions to a fractional boundary value problem with Sturm–Liouville boundary conditions when the data function is of Lipschitzian type. Moreover, they have presented an application of our result to the eigenvalues problem and its connection with a Lyapunov-type inequality.

In [17], Hristova and Tersian have studied Riemann–Liouville fractional differential equations with a constant delay and impulses. Also, they have studied the case when the lower limit of the fractional derivative is fixed on the whole interval of consideration and the case when the lower limit of the fractional derivative is changed at any point of im- pulse. The initial conditions as well as impulsive conditions are defined in an appropriate way for both cases. The explicit solutions are obtained and applied to the study of finite- time stability.

In [18], Kataria et al. have established the existence of mild solution for noninstan- taneous impulsive fractional-order integro–differential equations with local and nonlocal conditions in Banach space. Existence results with local and nonlocal conditions are ob- tained through operator semigroup theory using generalized Banach contraction theorem and Krasnoselskii's fixed point theorem, respectively. Finally, illustrations are added to validate derived results.

In [22], Nabil has studied the solvability of a coupled system of integral equations of Urysohn Volterra–Chandrasekhar mixed type. To realize the existence of a solution of those mixed systems, he has use the Perov's fixed point combined with the Leray– Schauder fixed point approach in generalized Banach algebra spaces.

Different real-life situations, which are modeled via FIEs, can be studied using FPT and measure of noncompactness (MNC) (see [2, 3, 5, 7, 10, 11, 13, 19, 21, 23–26]).

The following notations will be used $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ denotes a Banach space (BS); $B[\theta, \kappa] = \{\theta \in \mathbb{E} : \|\theta\|_{\mathbb{E}} \leqslant \kappa\}$; $\overline{\Omega}$ – the closure of Ω; Conv Ω - the convex closure of Ω; m_E - the family of all nonempty and bounded subsets of $\mathbb{E}$; $\mathfrak{N}_{\mathbb{E}}$ – the subfamily consisting of all relatively compact sets; $\mathbb{R}$ – the set of real numbers; $\mathbb{R}^+ = [0, \infty)$.

Definition 1. A mapping $\pi : \mathfrak{m}_{\mathbb{E}} \to \mathbb{R}^+$ is said to be an MNC in $\mathbb{E}$ if the following hold:

(i) If $\Omega \in \mathfrak{m}_{\mathbb{E}}$ and $\pi(\Omega) = 0$, then Ω is relatively compact;

(ii) $\ker \pi = \{\Omega \in \mathfrak{m}_{\mathbb{E}} : \pi(\Omega) = 0\} (\neq \emptyset)$ and $\ker \pi \subset \mathfrak{N}_{\mathbb{E}}$;

(iii) $\Omega \subseteq \Omega_1$ implies $\pi(\Omega) \leqslant \pi(\Omega_1)$;

(iv) $\pi(\overline{\Omega}) = \pi(\Omega)$;

(v) $\pi(\text{Conv }\Omega) = \pi(\Omega)$;

vi) $\pi(\iota\Omega + (1 - \iota)\Omega_1) \leqslant \iota\pi(\Omega) + (1 - \iota)\pi(\Omega_1)$ for $\iota \in [0, 1]$;

(vii) If $\Omega_\sigma \in \mathfrak{m}_{\mathbb{E}}$, $\Omega_\sigma = \overline{\Omega}_\sigma$, $\Omega_{\sigma+1} \subset \Omega_\sigma$ for $\sigma \in \mathbb{N}$ and $\lim_{\sigma \to \infty} \pi(\Omega_\sigma) = 0$, then $\Omega_\infty = \bigcap_{\sigma=1}^{\infty} \Omega_\sigma \neq \emptyset$.

Here $\ker \pi$ denotes the *kernel of* π. Also, $\Omega_\infty \in \ker \pi$ and $\pi(\Omega_\infty) \leqslant \pi(\Omega_\sigma)$ for $\sigma \in \mathbb{N}$ imply $\pi(\Omega_\infty) = 0$; hence, $\Omega_\infty \in \ker \pi$.

Theorem 1 [Schauder theorem]. (*See* [1].) *Let* $\mathbb{E}$ *be a BS, and let* $\Lambda(\neq \emptyset) \subseteq \mathbb{E}$ *be closed and convex. If* $\Delta : \Lambda \to \Lambda$ *is continuous and compact, the it admits at least one fixed point.*

Theorem 2 [Darbo theorem]. (*See* [9].) *Let* $\mathbb{E}$ *be a Banach space and* $\Lambda \subseteq \mathbb{E}$ *be nonempty, bounded, closed, and convex (NBCC). Let* $\Delta : \Lambda \to \Lambda$ *be continuous, and let there exist a constant* $0 \leqslant \tau < 1$ *with*

$$\pi(\Delta \Pi) \leqslant \tau \pi(\Pi), \quad \Pi \subseteq \Lambda.$$

Then Δ has a fixed point.

With the help of following concepts, we establish our fixed point theorem.

Definition 2. (See [24].) Let functions $\Delta_1, \Delta_2 : \mathbb{R}^+ \to \mathbb{R}$. Then the pair is called a pair (Δ_1, Δ_2) of shifting distance functions (SDF) if:

1. For $l, m \in \mathbb{R}_+$, if $\Delta_1(l) \leqslant \Delta_2(m)$, then $l \leqslant m$;
2. For $l_k, m_k \in \mathbb{R}_+$ with $\lim_{k \to \infty} l_k = \lim_{k \to \infty} m_k = w$, if $\Delta_1(l_k) \leqslant \Delta_2(m_k)$ for all k, then $w = 0$.

Examples of SDF are:
(i) $\Delta_1(\zeta) = \ln((1 + 2\zeta)/2), \Delta_2(\zeta) = \ln((1 + \zeta)/2)$;
(ii) $\Delta_1(\zeta) = \zeta, \ \Delta_2(\zeta) = \lambda\zeta, \ \lambda \in [0, 1)$.

Definition 3. (See [12].) $\check{K}$ will denote the family of all maps $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with:

1. $\max\{\tau, \varsigma\} \leqslant k(\tau, \varsigma)$ for $\tau, \varsigma \geqslant 0$;
2. k is continuous and nondecreasing;
3. $k(\tau_1 + \tau_2, \varsigma_1 + \varsigma_2) \leqslant k(\tau_1, \varsigma_1) + k(\tau_2, \varsigma_2)$.

For example: $k(\tau, \varsigma) = \tau + \varsigma, \tau, \varsigma \geqslant 0$.

2 New results

Theorem 3. *Let* $\mathbb{E}$ *be a BS and* $\mathbb{U} \subseteq \mathbb{E}$ *be NBCC. Also, let* $\mathfrak{F} : \mathbb{U} \to \mathbb{U}$ *be a continuous mapping with*

$$\begin{aligned}\Delta_1\big[\psi\{k(\vartheta(\mathfrak{F}\mathbb{V}), \alpha(\vartheta(\mathfrak{F}\mathbb{V})))\}\big] \\ \leqslant \Delta_2\big[\phi\big(\psi\{k(\vartheta(\mathbb{V}), \alpha(\vartheta(\mathbb{V})))\}\big)\psi\{k(\vartheta(\mathbb{V}), \alpha(\vartheta(\mathbb{V})))\}\big]\end{aligned} \tag{1}$$

for $\mathbb{V}(\neq \emptyset) \subseteq \mathbb{U}$, *where* $k \in \check{K}$, $\Delta_1, \Delta_2 \in \Delta$, $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous mapping, and* ϑ *is an arbitrary MNC. Moreover,* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing with* $\psi(t) = 0$ *iff* $t = 0$ *and* $\phi : \mathbb{R}^+ \to [0, 1)$. *Then* $\mathfrak{F}$ *admits a fixed point in* $\mathbb{U}$.

