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Existence, uniqueness, Ulam–Hyers–Rassias stability, well-posedness and data dependence property related to a fixed point problem in γ-complete metric spaces with application to integral equations

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Abstract: In this paper, we study a fixed point problem for certain rational contractions on .-complete metric spaces. Uniqueness of the fixed point is obtained under additional conditions. The Ulam–Hyers–Rassias stability of the problem is investigated. Well-posedness of the problem and the data dependence property are also explored. There is a corollary of the main result. Finally, our fixed point theorem is applied to solve a problem of integral equation. There is no continuity assumption on the mapping.
Keywords: fixed point, Ulam–Hyers stability, well-posedness, data dependence, Fredholm-type nonlinear integral equation.

1 Introduction and mathematical background

In this paper, we consider a rational contraction on metric spaces and investigate the fixed point problem associated with it. We assume that the metric space is γ-complete, which is a concept introduced by Kutbi and Sintunavarat in the paper [11]. The uniqueness of the fixed point is obtained under additional conditions. Rational contractions were first introduced by Dass et al. [6] and have been considered in fixed point theory in recent works like [4, 8]. Our investigation of the different aspects of the fixed point problem is performed in a metric space without completeness property. In most of the works on similar problems, the results are obtained by employing metric completeness. Instead, we assume the weaker concept of γ-completeness. There is a flexibility in such assumption since the choice of γ can be different subject to certain restrictions. This is one of the main motivations behind our considerations of the problems discussed in this paper. We impose an admissibility condition on the concerned mapping. The assumption of the contractive inequality is restricted to certain pairs of points. These assumptions are in tune with certain recent trends appearing in metric fixed point theory. Further, there are scopes of extending our present results, which are discussed at the end of the paper.

We investigate Ulam–Hyers–Rassias stability of the fixed point problem. It is a general type of stability, which is considered in several areas of mathematics. Introduced by Ulam [25] through a mathematical question posed in 1940 and later elaborated by Hyers [9] and Rassias [18], such stabilities have a very large literature today [10, 16, 19].

Well-posedness and data dependence property associated with this problem are also investigated.

Finally, we have an application of our results to a problem of a nonlinear integral equation.

Definition 1.(See [1].) An element $s \in Z$ is called a fixed point of a mapping $F : Z \to Z$ $s = Fs$.

Several sufficient conditions have been discussed for the existence of fixed points of $F : Z \to Z$, where Z has a metric d defined on it. The study is a part of the subject domain known as metric fixed point theory. The subject is widely recognized to have been originated in the work of Banach in 1922 [1], which is known as the Banach's contraction mapping principle and is instrumental to the proofs of many important results. In subsequent times, many metric fixed point results were proved and applied to different problem arising in mathematics. Today fixed point methods are recognized as strong mathematical methods. References [12, 13] describe this development to a considerable extent.

Definition 2.(See [22].) A function $\gamma : Z \times Z \to [0, \infty)$, where Z is a nonempty set, has triangular property if for $a, b, c \in Z$, $\gamma(a, b) \geqslant 1$ and $\gamma(b, c) \geqslant 1$ imply $\gamma(a, c) \geqslant 1$.

Admissibility conditions have recently been used for obtaining fixed point results. Various admissibility criteria were introduced in the study of fixed points of mappings. We refer the reader to [21–23] for some details on admissibility conditions.

Definition 3.(See [22].) Let Z be a nonempty set, $F : Z \to Z$ and $\gamma : Z \times Z \to [0, \infty)$ .The mapping F is called γ-admissible if $\gamma(a, b) \geqslant 1$ for $a, b \in Z$ implies that $\gamma(Fa, Fb) \geqslant 1$.

Definition 4. Let Z be a nonempty set and $F : Z \to Z$ A function $\gamma : Z \times Z \to [0, \infty)$ is said to have F -directed property if for every $a, b \in Z$, there exists $u \in Z$ with $\gamma(u, Fu) \geqslant 1$ such that $\gamma(a, u) \geqslant 1, \gamma(b, u) \geqslant 1$.

Definition 5. (See [23].) A metric space (Z, d) is said to have regular property with respect to a mapping $\gamma : Z \times Z \to [0, \infty)$ if for any sequence $\{a_n\}$ in Z with limit $a \in Z$, $\gamma(a_n, a_{n+1}) \geqslant 1$ implies for all n .

Example 1. Let $Z = (-2, 2)$ be equipped with usual metric. Let $F : Z \to Z$ and $\gamma : Z \times Z \to [0, \infty)$ be respectively defined as follows:

$$Fx = \frac{\sin^2 x}{16}, \quad x \in Z, \quad \text{and} \quad \gamma(x, y) = \begin{cases} e^{x+y} & \text{if } 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1/8, \\ 0 & \text{otherwise.} \end{cases}$$

Here (i) F is a γ-admissible mapping; (ii) γ has triangular property; (iii) Z has regular property with respect to γ.

Recently, Kutbi and Sintunavarat coined the concept of γ-complete metric space in the paper [11].

Definition 6. Let (Z, d) be a metric space and $\gamma : Z \times Z \to [0, \infty)$..A Cauchy sequence $\{a_n\}$ in Z is called a γ-Cauchy sequence if $\gamma(a_n, a_{n+1}) \geqslant 1$ for all n .

Definition 7. (See [11].) A metric space (Z,d) is said to be γ-complete, where $\gamma : Z \times Z \to [0,\infty)$., if every γ-Cauchy sequence in . converges to a point in Z.

Remark 1. If (Z,d) is a complete metric space, then Z is also a γ-complete metric space for any $\gamma : Z \times Z \to [0,\infty)$. but the converse is not true.

Example 2. Let $Z=(0,\infty)$ be equipped with usual metric d. Let $\gamma : Z \times Z \to [0,\infty)$ be defined as

$$\gamma(x,y) = \begin{cases} e^{x+y} & \text{if } x,y \in [2019,2020], \\ 0 & \text{otherwise;} \end{cases}$$

then (Z,d) is a γ-complete metric space. Here (Z,d) is not a complete metric space. Indeed, if $\{a_n\}$ is a Cauchy sequence in Z such that $\gamma(a_n,a_{n+1}) \geqslant 1$ for all $n \in N$, then $a_n \in$ [2019, 2020] for all $n \in N$.As [2019, 2020] is a closed subset of R, it follows that there exists $a \in$ [2019, 2020] such that $a_n \to a$ as $n \to \infty$.

Definition 8. Let $F : Z \to Y$ be a mapping and $\gamma : Z \times Z \to [0,\infty)$, where (Z,ρ), (Y,d) are two metric spaces. The mapping F is said to be γ-continuous at $c \in Z$ if for any sequence $\{t_n\}$ in Z, $\rho(c,t_n) \to 0$ as $n \to \infty$ and $\gamma(t_n,t_{n+1}) \geqslant 1$ for all n imply that Z, $\rho(c,t_n) \to 0$ as $n \to \infty$ and $\gamma(t_n,t_{n+1}) \geqslant 1$

Remark 2. The continuity of a mapping implies its γ-continuity for any $\gamma : Z \times Z \to [0,\infty)$.In general, the converse is not true.

Problem P. Let $F : Z \to Z$ be a mapping, where (Z,d) is a metric space. Consider the problem of finding a point $s \in Z$ satisfying $s = Fs$.

Our paper is characterized by the following features.

1. We consider rational contractions in our theorems.
2. We prove our main result with a generalized notion of completeness assumption of the underlying space and without the continuity assumption on the mapping.
3. We investigate Ulam–Hyers–Rassias stability of the fixed point problem.
4. We investigate well-posedness of the problem.
5. We investigate data dependence of fixed point set and solution of the integral equation.
6. We apply our theorem to a problem of an integral equation.

2 Main results

In this section, we establish some fixed point results and illustrate them with examples. We discuss the uniqueness of the fixed point under some additional assumptions. We deduce a corollary of the main result.

Let (Z,d) be a metric space and $\gamma : Z \times Z \to [0,\infty)$, $F : Z \to Z$ be two mappings.

We designate the following properties by (A1), (A2) and (A3):

(A1) Z has regular property with respect to γ;
(A2) γ has triangular property;
(A3) γ has F -directed property.

Theorem 1. *Let* (Z,d) *be a metric space and* $\gamma : Z \times Z \to [0,\infty)$, *be a function such that* (Z,d) *is* γ *-complete. Let* $F : Z \to Z$ *be a* γ*-admissible mapping and there exists* $k \in (0,1)$ *such that for* $x,y \in Z$ *with* $\gamma(x,y) \geqslant 1$,

$$d(Fx,Fy) \leqslant k \max\left\{d(x,y), \frac{d(x,Fx)+d(y,Fy)}{2}, \frac{d(x,Fy)+d(y,Fx)}{2}, \frac{d(x,Fx)d(y,Fy)}{1+d(Fx,Fy)}, \frac{d(y,Fx)d(x,Fy)}{1+d(Fx,Fy)}, \frac{d(y,Fy)d(y,Fx)}{1+d(Fx,Fy)}\right\}. \quad (1)$$

If there exists $z_0 \in Z$ *such that* $\gamma(z_0,Fz_0) \geqslant 1$ *and property* (A1) *holds, then* F *has a fixed point in* Z *.*

Proof. Let $z_0 \in Z$ be such that $\gamma(z_0,Fz_0) \geqslant 1$. We construct a sequence $\{z_n\}$ in Z such that

$$z_{n+1} = Fz_n \qquad \forall n \geqslant 0. \quad (2)$$

As $\gamma(z_0,Fz_0) = \gamma(z_0,z_1) \geqslant 1$ and F is γ-admissible, we have $\gamma(z_1,z_2) \geqslant 1$. Since F is γ $\gamma(Fz_1,Fz_2) = \gamma(z_2,z_3) \geqslant 1$

$$\gamma(z_n,z_{n+1}) \geqslant 1 \quad \forall n \geqslant 0. \quad (3)$$

Let

$$\delta_n = d(z_n,z_{n+1}) \quad \forall n \geqslant 0. \quad (4)$$

By (1)–(4) we have

$$\begin{aligned} d(z_{n+1}, z_{n+2}) &= d(Fz_n, Fz_{n+1}) \\ &\leqslant k \max\left\{d(z_n, z_{n+1}), \frac{d(z_n, Fz_n) + d(z_{n+1}, Fz_{n+1})}{2}, \right. \\ &\quad \frac{d(z_n, Fz_{n+1}) + d(z_{n+1}, Fz_n)}{2}, \frac{d(z_n, Fz_n)d(z_{n+1}, Fz_{n+1})}{1 + d(Fz_n, Fz_{n+1})}, \\ &\quad \left. \frac{d(z_{n+1}, Fz_n)d(z_n, Fz_{n+1})}{1 + d(Fz_n, Fz_{n+1})}, \frac{d(z_{n+1}, Fz_{n+1})d(z_{n+1}, Fz_n)}{1 + d(Fz_n, Fz_{n+1})}\right\} \\ &= k \max\left\{d(z_n, z_{n+1}), \frac{d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2})}{2}, \right. \\ &\quad \left. \frac{d(z_n, z_{n+2})}{2}, \frac{d(z_n, z_{n+1})d(z_{n+1}, z_{n+2})}{1 + d(z_{n+1}, z_{n+2})}, 0, 0\right\} \\ &\leqslant k \max\left\{d(z_n, z_{n+1}), \frac{d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2})}{2}, \right. \\ &\quad \left. \frac{d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2})}{2}, d(z_n, z_{n+1})\right\} \\ &= k \max\left\{\delta_n, \frac{\delta_n + \delta_{n+1}}{2}, \frac{\delta_n + \delta_{n+1}}{2}, \delta_n\right\} \\ &= k \max\{\delta_n, \delta_{n+1}\}. \end{aligned}$$