Proof. Define a sequence $(\mathbf{C}_s)$, where $\mathbf{C}_1 = \mathbb{U}$ and $\mathbf{C}_{s+1} = \mathrm{Conv}(\mathfrak{F}\mathbf{C}_s), s \geqslant 1$. Then $\mathfrak{F}\mathbf{C}_1 = \mathfrak{F}\mathbb{U} \subseteq \mathbb{U} = \mathbf{C}_1, \mathbf{C}_2 = \mathrm{Conv}(\mathfrak{F}\mathbf{C}_1) \subseteq \mathbb{U} = \mathbf{C}_1$. Similary $\mathbf{C}_1 \supseteq \mathbf{C}_2 \supseteq \mathbf{C}_3 \supseteq \ldots \supseteq \mathbf{C}_s \supseteq \mathbf{C}_{s+1} \supseteq \ldots$. Let $s_1 \in \mathbb{N}$ with $\vartheta(\mathbf{C}_{s_1}) = 0$, then $\mathbf{C}_{s_1}$ is compact. Applying Theorem 1, we observe that $\mathfrak{F}$ admits a fixed point.

If $\vartheta(\mathbf{C}_s) > 0$ for $s \geqslant 0$, by (1) we have

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$$\Delta_1\big[\psi\big\{k\big(\vartheta(\mathbf{C}_{s+1}),\alpha\big(\vartheta(\mathbf{C}_{s+1})\big)\big)\big\}\big]$$

$$= \Delta_1\big[\psi\big\{k\big(\vartheta(\mathrm{Conv}(\mathfrak{F}\mathbf{C}_s)),\alpha\big(\vartheta(\mathrm{Conv}(\mathfrak{F}\mathbf{C}_s))\big)\big)\big\}\big]$$

$$= \Delta_1\big[\psi\big\{k\big(\vartheta(\mathfrak{F}\mathbf{C}_s),\alpha\big(\vartheta(\mathfrak{F}\mathbf{C}_s)\big)\big)\big\}\big]$$

$$\leqslant \Delta_2\big[\phi\big(\psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}\big)\psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}\big],$$

which gives

$$\psi\big\{k\big(\vartheta(\mathbf{C}_{s+1}),\alpha\big(\vartheta(\mathbf{C}_{s+1})\big)\big)\big\}$$

$$\leqslant \phi\big(\psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}\big)\psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}.$$

Since $\phi:\mathbb{R}^+\to[0,1)$, we have

$$0\leqslant \psi\big\{k\big(\vartheta(\mathbf{C}_{s+1}),\alpha\big(\vartheta(\mathbf{C}_{s+1})\big)\big)\big\}\leqslant \psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}.$$

Clearly, the sequence $\{\psi\{k(\vartheta(\mathbf{C}_s),\alpha(\vartheta(\mathbf{C}_s)))\}\}_{s=1}^{\infty}$ is nonnegative and nonincreasing; thus, we can find an $r\geqslant 0$ such that

$$\lim_{s\to\infty}\psi\big\{k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\big\}=r.$$

We claim that $r=0$.

Let $z_s=\psi\{k(\vartheta(\mathbf{C}_s),\alpha(\vartheta(\mathbf{C}_s)))\}$ and $z=\sup_{s\in\mathbb{N}}\phi(z_s)$, which gives $z\in[0,1)$.

Therefore,

$$z_{s+1}\leqslant \phi(z_s)z_s\leqslant zz_s,$$

which gives

$$z_{s+1}\leqslant zz_s\leqslant z^2 z_{s-1}\leqslant\cdots\leqslant z^s z_1.$$

Letting $s\to\infty$, we get $z_{s+1}\to 0$. Hence, $\lim_{s\to\infty}\psi\{k(\vartheta(\mathbf{C}_s),\alpha(\vartheta(\mathbf{C}_s)))\}=0$, i.e., $r=0$, wich gives

$$\psi\Big\{\lim_{s\to\infty}k\big(\vartheta(\mathbf{C}_s),\alpha\big(\vartheta(\mathbf{C}_s)\big)\big)\Big\}=0.$$

which gives

By using the properties of ψ and k we get $\lim_{s\to\infty}\vartheta(\mathbf{C}_s)=0=\lim_{s\to\infty}\alpha[\vartheta(\mathbf{C}_s)]$. Since $\mathbf{C}_s\supseteq\mathbf{C}_{s+1}$, by Definition 1 we get $\mathbf{C}_{\infty}=\bigcap_{s=1}^{\infty}\mathbf{C}_s\subseteq\mathbb{U}$ is nonempty, closed, and convex. Also, C_{∞} is invariant under $\mathfrak{F}$. Thus, Theorem 1 implies that $\mathfrak{F}$ has a fixed point in $\mathbf{C}_{\infty}\subseteq\mathbb{U}$.

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Theorem 4. *Let* $\mathbb{E}$ *be a BS and* $\mathbb{U} \subseteq \mathbb{E}$ *be NBCC. Also, let* $\mathfrak{F} : \mathbb{U} \to \mathbb{U}$ *be a continuous mapping with*

$$\Delta_1\big[\psi\{\vartheta(\mathfrak{F}\mathbb{V}) + \alpha\big(\vartheta(\mathfrak{F}\mathbb{V})\big)\}\big]$$
$$\leqslant \Delta_2\big[\phi(\psi\{\vartheta(\mathbb{V}) + \alpha\big(\vartheta(\mathbb{V})\big)\})\psi\{\vartheta(\mathbb{V}) + \alpha\big(\vartheta(\mathbb{V})\big)\}\big] \qquad (2)$$

for $\mathbb{V}(\neq \emptyset) \subseteq \mathbb{U}$, *where* $\Delta_1, \Delta_2 \in \Delta$, $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous mapping, and* θ *is an arbitrary MNC. Moreover,* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing with* $\psi(t) = 0$ *iff* $t = 0$ *and* $\phi : \mathbb{R}^+ \to [0, 1)$. *Then* $\mathfrak{F}$ *admits a fixed point in* $\mathbb{U}$.

Proof. Taking $k(l, m) = l + m$ in Theorem 3, the result follows.

Corollary 1. *Let* $\mathbb{E}$ *be a BS and* $\mathbb{U} \subseteq \mathbb{E}$ *be NBCC. Also, let* $\mathfrak{F} : \mathbb{U} \to \mathbb{U}$ *be a continuous mapping with*

$$\Delta_1\big[\psi\{\vartheta(\mathfrak{F}\mathbb{V})\}\big] \leqslant \Delta_2\big[\phi(\psi\{\vartheta(\mathbb{V})\})\psi\{\vartheta(\mathbb{V})\}\big] \qquad (3)$$

for $\mathbb{V}(\neq \emptyset) \subseteq \mathbb{U}$, *where* $\Delta_1, \Delta_2 \in \Delta$, *and* ϑ *is an arbitrary MNC. Moreover,* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing with* $\psi(t) = 0$ *iff* $t = 0$ *and* $\phi : \mathbb{R}^+ \to [0, 1)$. *Then* $\mathfrak{F}$ *has a fixed point in* $\mathbb{U}$.