Therefore,

$$d(z_{n+1}, z_{n+2}) \leqslant k \max\{\delta_n, \delta_{n+1}\}.$$

Suppose that $0 \leqslant \delta_n < \delta_{n+1}$. From (4) and (5) we have

$$\delta_{n+1} = d(z_{n+1}, z_{n+2}) \leqslant k \max\{\delta_n, \delta_{n+1}\} = k\delta_{n+1} < \delta_{n+1},$$

which is a contradiction. Therefore, $\delta_{n+1} \leqslant \delta_n$ for all $n \geqslant 0$, that is, $\{d(z_n, z_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Then from (5) we have

$$d(z_{n+1}, z_{n+2}) = \delta_{n+1} \leqslant k\delta_n = kd(z_n, z_{n+1}) \quad \forall n \geqslant 0. \tag{6}$$

By repeated application of (6) we have

$$d(z_{n+1}, z_{n+2}) \leqslant kd(z_n, z_{n+1}) \leqslant k^2 d(z_{n-1}, z_n) \leqslant \cdots \leqslant k^{n+1} d(z_0, z_1). \tag{7}$$

With the help of (7), we have

$$\sum_{n=1}^{\infty} d(z_n, z_{n+1}) \leqslant \sum_{n=1}^{\infty} k^n d(z_0, z_1) = \frac{1}{1-k} d(z_0, z_1) < \infty,$$

which implies that $\{z_n\}$ is a γ-Cauchy sequence in Z. As Z is γ-complete, there exists $s \in Z$ such that

$$\lim_{n\to\infty} z_n = s. \tag{8}$$

By (3), (8) and property (A1) we have $\gamma(z_n, s) \geqslant 1$ for all $n \geqslant 0$.. Using (2), we have

$$\begin{aligned} d(z_{n+1}, Fs) &= d(Fz_n, Fs) \\ &\leqslant k \max\left\{d(z_n, s), \frac{d(z_n, Fz_n) + d(s, Fs)}{2}, \frac{d(z_n, Fs) + d(s, Fz_n)}{2},\right. \\ &\qquad \left.\frac{d(z_n, Fz_n)d(s, Fs)}{1 + d(Fz_n, Fs)}, \frac{d(s, Fz_n)d(z_n, Fs)}{1 + d(Fz_n, Fs)}, \frac{d(s, Fs)d(s, Fz_n)}{1 + d(Fz_n, Fs)}\right\} \\ &= k \max\left\{d(z_n, s), \frac{d(z_n, z_{n+1}) + d(s, Fs)}{2}, \frac{d(z_n, Fs) + d(s, z_{n+1})}{2},\right. \\ &\qquad \left.\frac{d(z_n, z_{n+1})d(s, Fs)}{1 + d(z_{n+1}, Fs)}, \frac{d(s, z_{n+1})d(z_n, Fs)}{1 + d(z_{n+1}, Fs)}, \frac{d(s, Fs)d(s, z_{n+1})}{1 + d(z_{n+1}, Fs)}\right\}. \end{aligned} \tag{9}$$

Taking limit as $n \to \infty$ in (9) and using (8), we have

$$d(s, Fs) \leqslant k \max\left\{0, \frac{d(s, Fs)}{2}, \frac{d(s, Fs)}{2}, 0, 0, 0\right\} = k\frac{d(s, Fs)}{2},$$

which implies that $d(s, Fs) = 0$, that is, $s = Fs$, that is, s is a fixed point of F.

Remark 3. By Remark 1, Theorem 1 is still valid if one considers (Z, d) to be a complete metric space instead of a γ-complete metric space.

We present the following illustrative example in support of Theorems 1.

Example 3. Using the metric space Z, mappings γ and F as in Example 1, we see that $Z = (-2, 2)$ is regular with respect to γ (see Example 1), that is, property (A1) holds, and F is a γ-admissible mapping. Let $k = 1/4$.

Let $x, y \in Z$ with $\gamma(x, y) \geqslant 1$. Then $x \in [0, 1]$ and $y \in [0, 1/8]$. Therefore, it is required to verify the inequality in Theorem 1 for $x \in [0, 1]$ and $y \in [0, 1/8]$. Now, $d(x, y) = |x - y|$ and

$$\begin{aligned} d(Fx, Fy) &= \left|\frac{\sin^2 x}{16} - \frac{\sin^2 y}{16}\right| = \frac{1}{16}\left|\sin(x-y)\sin(x+y)\right| \\ &\leqslant \frac{1}{16}\left|\sin(x-y)\right| \leqslant \frac{|x-y|}{16} = \frac{1}{4}\frac{|x-y|}{4} = \frac{1}{4}\frac{d(x,y)}{4} \\ &\leqslant \frac{1}{4}\max\left\{d(x, y), \frac{d(x, Fx) + d(y, Fy)}{2}, \frac{d(x, Fy) + d(y, Fx)}{2},\right. \\ &\qquad \left.\frac{d(x, Fx)d(y, Fy)}{1 + d(Fx, Fy)}, \frac{d(y, Fx)d(x, Fy)}{1 + d(Fx, Fy)}, \frac{d(y, Fy)d(y, Fx)}{1 + d(Fx, Fy)}\right\}. \end{aligned}$$

Hence, all the conditions of Theorem 1 are satisfied, and 0 is a fixed point of F.

Note 1. Theorem 1 is still valid if one considers the γ-continuity of F instead of taking property (A1). Then the portion of the proof just after (8) of Theorem 1 is replaced by the following portion:

Using the γ-continuity assumption of F , we have

$$d(s, Fs) = \lim_{n\to\infty} d(z_{n+1}, Fs) = \lim_{n\to\infty} d(Fz_n, Fs) = 0.$$

Hence, $s = Fs$, that is, s is a fixed point of F .

We present the following illustrative example in view of Note 1.