Proof. The result follows by taking $\alpha \equiv 0$ in Theorem 4.

Remark 1. If we take $\Delta_1(\varsigma) = \varsigma, \Delta_2(\varsigma) = \varsigma, \psi(\varsigma) = \varsigma, \phi(\varsigma) = \lambda \in [0, 1)$, then $\vartheta(\mathfrak{F}\mathbb{V}) \leqslant \lambda\vartheta(\mathbb{V})$, and Theorem 2 follows as a spaecial case.

Definition 4. (See [8].) An element $(\varpi, \iota) \in \Omega \times \Omega$ is called a coupled fixed point of a mapping $\mathfrak{F} : \Omega \times \Omega \to \Omega$ if $\mathfrak{F}(\varpi, \iota) = \varpi$ and $\mathfrak{F}(\varpi, \iota) = \iota$.

Theorem 5. (*See* [4].) *Suppose* $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ *are MNCs in* $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$, *respectively, and the function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is convex and* $\mathfrak{F}(p_1, p_2, \ldots, p_n) = 0$ *if and only if* $p_l = 0, l \in \mathbb{N}$. *Then* $\vartheta(\Upsilon) = \mathfrak{F}(\vartheta_1(\Upsilon_1), \vartheta_2(\Upsilon_2), \ldots, \vartheta_n(\Upsilon_n))$ *defines an MNC in* $\mathbb{E}_1 \times \mathbb{E}_2 \times \cdots \times \mathbb{E}_n$, *where* Υ_l *denotes the natural projection of* Υ *into* $\mathbb{E}_l, l \in \mathbb{N}$.

Example 1. (See [4].) Let ϑ be an MNC on $\mathbb{E}$ and $\mathfrak{F}(\varpi, \iota) = \varpi + \iota, \varpi, \iota \in \mathbb{R}^+$. Then $\vartheta^{cf}(\Upsilon) = \vartheta(\Upsilon_1) + \vartheta(\Upsilon_2)$ is an MNC in $\mathbb{E} \times \mathbb{E}$, where $\Upsilon_l, l = 1, 2$, denotes the natural projections of Υ.

Theorem 6. *Let* $\mathbb{E}$ *be a BS and* $\mathbb{U} \subseteq \mathbb{E}$ *be NBCC. Also, let* $\mathfrak{F} : \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ *be a continuous with*

$$\Delta_1\big[\psi\big[k\{\vartheta\big(\mathfrak{F}(\mathbb{V}_1 \times \mathbb{V}_2)\big), \alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1 \times \mathbb{V}_2)\big)\big)\}\big]\big]$$
$$\leqslant \frac{1}{2}\Delta_2\big[\phi\{\psi\big(k\big(\vartheta(\mathbb{V}_1) + \vartheta(\mathbb{V}_2), \alpha\big(\vartheta(\mathbb{V}_1) + \vartheta(\mathbb{V}_2)\big)\big)\big)\}\big]$$
$$\times \psi\big(k\big(\vartheta(\mathbb{V}_1) + \vartheta(\mathbb{V}_2), \alpha\big(\vartheta(\mathbb{V}_1) + \vartheta(\mathbb{V}_2)\big)\big)\big)$$

for all $\mathbb{V}_1, \mathbb{V}_2 \subseteq \mathbb{U}$, *where* $k \in \hat{K}$, $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function satisfying* $\alpha(\varpi_1 + \varpi_2) \leqslant \alpha(\varpi_1) + \alpha(\varpi_2), \varpi_1, \varpi_2 \geqslant 0$, *and* ϑ *is an arbitrary MNC. Moreover,* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing such that* $\psi(t) = 0$ *iff* $t = 0, \phi : \mathbb{R}^+ \to [0, 1)$, *and* $\Delta_1, \Delta_2 \in \Delta$ *satisfy* $\psi(\varpi_1 + \varpi_2) \leqslant \psi(\varpi_1) + \psi(\varpi_2)$ *and* $\Delta_1(\varpi_1 + \varpi_2) \leqslant \Delta_1(\varpi_1) + \Delta_1(\varpi_2)$. *Then* $\mathfrak{F}$ *admits a coupled fixed point in* $\mathbb{U} \times \mathbb{U}$.

Proof. We observe that $\vartheta^{cf}(\mathbb{V}) = \vartheta(\mathbb{V}_1) + \vartheta(\mathbb{V}_2)$ is an MNC on $\mathbb{E} \times \mathbb{E}$ for any bounded subset $\mathbb{V} \subseteq \mathbb{E} \times \mathbb{E}$, where $\mathbb{V}_1, \mathbb{V}_2$ are natural projections of V. Consider $\mathfrak{F}^{cf} : \mathbb{U} \times \mathbb{U} \to$ defined by $\mathfrak{F}^{cf}(l, m) = (\mathfrak{F}(l, m), \mathfrak{F}(m, l))$. It is trivial to see that $\mathfrak{F}^{cf}$ is continuous. Let $\mathbb{V} \subseteq \mathbb{U} \times \mathbb{U}$. We obtain

$$\Delta_1\big[\psi\big[k\big\{\vartheta^{cf}\big(\mathfrak{F}^{cf}(\mathbb{V})\big),\alpha\big(\vartheta^{cf}\big(\mathfrak{F}^{cf}(\mathbb{V})\big)\big)\big\}\big]\big]$$
$$\leqslant \Delta_1\big[\psi\big[k\big\{\vartheta^{cf}\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\times\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big),$$
$$\alpha\big(\vartheta^{cf}\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\times\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big)\big)\big\}\big]\big]$$
$$= \Delta_1\big[\psi\big[k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)+\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big),$$
$$\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)+\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big)\big)\big\}\big]\big]$$

$$\leqslant \Delta_1\big[\psi\big[k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)+\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big),$$
$$\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)\big)+\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big)\big)\big\}\big]\big]$$
$$\leqslant \Delta_1\big[\psi\big[k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big),\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)\big)\big\}$$
$$+ k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big),\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big)\big)\big\}\big]\big]$$
$$\leqslant \Delta_1\big[\psi\big[k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big),\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)\big)\big\}\big]\big]$$

$$+ \Delta_1\big[\big[k\big\{\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big),\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_2\times\mathbb{V}_1)\big)\big)\big\}\big]\big]$$
$$\leqslant \Delta_2\big[\phi\big\{\psi\big(k\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2),\alpha\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)\big)\big)\big)\big\}\big]$$
$$\times \psi\big(k\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2),\alpha\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)\big)\big)\big)$$
$$= \Delta_2\big[\phi\big\{\psi\big(k\big(\vartheta^{cf}(\mathbb{V}),\alpha\big(\vartheta^{cf}(\mathbb{V})\big)\big)\big)\big\}\big]\psi\big(k\big(\vartheta^{cf}(\mathbb{V}),\alpha\big(\vartheta^{cf}(\mathbb{V})\big)\big)\big).$$

By Theorem 3 we conclude that $\mathfrak{F}^{cf}$ has a fixed point in $\mathbb{U}\times\mathbb{U}$, i.e., $\mathfrak{F}$ has a coupled fixed

Corollary 2. *Let* $\mathbb{E}$ *be a BS and* $\mathbb{U}\subseteq\mathbb{E}$ *be NBCC. Also, let* $\mathfrak{F}^{cf}:\mathbb{U}\times\mathbb{U}\to$ *be a continuous function satisfying*

$$\Delta_1\big[\psi\big[\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)+\alpha\big(\vartheta\big(\mathfrak{F}(\mathbb{V}_1\times\mathbb{V}_2)\big)\big)\big]\big]$$
$$\leqslant \frac{1}{2}\Delta_2\big[\phi\big\{\psi\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)+\alpha\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)\big)\big)\big\}\big]$$
$$\times \psi\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)+\alpha\big(\vartheta(\mathbb{V}_1)+\vartheta(\mathbb{V}_2)\big)\big)$$

for all $\mathbb{V}_1,\mathbb{V}_2\subseteq\mathbb{U}$, *where* $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ *is a continuous function satisfying* $\alpha(\varpi_1+\varpi_2)\leqslant\alpha(\varpi_1)+\alpha(\varpi_2)$, $\varpi_1,\varpi_2\geqslant 0$, *and* ϑ *is an arbitrary MNC. Also,* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ *is nondecreasing such that* $\psi(t)=0$ *if and only if* $t=0$, $\phi:\mathbb{R}^+\to[0,1)$ *and* $\Delta_1,\Delta_2\in\Delta$ *and* $\Delta_1,\Delta_2\in\Delta$ *satisfy* $\psi(\varpi_1+\varpi_2)\leqslant\psi(\varpi_1)+\psi(\varpi_2)$ *and* $\Delta_1(\varpi_1+\varpi_2)\leqslant\Delta_1(\varpi_1)+\Delta_1(\varpi_2)$. *Then* $\mathfrak{F}$ *has a coupled fixed point in* $\mathbb{U}\to\mathbb{U}$.

Proof. We obtain the desired result by choosing $k(l,m)=l+m$ in Theorem 6.

2.1 Measure of noncompactness

Banaś' and Krajewska [6] introduced the notions of tempering sequence and space of tempered sequences. Namely, a fixed positive nonincreasing real sequence $\alpha = (\alpha_\sigma)_{\sigma=1}^\infty$ is called a tempering sequence.

Recently, Rabbani et al. [26] denoted by W a collection of all real or complex sequence $\nu = (\nu_\sigma)_{\sigma=1}^\infty$ with $\sum_{\sigma=1}^\infty \alpha_\sigma^p |\nu_\sigma|^p < \infty \, (1 \leqslant p < \infty)$. Clearly, W space over R or C, and this space is denoted by $W := \ell_p^\alpha$ for $1 \leqslant p < \infty$. It is easy to observe that ℓ_p^α for $1 \leqslant p < \infty$ is a Banach space with the norm

$$\|\nu\|_{\ell_p^\alpha} = \left(\sum_{\sigma=1}^\infty \alpha_\sigma^p |\nu_\sigma|^p \right)^{1/p}.$$

If $\alpha_\sigma = 1$ for all $\sigma \in \mathbb{N}$, then $\ell_p^\alpha = \ell_p$ for $1 \leqslant p < \infty$.

A Hausdorff MNC $\chi_{\ell_p^\alpha}$ for a nonempty bounded set B^α of $\ell_p^\alpha \, (1 \leqslant p < \infty)$ can be given by (see [26])

$$\chi_{\ell_p^\alpha}(B^\alpha) = \lim_{\sigma \to \infty} \left[\sup_{\nu \in B^\alpha} \left(\sum_{k \geqslant \sigma} \alpha_k^p |\nu_k|^p \right)^{1/p} \right]. \tag{4}$$

Let us denote by $C(I, \ell_p^\alpha)$ the space of all continuous functions on $I = [0, a], a > 0$, with the values in $\ell_p^\alpha \, (1 \leqslant p < \infty)$, which is also a Banach space with the norm

$$\|\nu\|_{C(I, \ell_p^\alpha)} = \sup_{\varsigma \in I} \|\nu(\varsigma)\|_{\ell_p^\alpha},$$

where $\nu(\varsigma) = (\nu_\sigma(\varsigma))_{\sigma=1}^\infty \in C(I, \ell_p^\alpha)$.

Let $E^\alpha (\neq \emptyset)$ be a bounded subset of $C(I, \ell_p^\alpha)$ and $\varsigma \in I$,

$$E^\alpha(\varsigma) = \left\{ \nu(\varsigma) \colon \nu(\varsigma) \in E^\alpha \right\}.$$

Thus, an MNC for $E^\alpha \subset C(I, \ell_p^\alpha)$ can be defined by

$$\chi_{C(I, \ell_p^\alpha)}(E^\alpha) = \sup_{t \in I} \chi_{\ell_p^\alpha}(E^\alpha(t)).$$

3 Infinite systems of mixed type fractional integral equations

Let $\varsigma \in \mathbb{R}_+$ and $\mathrm{Re}(\varpi) > 0$. The Hadamard fractional integral of order ϖ, applied to the function $f \in L^p[a,b], 1 \leqslant p < \infty, 1 < a < b < \infty$, for $\varsigma \in [a,b]$, is defined by [15]

$$J^\alpha f(\varsigma) = \frac{1}{\Gamma(\varpi)} \int\limits_a^\varsigma \left(\ln \frac{\varsigma}{\xi} \right)^{\varpi-1} f(\xi)\, \frac{\mathrm{d}\xi}{\xi}.$$

Therefore, we have

$$J^\alpha f(\varsigma) = \frac{1}{\Gamma(\varpi)} \int\limits_1^\varsigma \left(\ln \frac{\varsigma}{\xi} \right)^{\varpi-1} f(\xi)\, \frac{\mathrm{d}\xi}{\xi}, \quad \alpha > 0,\ \varsigma > 1.$$

Let $f \in L^1[a,b], 0 \leqslant a < b < \infty$, and $\varpi > 0$ be a real number. The Riemann–Liouville fractional integral of order ϖ is defined by [20]

$$I^\alpha f(\varsigma) = \frac{1}{\Gamma(\varpi)} \int\limits_a^\varsigma \frac{f(\xi)}{(\varsigma - \xi)^{1-\varpi}}\, \mathrm{d}\xi, \quad \varsigma \in (a,b).$$

Consider the following infinite system of mixed-type fractional integral equations:

$$\Theta_\rho(\varsigma) = \Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \Theta(\varsigma)), I^\varpi P_\rho(\varsigma, \Theta(\varsigma)), J^\varpi Q_\rho(\varsigma, \Theta(\varsigma))\big), \quad \rho \in \mathbb{N}, \tag{5}$$

where $0 < \varpi < 1, \varsigma \in I = [1,T]$, and $\Theta(\varsigma)) = \{\Theta_\rho(\varsigma)\}_{\rho=1}^\infty \in \mathbb{E}$, where E is a Banach sequence space.