Example 4. Let $Z = (-2, 2)$ be equipped with usual metric d . Let $F : Z \to Z$ and $\gamma : Z \times Z \to [0, \infty)$ be respectively defined as follows:

$$Fx = \begin{cases} x/16 & \text{if } x \in [0,1], \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma(x,y) = \begin{cases} 1 & \text{if } x, y \in (0,1), \\ 0 & \text{otherwise;} \end{cases}$$

then (Z, d) is a γ-complete metric space. Here (Z, d) is not a complete metric space. Let us consider the sequence $\{t_n\}$, where where $t_n = 1/(2n)$. Here $t_n \to 0$ as $n \to \infty$., and $\gamma(t_n, t_{n+1}) \geqslant 1$ for all n . But $\gamma(t_n, 0) = 0$, and hence, Z is not regular with respect to γ. Also, F is γ-admissible. Here the function F is not continuous but γ-continuous. Choose $k = 1/4$.

Let $x, y \in Z$ with $\gamma(x, y) \geqslant 1$. Then $x, y \in (0, 1)$. In view of the above and Note 1, it only remains to be verified that the inequality in Theorem 1 is valid for all $x, y \in (0, 1)$. Now, $d(x, y) = |x - y|$ and

$$\begin{aligned} d(Fx, Fy) &= \left|\frac{x}{16} - \frac{y}{16}\right| = \frac{1}{16}|x-y| = \frac{1}{4}\frac{|x-y|}{4} = \frac{1}{4}\frac{d(x,y)}{4} \\ &\leqslant \frac{1}{4}\max\Bigg\{d(x,y), \frac{d(x,Fx)+d(y,Fy)}{2}, \frac{d(x,Fy)+d(y,Fx)}{2}, \\ &\qquad \frac{d(x,Fx)d(y,Fy)}{1+d(Fx,Fy)}, \frac{d(y,Fx)d(x,Fy)}{1+d(Fx,Fy)}, \frac{d(y,Fy)d(y,Fx)}{1+d(Fx,Fy)}\Bigg\}. \end{aligned}$$

Here 0 is a fixed point of F.

Remark 4. By Remark 2, Theorem 1 is still valid if one considers the continuity of F instead of taking property (A1).

Theorem 2. *In addition to the hypothesis of Theorem ., suppose that properties* (A2) *and* (A3) *hold. Then* F *has a unique fixed point.*

Proof. By Theorem 1 the set of fixed points of F is nonempty. If possible, let x and x^* be two fixed points of F . Then $x = Fx$ and $x^* = Fx^*$. Our aim is to show that $x = x^*$. By property (A3) there exists $u \in Z$ with $\gamma(u, Fu) \geqslant 1$ such that $\gamma(x, u) \geqslant 1, \gamma(x^*, u) \geqslant 1$. Put $u_0 = u$ and let $u_1 = Fu_0$. Then $\gamma(x, u_0) \geqslant 1$ and $\gamma(u_0, u_1) \geqslant 1$. Similarly, as in the proof of Theorem 1, we define a sequence $\{u_n\}$ such that

$$u_{n+1} = Fu_n \quad \forall n \geqslant 1.$$

As F is a γ-admissible mapping, we have

$$\gamma(u_n, u_{n+1}) \geqslant 1 \quad \forall n \geqslant 1. \tag{10}$$

Arguing similarly as in proof of Theorem 1, we prove that $\{u_n\}$ is a γ-Cauchy sequence in Z, and there exists $p \in Z$ such that

$$\lim_{n\to\infty} u_n = p. \tag{11}$$

We claim that

$$\gamma(Fa, Fb) \geqslant 1.$$

$\gamma(u_0, u_1) \geqslant 1$. As $\gamma(x, u_0) \geqslant 1$ and , by property (A2) we have $\gamma(x, u_1) \geqslant 1$. Therefore, our claim is true for $n = 1$. We assume that $\gamma(x, u_m) \geqslant 1$ holds for some $m > 1$. By (10), $\gamma(u_m, u_{m+1}) \geqslant 1$. Applying property (A2), we have $\gamma(x, u_{m+1}) \geqslant 1$, and this proves our claim.

By (1) and (12) we have, for all $n \geqslant 0$,

$$\begin{aligned}
&d(x, u_{n+1})\\
&\quad = d(Fx, Fu_n)\\
&\quad \leqslant k\max\bigg\{d(x, u_n), \frac{d(x, Fx) + d(u_n, Fu_n)}{2}, \frac{d(x, Fu_n) + d(u_n, Fx)}{2},\\
&\qquad\qquad \frac{d(x, Fx)d(u_n, Fu_n)}{1 + d(Fx, Fu_n)}, \frac{d(x, Fu_n)d(u_n, Fx)}{1 + d(Fx, Fu_n)}, \frac{d(u_n, Fu_n)d(u_n, Fx)}{1 + d(Fx, Fu_n)}\bigg\},\\
&\quad = k\max\bigg\{d(x, u_n), \frac{d(u_n, u_{n+1})}{2}, \frac{d(x, u_{n+1}) + d(u_n, x)}{2}, 0,\\
&\qquad\qquad \frac{d(x, u_{n+1})d(u_n, x)}{1 + d(x, u_{n+1})}, \frac{d(u_n, u_{n+1})d(u_n, x)}{1 + d(x, u_{n+1})}\bigg\}.
\end{aligned} \tag{13}$$

Taking limit as $n \to \infty$ in (13) and using (11), we have

$$\begin{aligned}
d(x, p) &\leqslant k\max\bigg\{d(x, p), 0, \frac{d(x, p) + d(p, x)}{2}, 0, \frac{d(x, p)d(p, x)}{1 + d(x, p)}, 0\bigg\}\\
&\leqslant k\max\big\{d(x, p), 0, d(x, p), 0, d(x, p), 0\big\} = kd(x, p),
\end{aligned}$$

which is a contradiction unless $d(x, p) = 0$, that is, $d(x, p) = 0$, that is,

$$x = p. \tag{14}$$

Similarly, we can show that

$$x^* = p. \tag{15}$$

From (14) and (15) we have $x = x^*$. Therefore, fixed point of F is unique.

We present some special cases illustrating the applicability of Theorem 1.

Remark 5. Choosing $\gamma(x, y) = 1$ for all $(x, y) \in Z \times Z$, we have a corollary.