Assume:

(i) The fuctions $\Lambda_\rho : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

$$\begin{aligned}
\big|\Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \Theta(\varsigma)), I, J\big) &- \Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \bar{\Theta}(\varsigma)), \bar{I}, \bar{J}\big)\big|^p \\
&\leqslant \gamma_1(\varsigma)\big|l_\rho\big(\varsigma, \Theta(\varsigma)\big) - l_\rho\big(\varsigma, \bar{\Theta}(\varsigma)\big)\big|^p + \gamma_2(\varsigma)|I - \bar{I}|^p + \gamma_3(\varsigma)|J - \bar{J}|^p,
\end{aligned}$$

And $l_\rho : I \times C(I, \ell_p^\alpha) \to \mathbb{R}$ are continuous with

$$\big|l_\rho\big(\varsigma, \Theta(\varsigma)\big) - l_\rho\big(\varsigma, \bar{\Theta}(\varsigma)\big)\big|^p \leqslant \gamma_4(\varsigma)\big|\Theta(\varsigma) - \bar{\Theta}(\varsigma)\big|^p$$

for $\Theta(\varsigma) = (\Theta_\rho(\varsigma))_{n=1}^\infty, \bar{\Theta}(\varsigma) = (\bar{\Theta}_\rho(\varsigma))_{\rho=1}^\infty \in C(I, \ell_p^\alpha); l_\rho(\varsigma, \Theta(\varsigma)), l_\rho(\varsigma, \bar{\Theta}(\varsigma)),$
$I, J, \bar{I}, \bar{J} \in \mathbb{R}$, and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 : I \to \mathbb{R}_+^{'}; (\rho \in \mathbb{N})$ are continuous fuction
Also,

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$$\sum_{\rho=1}^{\infty} \alpha_\rho^p \big|\Lambda_\rho(\varsigma, 0, 0, 0)\big|^p$$

converger to zero for all

$$\sup_{\varsigma \in I} \gamma_2(\varsigma) = \hat{\gamma}_2, \qquad \sup_{\varsigma \in I} \gamma_3(\varsigma) = \hat{\gamma}_3, \qquad \sup_{\varsigma \in I} \gamma_1(\varsigma)\gamma_4(\varsigma) = \Gamma_1$$

with $2^{1-1/p}\Gamma_1^{1/p} < 1$. Let $\sum_{\rho=1}^{\infty} \alpha_\rho^p \gamma_2(\varsigma)$ and $\sum_{\rho=1}^{\infty} \alpha_\rho^p \gamma_3(\varsigma)$ be converger for all $\varsigma \in I$

(ii) The fuctions $P_\rho, Q_\rho : I \times C(I, \ell_p^\alpha) \to \mathbb{R} \, (\rho \in \mathbb{N})$ are continuos, and

$$\hat{P}_\rho = \sup\big\{\big|P_\rho(\varsigma, \Theta(\varsigma))\big| : \varsigma \in I; \, \Theta(\varsigma) \in C(I, \ell_p^\alpha)\big\},$$
$$\hat{Q}_\rho = \sup\big\{\big|Q_\rho(\varsigma, \Theta(\varsigma))\big| : \varsigma \in I; \, \Theta(\varsigma) \in C(I, \ell_p^\alpha)\big\}.$$

Also, $\sup_{\rho \in \mathbb{N}} \hat{P}_\rho = \hat{P}, \sup_{\rho \in \mathbb{N}} \hat{Q}_\rho = \hat{Q}$, and $\lim_{\rho \to \infty} \hat{P}_\rho = \lim_{\rho \to \infty} \hat{Q}_\rho = 0$.

(iii) Define an operator $\mathfrak{T}$ from $I \times C(I, \ell_p^\alpha)$ to $C(I, \ell_p^\alpha)$ as $(\varsigma, \Theta(\varsigma)) \to (\mathfrak{T}\Theta)(\varsigma)$, were

$$(\mathfrak{T}\Theta)(\varsigma) = \big\{\Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \Theta(\varsigma)), I^\varpi P_\rho(\varsigma, \Theta(\varsigma)), J^\varpi Q_\rho(\varsigma, \Theta(\varsigma))\big)\big\}_{\rho=1}^{\infty}.$$

Finally, let $B_{p,\alpha} = \{z \in C(I, \ell_p^\alpha) : \|z\|_{C(I,\ell_p^\alpha)} \leqslant r\}$.

Theorem 7. *If conditions (1).(3) hold, then equation (5) admits a solution in* $C(I, \ell_p^\alpha) \, p > 1$.

Proof. For arbitrary fixed

$$\|\Theta(\varsigma)\|_{\ell_p^\alpha}^p = \sum_{\rho \geqslant 1} \alpha_n^p \big|\Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \Theta(\varsigma)), I^\varpi P_\rho(\varsigma, \Theta(\varsigma)), J^\varpi Q_\rho(\varsigma, \Theta(\varsigma))\big)\big|^p$$

$$\leqslant 2^{p-1} \sum_{\rho \geqslant 1} \alpha_\rho^p \big|\Lambda_\rho\big(\varsigma, l_\rho(\varsigma, \Theta(\varsigma)), I^\varpi P_\rho(\varsigma, \Theta(\varsigma)), J^\varpi Q_\rho(\varsigma, \Theta(\varsigma))\big)$$
$$- \Lambda_\rho(\varsigma, 0, 0, 0)\big|^p + 2^{p-1} \sum_{n \geqslant 1} \alpha_\rho^p \big|\Lambda_\rho(\varsigma, 0, 0, 0)\big|^p$$

$$\leqslant 2^{p-1} \sum_{\rho \geqslant 1} \alpha_\rho^p \big[\gamma_1(\varsigma)\gamma_4(\varsigma)\big|\Theta_\rho(\varsigma)\big|^p + \gamma_2(\varsigma)\big|I^\varpi P_\rho(\varsigma, \Theta(\varsigma))\big|^p$$
$$+ \gamma_3(\varsigma)\big|I^\varpi Q_\rho(\varsigma, \Theta(\varsigma))\big|^p\big]$$

$$\leqslant 2^{p-1} \sum_{\rho \geqslant 1} \alpha_\rho^p \big[\gamma_1(\varsigma)\gamma_4(\varsigma)\big|\Theta_\rho(\varsigma)\big|^p$$

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$$+ \gamma_2(\varsigma)\left\{\frac{\hat{P}(T-1)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p + \gamma_3(\varsigma)\left\{\frac{\hat{Q}(\ln T)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p\right]$$

$$\leqslant 2^{p-1}\left[\Gamma_1\|\Theta(\varsigma)\|_{\ell_p^{\alpha}}^p + \Gamma_2\left\{\frac{\hat{P}(T-1)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p + \Gamma_3\left\{\frac{\hat{Q}(\ln T)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p\right].$$

Therefore,

$$(1 - 2^{p-1}\Gamma_1)\|\Theta(\varsigma)\|_{\ell_p^{\alpha}}^p \leqslant 2^{p-1}\left[\Gamma_2\left\{\frac{\hat{P}(T-1)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p + \Gamma_3\left\{\frac{\hat{Q}(\ln T)^{\varpi}}{\Gamma(\varpi+1)}\right\}^p\right]$$

implies

$$\|\Theta(\varsigma)\|_{\ell_p^{\alpha}}^p \leqslant \frac{2^{p-1}[\Gamma_2\hat{P}^p(T-1)^{p\varpi} + \Gamma_3\hat{Q}^p(\ln T)^{p\varpi}]}{(1 - 2^{p-1}\Gamma_1)\{\Gamma(\varpi+1)\}^p} = r^p \quad \text{(say)}.$$

Hence, $\|\Theta\|_{C(I,\ell_p^{\alpha})} \leqslant r$.

Consider the operator $\mathfrak{T}: I \times B_{p,\alpha} \to B_{p,\alpha}$ given by

$$(\mathfrak{T}\Theta)(\varsigma) = \left(\Lambda_{\rho}\big(\varsigma, l_{\rho}\big(\varsigma, \Theta(\varsigma)\big), I^{\varpi}P_{\rho}\big(\varsigma, \Theta(\varsigma)\big), J^{\varpi}Q_{\rho}\big(\varsigma, \Theta(\varsigma)\big)\big)\right)_{\rho=1}^{\infty}$$

$$= \left((\mathfrak{T}_n\Theta)(\varsigma)\right)_{\rho=1}^{\infty},$$

where $\Theta(\varsigma) \in B_{p,\alpha}, \varsigma \in I$.