Corollary 1. *Let* (Z, d) *be a complete metric space. Then* $F : Z \to Z$ *has a unique fixed point if for some* $k \in (0, 1)$ *and for all* $x, y \in Z$ *one of the following inequalities holds*:

(i) $d(Fx, Fy) \leqslant kd(x, y)$;

(ii) $d(Fx, Fy) \leqslant (k/2)[d(x, Fx) + d(y, Fy)]$;

(iii) $d(Fx, Fy) \leqslant (k/2)[d(x, Fy) + d(y, Fx)]$;

(iv) $d(Fx, Fy) \leqslant k \max\{d(x, y), (d(x, Fx)+d(y, Fy))/2, (d(x, Fy)+d(y, Fx))/2\}$.

3 Ulam–Hyers stability

In [19], one can find the following definition as well as some related notions concerning the Ulam–Hyers stability, which is relevant to the present considerations. Let (Z, d) be a metric space and $T : Z \to Z$ be a mapping. We say that the fixed point problem $x = Tx$ Ulam–Hyers stable if there is $\epsilon > 0$ such that for $y \in Z$ with $d(y, Ty) \leqslant \epsilon$, there exists $x_0 \in Z$ satisfying $x = Tx$ and $d(y, x_0) \leqslant \epsilon$.

Definition 9.(See [24].) Problem P is called Ulam–Hyers stable if there exists a function $\phi : [0, \infty) \to [0, \infty)$, which is monotone increasing and continuous at 0 with $\phi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u^* \in Z$ of the inequality $d(x, Fx) \leqslant \epsilon$, there exists a solution $x^* \in Z$ of $x = Fx$ such that $d(u^*, x^*) \leqslant \phi(\epsilon)$.

Remark 6. If $\phi : [0, \infty) \to [0, \infty)$ is defined as $\phi(t) = ct$ for $t \geqslant 0$., where $c > 0$ is a constant, then Definition 9 reduces to Definition in [10].

Let us consider the fixed point Problem $\mathrm{P}\,(x = Fx)$ and the following inequation:

$$d(x, Fx) \leqslant \epsilon, \quad \epsilon > 0. \tag{16}$$

In the next theorem, we take the following additional condition to assure the Ulam– Hyers stablity via γ-admissible mapping.

(A4) For any solution x^* of Problem P and any solution u^* of (16), one has $\gamma(u^*, x^*) \geqslant 1$.

Theorem 3. *In addition to the hypothesis of Theorem ., suppose that* (A4) *holds. Then the fixed point Problem . is Ulam–Hyers stable.*

Proof. By Theorem 2 there exists unique $x^* \in Z$ such that $x^* = Fx^*$ So, x^* is a solution of Problem P. Let $u^* \in Z$ be a solution of (16). Then $d(u^*, Fu^*) \leqslant \epsilon$. By property (A4) we have $\gamma(u^*, x^*) \geqslant 1$. With the help of (1), we have

$$\begin{aligned}
&d(u^*, x^*)\\
&\quad = d(u^*, Fx^*) \leqslant d(u^*, Fu^*) + d(Fu^*, Fx^*)\\
&\quad \leqslant k\max\biggl\{d(u^*, x^*), \frac{d(u^*, Fu^*) + d(x^*, Fx^*)}{2}, \frac{d(u^*, Fx^*) + d(x^*, Fu^*)}{2},\\
&\qquad\qquad \frac{d(u^*, Fu^*)d(x^*, Fx^*)}{1 + d(Fu^*, Fx^*)}, \frac{d(x^*, Fu^*)d(u^*, Fx^*)}{1 + d(Fu^*, Fx^*)},\\
&\qquad\qquad \frac{d(x^*, Fx^*)d(x^*, Fu^*)}{1 + d(Fu^*, Fx^*)}\biggr\} + d(u^*, Fu^*)
\end{aligned}$$

$$\begin{aligned}
&\leqslant k\max\biggl\{d(u^*, x^*), \frac{\epsilon}{2}, \frac{d(u^*, x^*) + d(x^*, u^*) + d(u^*, Fu^*)}{2}, 0,\\
&\qquad\qquad \frac{d(x^*, Fu^*)d(u^*, x^*)}{1 + d(x^*, Fu^*)}, 0\biggr\} + \epsilon\\
&\leqslant k\max\biggl\{d(x^*, u^*), \frac{\epsilon}{2}, \frac{d(x^*, u^*) + \epsilon + d(u^*, x^*)}{2}, 0, \frac{d(x^*, Fu^*)d(u^*, x^*)}{1 + d(x^*, Fu^*)}\biggr\} + \epsilon\\
&\leqslant k\max\biggl\{d(x^*, u^*), \frac{\epsilon}{2}, \frac{2d(x^*, u^*) + \epsilon}{2}, 0, d(x^*, u^*)\biggr\} + \epsilon\\
&= k\left[d(x^*, u^*) + \frac{\epsilon}{2}\right] + \epsilon,
\end{aligned}$$

which implies that

$$d(x^*, u^*) \leqslant \frac{(k+2)\epsilon}{2(1-k)}. \tag{17}$$

Let $\phi : [0,\infty) \to [0,\infty)$ be defined by

$$\phi(t) = \frac{(k+2)t}{2(1-k)}, \quad 0 < k < 1.$$

The function ϕ is monotone increasing, continuous, and $\phi(0) = 0$. By (17) we have

$$d(x^*, u^*) \leqslant \frac{(k+2)\epsilon}{2(1-k)} = \phi(\epsilon).$$

Therefore, the fixed point Problem P is Ulam–Hyers stable.

4 Well-posedness

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians (see, for example, [16, 17]). Let (Z,d) be a metric space and $T:Z\to Z$ be a mapping. The fixed point problem of T is said to be well-posed if T has a unique fixed point $x\in Z$ and for any sequence $\{x_n\}$ in Z, $d(x_n,Tx_n)\to 0$ as $n\to\infty$ implies $d(x_n,x)\to 0$ as $n\to\infty$.

Definition 10.(See [10].) Problem P is called well-posed if (i) F has a unique fixed point x^*, (ii) $d(x_n,x^*)\to 0$ as $n\to\infty$, whenever $\{x_n\}$ is a sequence in Z with $d(x_n,Fx_n)\to 0$ as $n\to\infty$.

In the next theorem, we take the following condition to assure the well-posedness via γ- admissible mapping.

(A5) If x^* is any solution of Problem P and $\{x_n\}$ is any sequence in Z for which $d(x_n,Fx_n)\to 0$ as $n\to\infty$, then $\gamma(x_n,x^*)\geqslant 1$ for all n.

Theorem 4. *In addition to the hypothesis of Theorem* P, *suppose that* (A5) *holds. Then the fixed point Problem . is well-posed.*

Proof. By Theorem 2 there exists unique $x^*\in Z$ such that $x^*=Fx^*$. So, x^* is a solution of Problem P. Let $\{x_n\}$ be a sequence in Z for which $d(x_n,Fx_n)\to 0$ as $n\to\infty$. As (A5) holds, we have $\gamma(x_n,x^*)\geqslant 1$ for all n. By (1) we have

$$\begin{aligned}
d(x_n,x^*) &\leqslant d(x_n,Fx_n)+d(Fx_n,Fx^*)\\
&\leqslant d(x_n,Fx_n)\\
&\quad + k\max\Big\{d(x_n,x^*),\frac{d(x_n,Fx_n)+d(x^*,Fx^*)}{2},\\
&\qquad\frac{d(x_n,Fx^*)+d(x^*,Fx_n)}{2},\frac{d(x_n,Fx_n)d(x^*,Fx^*)}{1+d(Fx_n,Fx^*)},\\
&\qquad\frac{d(x^*,Fx_n)d(x_n,Fx^*)}{1+d(Fx_n,Fx^*)},\frac{d(x^*,Fx^*)d(x^*,Fx_n)}{1+d(Fx_n,Fx^*)}\Big\}\\
&= k\max\Big\{d(x_n,x^*),\frac{d(x_n,Fx_n)}{2},\frac{d(x_n,x^*)+d(x^*,Fx_n)}{2},0,\\
&\qquad\frac{d(x^*,Fx_n)d(x_n,x^*)}{1+d(Fx_n,x^*)},0\Big\}+d(x_n,Fx_n)\\
&\leqslant k\max\Big\{d(x_n,x^*),\frac{d(x_n,Fx_n)}{2},\frac{d(x_n,x^*)+d(x^*,Fx_n)}{2},0,d(x_n,x^*),0\Big\}\\
&\quad + d(x_n,Fx_n)\\
&\leqslant k\max\Big\{d(x_n,x^*),\frac{d(x_n,Fx_n)}{2},\frac{d(x_n,x^*)+d(x^*,x_n)+d(x_n,Fx_n)}{2},\\
&\qquad 0,d(x_n,x^*),0\Big\}+d(x_n,Fx_n)\\
&= k\max\Big\{d(x_n,x^*),\frac{d(x_n,Fx_n)}{2},\frac{2d(x_n,x^*)+d(x_n,Fx_n)}{2},d(x_n,x^*)\Big\}\\
&\quad + d(x_n,Fx_n)\\
&\leqslant k\big[d(x^*,x_n)+d(x_n,Fx_n)\big]+d(x_n,Fx_n),
\end{aligned}$$

which implies that

$$d(x^*,x_n)\leqslant\frac{1+k}{1-k}d(x_n,Fx_n).$$

Thus, $\lim_{n\to\infty} d(x_n, x^*) = 0$, and hence, the fixed point Problem P is well-posed.

5 Data dependence result

In this section, we investigate the data dependence of fixed points.

Definition 11. Let $T_1, T_2 : Z \to Z$ be two mappings, where (Z, d) is a metric space such that $d(T_1x, T_2x) \leqslant \eta$ for all $x \in Z$, where η is some positive number. Then the problem of data dependence is to estimate the distance between the fixed points of these two mappings.

Several research papers on data dependence have been published in the recent literature, some of which we mention in references [3, 5, 20].

Theorem 5. *In addition to the hypothesis of Theorem ., suppose that* $T : Z \to Z$ *be a mapping with nonempty fixed point set. If for each fixed point u of* T, $\gamma(u, Fu) \geqslant 1$ *and there exists* $M > 0$ *such that* $d(Fx, Tx) \leqslant M$ *for all* $x \in Z$ *, then* $d(s,t) \leqslant M/(1-k)$. *where s and t are fixed points of F and T, respectively.*

$x_0 = Tx_0$. *Proof.* By Theorem 2 there exists unique $s \in Z$ such that $s = Fs$. Suppose t is a fixed point of T . Take $x_0 = t$. Then $x_0 = Tx_0$. Let $x_1 = Fx_0$. Then by definition of M we have

$$d(x_0, x_1) = d(Tx_0, Fx_0) \leqslant M. \tag{18}$$

Applying the assumption of the theorem, we have $\gamma(x_0, x_1) \geqslant 1$. Let $x_2 = Fx_1$, then by admissibility property of F we have $\gamma(x_1, x_2) \geqslant 1$. Inductively, arguing similarly as in the proof of Theorem 1, we have a sequence $\{x_n\}$ in Z such that

$$x_{n+1} = Fx_n \quad \text{and} \quad \gamma(x_n, x_{n+1}) \geqslant 1 \quad \forall n \geqslant 0.$$

Arguing similarly as in proof of Theorem 1, we can prove that

- (7) is satisfied;
- $\{x_n\}$ is a γ-Cauchy sequence in the metric space (Z, d), and there exists $u \in Z$ such that $\lim_{n\to\infty} x_n = u$;
- u is a fixed point of F , that is, $u = Fu$; as fixed point of F is unique, we have $u = s$ and $Fs = s$. Using triangular property, we have

$$d(t,s) = d(x_0, u) \leqslant \sum_{i=0}^{n} d(x_i, x_{i+1}) + d(x_{n+1}, u)$$
$$\leqslant \sum_{i=0}^{n} \delta_i + d(x_{n+1}, u) \leqslant \sum_{i=0}^{n} k^i \delta_0 + d(x_{n+1}, u).$$

Taking limit as $n \to \infty$ in the above inequality and using (18), we have

$$d(s,t) = d(t,s) \leqslant \sum_{i=0}^{\infty} k^i \delta_0 = \frac{\delta_0}{1-k} = \frac{d(x_0,x_1)}{1-k} \leqslant \frac{M}{1-k}.$$

6 Application

We have already mentioned in introduction that fixed point theorems in metric spaces are widely investigated and have applications in differential and integral equations (see [2,15, 22]). In this section, we deal with a nonlinear integral equation. In the first part, we apply Theorems 1 and 2 to prove the existence and uniqueness of solution of Fredholm-type nonlinear integral equations. In the remaining part, we discuss three aspects of the same integral equation, namely, Ulam–Hyers stability, well-posedness and data dependence.