By assumption (iii), $(\mathfrak{T}\Theta)(\varsigma) \in C(I,\ell_p^{\alpha})$ for $\Theta(\varsigma) \in C(I,\ell_p^{\alpha})$. Also, $\|\mathfrak{T}\Theta\|_{C(I,\ell_p^{\alpha})} \leqslant r$. hence, $\mathfrak{T}$ is a self-mapping on $B_{p,\alpha}$

Let $\bar{\Theta}(\varsigma) = (\bar{\Theta}_{\rho}(\varsigma))_{\rho=1}^{\infty} \in B_{p,\alpha}$ and $\epsilon > 0$ be such that

$$\|\Theta - \bar{\Theta}\|_{C(I,\ell_p^{\alpha})} < \frac{\epsilon}{3^{1/p}\Gamma_1^{1/p}} = \delta.$$

Then, for arbitrary fixed $\varsigma \in I$

$$\left|(\mathfrak{T}_{\rho}\Theta)(\varsigma) - (\mathfrak{T}_{\rho}\bar{\Theta})(\varsigma)\right|^p$$

$$= \left|\Lambda_{\rho}\big(\varsigma, l_{\rho}(\varsigma, \Theta(\varsigma)), I^{\varpi}P_{\rho}\big(\varsigma, \Theta(\varsigma)\big), J^{\varpi}Q_{\rho}\big(\varsigma, \Theta(\varsigma)\big)\big)\right.$$

$$\left. - \Lambda_{\rho}\big(\varsigma, l_{\rho}\big(\varsigma, \bar{\Theta}(\varsigma)\big), I^{\varpi}P_{\rho}\big(\varsigma, \bar{\Theta}(\varsigma)\big), J^{\varpi}Q_{\rho}\big(\varsigma, \bar{\Theta}(\varsigma)\big)\big)\right|^p$$

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$$\leqslant \gamma_1(\varsigma)\big|l_\rho\big(\varsigma,\Theta(\varsigma)\big) - l_\rho\big(\varsigma,\bar{\Theta}(\varsigma)\big)\big|^p$$

$$+ \gamma_2(\varsigma)\big|I^\varpi P_\rho\big(\varsigma,\Theta(\varsigma)\big) - I^\varpi P_\rho\big(\varsigma,\bar{\Theta}(\varsigma)\big)\big|^p$$

$$+ \gamma_3(\varsigma)\big|J^\varpi Q_\rho\big(\varsigma,\Theta(\varsigma)\big) - J^\varpi Q_\rho\big(\varsigma,\bar{\Theta}(\varsigma)\big)\big|^p$$

$$\leqslant \gamma_1(\varsigma)\gamma_4(\varsigma)\big|\Theta_\rho(\varsigma) - \bar{\Theta}_\rho(\varsigma)\big|^p$$

$$+ \gamma_2(\varsigma)\left\{\frac{1}{\Gamma(\varpi)}\int_1^t \frac{|P_\rho(\varsigma,\Theta(\rho)) - P_\rho(\varsigma,\bar{\Theta}(\rho))|}{(t-s)^{1-\varpi}}\,\mathrm{d}s\right\}^p$$

$$+ \gamma_3(\varsigma)\left\{\frac{1}{\Gamma(\varpi)}\int_1^t \frac{|Q_\rho(\varsigma,\Theta(\rho)) - Q_\rho(\varsigma,\bar{\Theta}(\rho))|}{s(\ln t - \ln s)^{1-\varpi}}\,\mathrm{d}s\right\}^p.$$

Since P_ρ, Q_ρ are continuous for all $\rho \in \mathbb{N}$, so we have $\|\Theta - \bar{\Theta}\|_{C(I,\ell_p^\alpha)} < \epsilon/(3^{1/p}\Gamma_1^{1/p})$ for all $\rho \in \mathbb{N}, \varsigma \in I$,

$$\alpha_\rho\big|P_\rho\big(\varsigma,\Theta(\rho)\big) - P_\rho\big(\varsigma,\bar{\Theta}(\rho)\big)\big| < \frac{\epsilon\Gamma(1+\varpi)}{3^{1/p}\hat{\gamma}_2^{1/p}(T-1)^\varpi}$$

and

$$\alpha_\rho\big|Q_\rho\big(\varsigma,\Theta(\rho)\big) - Q_\rho\big(\varsigma,\bar{\Theta}(\rho)\big)\big| < \frac{\epsilon\Gamma(1+\varpi)}{3^{1/p}\hat{\gamma}_3^{1/p}(\ln T)^\varpi}.$$

Therefore

$$\sum_{n\geqslant 1}\alpha_\rho^p\big|(\mathfrak{T}_\rho\Theta)(\varsigma) - (\mathfrak{T}_\rho\bar{\Theta})(\varsigma)\big|^p$$

$$\leqslant \Gamma_1\sum_{\rho\geqslant 1}\alpha_\rho^p\big|\Theta_\rho(\varsigma) - \bar{\Theta}_\rho(\varsigma)\big|^p + \hat{\gamma}_2\left\{\frac{1}{\Gamma(\varpi)}\frac{\epsilon\Gamma(1+\varpi)}{3^{1/p}\hat{\gamma}_2^{1/p}(T-1)^\varpi}\frac{(T-1)^\varpi}{\varpi}\right\}^p$$

$$+ \hat{\gamma}_3\left\{\frac{1}{\Gamma(\varpi)}\frac{\epsilon\Gamma(1+\varpi)}{3^{1/p}\hat{\gamma}_2^{1/p}(\ln T)^\varpi}\frac{(\ln T)^\varpi}{\varpi}\right\}^p$$

$$\Gamma_1\|\Theta - \bar{\Theta}\|_{C(I,\ell_p^\alpha)}^p + \frac{\epsilon^p}{3} + \frac{\epsilon^p}{3} < \Gamma_1\frac{\epsilon^p}{3\Gamma_1} + \frac{2\epsilon^p}{3} = \epsilon^p.$$

Thus, $\|\mathfrak{T}\Theta - \mathfrak{T}\bar{\Theta}\|_{C(I,\ell_p^\alpha)}^p < \epsilon^p$ when $\|\Theta - \bar{\Theta}\|_{C(I,\ell_p^\alpha)}^p < \epsilon^p/(2\hat{\delta})$; hence, $\mathfrak{T}$ is continuous on $B_{p,\alpha}$
Finally,