We consider the following Fredholm-type nonlinear integral equation:

$$x(t) = g(t) + \lambda \int_a^b K(t,s) h\big(s, x(s)\big)\, \mathrm{d}s \quad \forall t \in [a,b],\ \lambda > 0, \tag{19}$$

where the unknown function $x(t)$ takes real values.

The space $Z = C([a,b])$ of all real valued continuous functions on [*a, b*] endowed with the metric $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$ is complete. Let this metric space be endowed with a partial ordered relation $\prec$ defined as $x \prec y$ if $x(t) \leqslant y(t)$ and only if for all $t \in [a,b]$.

Problem I. To find out a solution of the Fredholm-type integral equation

$$x(t) = g(t) + \lambda \int_a^b K(t,s) h\big(s, x(s)\big)\, \mathrm{d}s \quad \forall t \in [a,b],\ \lambda > 0,$$

under some appropriate conditions on *g, h* and *K.*

We take the following assumptions:

(I1) $[a,b] \times \mathbb{R} \to [0,\infty)$, $K : [a,b] \times [a,b] \to [0,\infty)$ are continuous mappings

(I2) $r_1, r_2 \in \mathbb{R}$ and $r_1 \leqslant r_2$ implies $h(s,r_1) \leqslant h(s,r_2)$ for all $s \in [a,b]$;

(I3) $|h(s,r_1) - h(s,r_2)| \leqslant |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$ with $r_1 \leqslant r_2$ and for all $s \in [a,b]$;

(I4) $|K(t,s)| \leqslant m$, where $0 \leqslant k = \lambda(b-a)m < 1$;

(I5) There exists $x_0 \in C([a,b])$ such that $x_0(t) \leqslant g(t) + \lambda \int_a^b K(t,s) h(s, x_0(s))\, \mathrm{d}s$.

(I6) For every $x, y \in C([a,b])$, ([*a, b*]), there exists $z \in C([a,b])$ such that $x \prec z, y \prec z$ and $z(t) \leqslant g(t) + \lambda \int_a^b K(t,s) h(s, z(s))\, \mathrm{d}s$ for all $t \in [a,b]$.

Theorem 6. *Let* $(Z,d) = (C([a,b]),d)$, *and let* $g, h(s,\cdot), K(t,s)$ *satisfy assumptions* (I1). (I5). *Then nonlinear integral equation* (19) *has a solution in* Z.

Proof. Define a mapping $F : Z \to Z$ by

$$F(x)(t) = g(t) + \lambda \int_a^b K(t,s)h\big(s, x(s)\big)\,\mathrm{d}s \quad \forall t \in [a,b].$$

Let $x, y \in C([a,b])$ ([*a, b*]) and $x \prec y$. Then $x(s) \leqslant y(s)$, hence, by (I2) we have

$$F(x)(t) = g(t) + \lambda \int_a^b K(t,s)h\big(s, x(s)\big)\,\mathrm{d}s \leqslant g(t) + \lambda \int_a^b K(t,s)h\big(s, y(s)\big)\,\mathrm{d}s = F(y)(t), \tag{21}$$

which implies $Fx \prec Fy$.

Let $x, y \in C([a,b])$ and $x \prec y$. Then $x(s) \leqslant y(s)$ $s \in [a,b]$ .Hence, by (13) we have for all $t \in [a,b]$,

$$\begin{aligned}
&\big|Fx(t) - Fy(t)\big| \\
&= \lambda\left|\int_a^b K(t,s)\big[h\big(s,x(s)\big) - h\big(s,y(s)\big)\big]\,\mathrm{d}s\right| \\
&= \lambda \int_a^b \big|K(t,s)\big|\big|\big[h\big(s,x(s)\big) - h\big(s,y(s)\big)\big]\big|\,\mathrm{d}s \\
&\leqslant \lambda \int_a^b m\big|\big[h\big(s,x(s)\big) - h\big(s,y(s)\big)\big]\big|\,\mathrm{d}s \\
&= \lambda m \int_a^b \big|\big[h\big(s,x(s)\big) - h\big(s,y(s)\big)\big]\big|\,\mathrm{d}s \leqslant \lambda m \int_a^b \big|x(s) - y(s)\big|\,\mathrm{d}s \\
&\leqslant \lambda m d(x,y) \int_a^b \mathrm{d}s = \lambda m(b-a)d(x,y) \leqslant kM(x,y),
\end{aligned} \tag{22}$$

Let $\gamma : Z \times Z \to [0,\infty)$ be defined by

$$\gamma(x,y) = \begin{cases} 1 & \text{if and only if } x \prec y, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $x \prec y$ for $x, y \in Z$ implies $\gamma(x,y)=1$ for $x, y \in Z$. Thus, by (22) the contraction condition holds for all $x, y \in Z$ with $\gamma(x,y)=1$.