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$$\chi_{\ell_p^\alpha}(\mathfrak{T}B_{p,\alpha})$$

$$= \lim_{n\to\infty} \sup_{\Theta\in B_{p,\alpha}} \left\{ \sum_{\rho\geqslant n} \alpha_\rho^p \big| \Lambda_\rho\big(\varsigma, l_\rho(\varsigma,\Theta(\varsigma)), I^\varpi P_\rho(\varsigma,\Theta(\varsigma)), J^\varpi Q_\rho(\varsigma,\Theta(\varsigma))\big) \big|^p \right\}^{1/p}$$

$$\leqslant \lim_{n\to\infty} \sup_{\Theta\in B_{p,\alpha}} \left\{ 2^{p-1} \sum_{\rho\geqslant n} \alpha_\rho^p \Big[\gamma_1(\varsigma)\gamma_4(\varsigma) |\Theta_\rho(\varsigma)|^p \right.$$

$$\left. + \gamma_2(\varsigma)\left(\frac{\hat{P}(T-1)^\varpi}{\Gamma(\varpi+1)} \right)^p + \gamma_3(\varsigma)\left(\frac{\hat{Q}(\ln T)^\varpi}{\Gamma(\varpi+1)} \right)^p \Big] \right\}^{1/p}$$

$$\leqslant 2^{1-1/p} \lim_{n\to\infty} \sup_{\Theta\in B_{p,\alpha}} \left\{ \left[\Gamma_1 \sum_{\rho\geqslant n} \alpha_\rho^p |\Theta_\rho(\varsigma)|^p + \left(\frac{\hat{P}(T-1)^\varpi}{\Gamma(\varpi+1)} \right)^p \sum_{\rho\geqslant n} \alpha_\rho^p \gamma_2(\varsigma) \right. \right.$$

$$\left. \left. + \left(\frac{\hat{Q}(\ln T)^\varpi}{\Gamma(\varpi+1)} \right)^p \sum_{\rho\geqslant n} \alpha_\rho^p \gamma_3(\varsigma) \right] \right\}^{1/p},$$

i.e.,

$$\chi_{\ell_p^\alpha}(\mathfrak{T}B_{p,\alpha}) \leqslant 2^{1-1/p} \Gamma_1^{1/p} \chi_{\ell_p^\alpha}(B_{p,\alpha}).$$

Therefore,

$$\chi_{C(I,\ell_p^\alpha)}(\mathfrak{T}B_{p,\alpha}) \leqslant 2^{1-1/p} \Gamma_1^{1/p} \chi_{C(I,\ell_p^\alpha)}(B_{p,\alpha}).$$

Thus, by assumption (i) and Remark 1 one gets that $\mathfrak{T}$ admits a fixed point in $B_{p,\alpha}$

Example 2. Consider

$$\Theta_\rho(\varsigma) = \frac{\Theta_\rho(\varsigma)}{12\varsigma^2} + \frac{1}{\Gamma(\frac{1}{2})\rho^2} \int_0^\varsigma \frac{\cos(\Theta_\rho(\varsigma))}{(\varsigma+\rho)(\varsigma-w)^{1/2}} \, \mathrm{d}w$$

$$+ \frac{1}{\Gamma(\frac{1}{2})\rho^4} \int_0^\varsigma \left(\ln \frac{\varsigma}{w} \right)^{1/2} \frac{\sin^2(\Theta_\rho(\varsigma))}{\varsigma+\rho^2} \frac{\mathrm{d}w}{w}, \tag{6}$$

where $\varsigma \in I = [1,2], \rho \in \mathbb{N}$. Here

$$\Lambda_\rho\big(\varsigma, l_\rho(\Theta), I(\Theta), J(\Theta)\big) = \frac{l_\rho(\Theta)}{2} + I(\Theta) + J(\Theta), \quad l_\rho(\Theta) = \frac{\Theta_\rho(\varsigma)}{6\varsigma^2},$$

$$I(\Theta) = \frac{1}{\Gamma(\frac{1}{2})\rho^2} \int_0^\varsigma \frac{\cos(\Theta_\rho(\varsigma))}{(\varsigma-w)^{1/2}} \, \mathrm{d}w, \quad J(\Theta) = \frac{1}{\Gamma(\frac{1}{2})\rho^4} \int_0^\varsigma \left(\ln \frac{\varsigma}{w} \right)^{1/2} \sin^2(\Theta_\rho(\varsigma)) \frac{\mathrm{d}w}{w},$$

$$P_\rho\big(\varsigma,\Theta(\varsigma)\big) = \frac{\cos(\Theta_\rho(\varsigma))}{\varsigma+\rho}, \quad Q_\rho\big(\varsigma,\Theta(\varsigma)\big) = \frac{\sin^2(\Theta_\rho(\varsigma))}{\varsigma+\rho^2}, \quad \varpi = \frac{1}{2}, \quad \text{and} \quad T = 2.$$

Let $\alpha_\rho = 1/\rho$. We have $\Lambda(\varsigma,0,0,0) = 0$, so $\sum_{\rho=1}^\infty \alpha_\rho^p |\Lambda(\varsigma,0,0,0)|^p$ *converges to zero.*
Let $\Theta(\varsigma) \in \ell_p^\alpha$ for some fixed $\varsigma \in I$. Then

$$\sum_{n\geqslant 1} \alpha_\rho^p \big| \Lambda_\rho\big(\varsigma, l_\rho(\Theta), I(\Theta), J(\Theta)\big)\big|^p$$

$$= \sum_{n\geqslant 1} \frac{1}{\rho^p} \left| \frac{\Theta_\rho(\varsigma)}{12\varsigma^2} + \frac{1}{\Gamma(\frac{1}{2})\rho^2} \int_0^\varsigma \frac{\cos(\Theta_\rho(\varsigma))}{(\varsigma+\rho)(\varsigma-w)^{1/2}}\, \mathrm{d}w \right.$$

$$\left. + \frac{1}{\Gamma(\frac{1}{2})\rho^4} \int_0^\varsigma \left(\ln\frac{\varsigma}{w}\right)^{1/2} \frac{\sin^2(\Theta_\rho(\varsigma))}{\varsigma+\rho^2} \frac{\mathrm{d}w}{w} \right|^p$$

$$\leqslant \frac{2^{p-1}}{12^p} \sum_{\rho\geqslant 1} \frac{1}{\rho^p} |\Theta_\rho(\varsigma)|^p + \frac{2^{p-1}}{(\Gamma(\frac{1}{2}))^p}\big[(T-1)^{1/2}+(\ln T)^{1/2}\big]^p \sum_{\rho\geqslant 1}\frac{1}{\rho^p}$$

is convergent as $\Theta(\varsigma)\in C(I,\ell_p^\alpha)$, and $\sum_{\rho\geqslant 1} 1/\rho^p$ is convergent for $p>1$. Therefore for fixed $\varsigma\in I$

$$\big\{\Lambda_\rho\big(\varsigma, l_\rho(\Theta), I(\Theta), J(\Theta)\big)\big\}_{\rho=1}^\infty \in \ell_p^\alpha,$$

i.e.,

$$\big\{\Lambda_\rho\big(\varsigma, l_\rho(\Theta), I(\Theta), J(\Theta)\big)\big\}_{\rho=1}^\infty \in C\big(I, \ell_p^\alpha\big).$$

It is obvious that Λ_ρ is continuous for all $\rho\in\mathbb{N}$ and

$$\big|\Lambda_\rho\big(\varsigma, l_\rho(\Theta), I(\Theta), J(\Theta)\big) - \Lambda_\rho\big(\varsigma, \bar{l}_\rho(\Theta), \bar{I}(\Theta), \bar{J}(\Theta)\big)\big|^p$$
$$\leqslant \frac{1}{2}|l_\rho(\Theta)-\bar{l}_\rho(\Theta)|^p + 2^{2p-2}|I(\Theta)-\bar{I}(\Theta)|^p + 2^{2p-2}|J(\Theta)-\bar{J}(\Theta)|^p;$$

also, $\gamma_1(\varsigma)=1/2, \gamma_2(\varsigma)=\gamma_3(\varsigma)=2^{2p-2}$. Moreover,

$$\big|l_\rho(\Theta)-l_\rho(\bar\Theta)\big|^p = \frac{1}{6^p\varsigma^{2p}}\big|\Theta_\rho(\varsigma)-\bar\Theta_\rho(\varsigma)\big|^p,$$

i.e., $\gamma_4(\varsigma)=1/(6^p\varsigma^{2p})$, and also, l_ρ is continuous for all $\rho\in\mathbb{N}$ It can be observed that $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ are all continuous, and

$$\hat\gamma_2 = \hat\gamma_3 = 2^{2p-2} \quad \text{and} \quad \Gamma_1 = \frac{1}{2.6^p},$$

which gives

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$$2^{1-1/p} \Gamma_1^{1/p} = 3^{-1} 2^{-2/p} < 1.$$

The functions P_ρ, Q_ρ are continuous, and

$$\hat{P}_\rho = \frac{1}{1+\rho}, \quad \hat{Q}_\rho = \frac{1}{1+\rho^2}, \quad \hat{P} = \hat{Q} = \frac{1}{2}, \quad \text{and} \quad \lim_{\rho \to \infty} \hat{P}_\rho = \lim_{\rho \to \infty} \hat{Q}_\rho = 0$$

Also,

$$\sum_{\rho \geqslant 1} \alpha_\rho^p \gamma_2(\varsigma) = \sum_{\rho \geqslant 1} \frac{2^{2p-2}}{\rho^p} \quad \text{and} \quad \sum_{\rho \geqslant 1} \alpha_\rho^p \gamma_3(\varsigma) = \sum_{\rho \geqslant 1} \frac{2^{2p-2}}{\rho^p}$$

are convergent for $p > 1$.

Thus, all conditions (1)–(3) of Theorem 7 are satisfied, hence, equation (6) admits a solution in $C(I, \ell_p^c)$.

Acknowledgments

References

1 R.P. Agarwal, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Univ. Press, Cambridge, 2004, https://doi.org/10.1017/CBO97805115 43005.

2 I. Altun, D. Turkoglu, A fixed point theorem for mappings satisfying a general contractive condition of operator type, *J. Comput. Anal. Appl.*, .(1):9–14, 2007.

3 R. Arab, H.K. Nashine, N.H. Can, T.T. Binh, Solvability of functional-integral equations (fractional order) using measure of noncompactness, *Adv. Difference Equ.*, **2020**:12, 2020, https://doi.org/10.1186/s13662-0 19-2487-4.

4 J. Banas′, K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lect. Notes Pure Appl. Math., Vol. 60, Marcel Dekker, New York, 1980.

5 J. Banas′, M. Jleli, M. Mursaleen, B. Samet, C. Vetro (Eds.), *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Springer, Singapore, 2017, https://doi. org/10.1007/978-981-10-3722-1.

6 J. Banas′, M. Krajewska, Existence of solutions for infinite systems of differential equations in spaces of tempered sequences, *Electron. J. Differ. Equ.,* **2017**:60, 2017.

7 J. Banas′, M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.

8 S.S. Chang, Y. J. Huang, Coupled fixed point theorems with applications, *J. Korean Math. Soc.*, **33**(3):575–585, 1996.

9 G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Sem. Mat. Univ. Padova*, **24**:84–92, 1955.

10 M.A. Darwish, K. Sadarangani, On a quadratic integral equation with supremum involving Erdélyi-Kober fractional order, *Math. Nachr.,* **288**(5–6):566–576, 2015, https://doi. org/10.1002/mana.201400063.

11 A. Das, B. Hazarika, R. Arab, R.P. Agarwal, H.K. Nashine, Solvability of infinite systems of fractional differential equations in the space of tempered sequences, *Filomat*, **33**(17):5519– 5530, 2019, https://doi.org /10.2298/FIL1917519D.

12 A. Das, B. Hazarika, V.Parvaneh, M. Mursaleen, Solvability of generalized fractional order integral equations via measures of noncompactness, *Math. Sci.*, **15**:241–251, 2021, https://doi.org/10.1007/s40096-020-003 59-0.

13 A. Das, B. Hazarkia, M. Mursaleen, Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in *lp* (1 < *p* <), *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **113**(1):31–40, 2019, https://doi.org/ 10.1007/ s13398-017-0452-1.

14 M. Gabeleh, H.A. Künzi, Equivalence of the existence of best proximity points and best proximity pairs for cyclic and noncyclic nonexpansive mappings, *Demonstr. Math.*, **53**:38– 43, 2020, https://doi.org/10.1515/ dema-2020-0005.

15 R. Garra, E. Orsingher, F. Polito, A note on Hadamard fractional differential equations with varying coefficients and their applications in probability, *Mathematics*, .(1):4, 2018, https://doi.org/10.3390/math6010004.

16 J. Harjani, B. López, K. Sadarangani, Existence and uniqueness of mild solutions for a fractional differential equation under Sturm-Liouville boundary conditions when the data function is of lipschitzian type, *Demonstr. Math.*, **53**:167–173, 2020, https://doi.org/ 10.1515/ dema-2020-0014.

17 S.G. Hristova, S.A. Tersian, Scalar linear impulsive Riemann-Liouville fractional differential equations with constant delay-explicit solutions and finite time stability, *Demonstr. Math.*, **53**:121–130, 2020, https://doi.org /10.1515/dema-2020-0012.

18 H.R. Kataria, P.H. Patel, V. Shah, Existence results of noninstantaneous impulsive fractional integro-differential equation, *Demonstr. Math.*, **53**:373–384, 2020, https://doi.org/ 10.1515/dema-2020-0029.

19 K. Kuratowski, Sur les espaces complets, *Fundamenta*, **15**:301–309, 1930, h ttps://doi. org/10.4064/fm-15-1-301-309.

20 R. Mollapourasl, A. Ostadi, On solution of functional integral equation of fractional order, *Appl. Math. Comput.,* **270**:631–643, 2015, https://doi.o rg/10.1016/j.amc. 2015.08.068.

21 M. Mursaleen, S.A. Mohiuddine, Applications of measures of noncompactness to the infinite system of differential equations in lp spaces, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2111–2115, 2012, https://doi.org/10.1016/j.na.2011.10.011.

22 T. Nabil, Existence results for nonlinear coupled system of integral equations of Urysohn Volterra-Chandrasekhar mixed type, *Demonstr. Math.*, **53**:236–248, 2020, https://doi. org/10.1515/dema-2020-0017.

23 H.K. Nashine, R. Arab, R.P. Agarwal, A.S. Haghigh, Darbo type fixed and coupled fixed point results and its application to integral equation, *Period. Math. Hung.*, 77(1):94–107, 2018, https://doi.org/10.1007/s10998-017 -0223-y.

24 H.K. Nashine, R. Arab, R.P. Agarwal, M. De la Sen, Positive solutions of fractional integral equations by the technique of measure of noncompactness, *J. Inequal. Appl.*, **2017**:225, 2017, https://doi.org/10.1 186/s13660-017-1497-6.

25 M. Rabbani, A. Das, B. Hazarika, R. Arab, Existence of solution for two dimensional nonlinear fractional integral equation by measure of noncompactness and iterative algorithm to solve it, *J. Comput. Appl. Math.*, **370**:112654, 2020, https://doi.org/10.1016/j.cam. 2019.112654.

26 M. Rabbani, A. Das, B. Hazarika, R. Arab, Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations, *Chaos Solitons Fractals*, **140**:110221, 2020, https:/ /doi.org/10.1016/j.chaos.2020. 110221.

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