By (21), for $x, y \in X$ with $x \prec y$, we have $Fx \prec Fy$. It follows that for $\gamma(x,y)=1$ implies $\gamma(Fx,Fy)=1$. Hence, F γ-admissible.

n Suppose that $\{x_n\}$ is a convergent sequence in $C([a,b])$ with limit $x \in C([a,b])$ and $x_n \prec x_{n+1}$ for all n. Then $x_n(s) \leqslant x_{n+1}(s)$ for all n and for all $s \in [a,b]$, which implies that $x_n(s) \leqslant x(s)$ for all n and for all $s \in [a,b]$, that is, $x_n \prec x$ for all n. It follows that if $\{x_n\}$ is a convergent sequence in $C([a,b])$ with limit $x \in C([a,b])$ and $\gamma(x_n, x_{n+1})=1$, then $\gamma(x_n,x)=1$ for all n. Therefore, Z has γ-regular property, that is, property (A1) holds.

By (I5) there exists $x_0 \in Z$ such that

$$x_0(t) \leqslant g(t) + \lambda \int_a^b K(t,s)h\big(s, x_0(s)\big)\, \mathrm{d}s = Fx_0(t) \quad \forall t \in [a,b].$$

So, $x_0 \prec Fx_0$. This implies that there exists $x_0 \in Z$ such that $\gamma(x_0, Fx_0)=1$.

$C([a,b])$ being complete, is a γ-complete metric space (see Remark 1).

All the assumptions of Theorem 1 are satisfied. Therefore, F has a fixed point, that is, the integral equation (19) has a solution in Z.

Example 5. Consider the integral equation

$$x(t) = \frac{t}{1+t^2} + \frac{1}{16}\int_0^1 \frac{s\cos t}{36(1+t)}\frac{|x(s)|}{1+|x(s)|}\, \mathrm{d}s \quad \forall t \in [0,1]. \tag{23}$$

Observe that this equation is a special case of (19) with $\lambda = 1/16$, $g(t) = t/(1+t^2)$, $h(s,x(s)) = |x(s)|/(36(1+|x(s)|))$ and $a=0$, $b=1$.

- $g \in C([0,1])$ and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$, $K : [0,1] \times [0,1] \to [0,\infty)$ are continuous mappings, and hence, assumption (I1)holds.
- Assumption (I2) holds. To check, let $r_1, r_2 \in \mathbb{R}$ with $r_1 \leqslant r_2$. Then

$$h(s,r_1) = \frac{|r_1|}{36(1+|r_1|)} \leqslant \frac{|r_2|}{36(1+|r_2|)} = h(s,r_2) \quad \forall s \in [0,1]$$

since the function $\psi(s) = s/(1+s)$ is increasing in [0, 1].

Assumption (I3) holds. To check, let $r_1, r_2 \in \mathbb{R}$ with $r_1 \leqslant r_2$. Then for all $s \in$ [0,1], we have

$$\begin{aligned}\big|h(s,r_1) - h(s,r_2)\big| &= \left|\frac{|r_1|}{36(1+|r_1|)} - \frac{|r_2|}{36(1+|r_2|)}\right| = \left|\frac{|r_1|-|r_2|}{36(1+|r_1|)(1+|r_2|)}\right| \\ &\leqslant \left|\frac{|r_1-r_2|}{36(1+|r_1|)(1+|r_2|)}\right| \leqslant |r_1-r_2|.\end{aligned}$$

- Assumption (I4) holds. To check, let $t, s \in$ [0, 1]. Then

$$|K(t,s)| = \frac{s\cos t}{1+t} \leqslant 1 = m,$$

where $0 \leqslant k = (1/16)(1-0)\cdot 1 = 1/16 < 1$.

• Assumption (I5) holds. To check, let $x_0(s)=0$ for all $s\in$ [0, 1],Then $x_0 \in Z$ such that for all $t\in[0,1]$, $0 = x_0(t) \leqslant t/(1+t^2)+0 = t/(1+t^2)$., that is,

$$x_0 \prec g(t) + \lambda \int_a^b K(t,s)h(s,x_0(s))\,\mathrm{d}s.$$

Therefore, all the assumptions of Theorem 6 are satisfied. Hence, integral equation (23) has a solution x^* in $C([0, 1])$.

Theorem 7. In addition to the hypothesis of Theorem 6, suppose that assumption (I6) holds. Then nonlinear integral equation (19) has a unique solution.

Proof. First, we show that Z has γ-triangular property. Let $x,y,z\in Z$ and $\gamma(x,y)\geqslant 1$ and $\gamma(y,z)\geqslant 1$.. By definition of γ we have $x\prec y$ and $y\prec z$, that is, $x(t)\leqslant y(t)$ and $y(t)\leqslant z(t)$ for all $t\in[a, b]$, which imply that $x(t)\leqslant z(t)$ for all $t\in[a, b]$, that is, $x\prec z$., that is, $\gamma(x,z)\geqslant 1$. Hence, Z has γ-triangular property. Therefore, property (A2) holds.

By assumption (I6), for $x,y\in C([a,b])$., there exists $z\in C([a,b])$ such that $x\prec z, y\prec z$ and

$$z(t) \leqslant g(t) + \lambda \int_a^b K(t,s)h(s,z(s))\,\mathrm{d}s \quad \forall t\in[a,b].$$

Hence, $z(t)\prec Fz(t)$ for all $t\in[a, b]$, that is, $z\prec Fz$. Thus, for $x,y\in C([a,b])$. there exists $z\in Z$ with $\gamma(z,Fz)\geqslant 1$ such that $\gamma(x,z)\geqslant 1$ and $\gamma(y,z)\geqslant 1$. Therefore, γ has F-directed property, that is, property (A3) holds.

All the assumptions of Theorem 2 are satisfied. Thus, by Theorems 2 and 6 . has a unique fixed point, that is, the nonlinear integral equation (19) has a unique solution in $C([a,b])$.

Being motivated by Definition 9, we give definitions of Ulam–Hyers stability for the case of integral equation (19).

Definition 12. Problem I is called Ulam–Hyers stable if there exists a function $\phi:[0,\infty)\to[0,\infty)$., which is monotone increasing, continuous at 0 with $\phi(0)=0$ such that for each $\epsilon>0$ and for each solution $u^*\in C([a,b])$ of the inequality

$$\sup_{t\in[a,b]}\left|x(t)-g(t)-\lambda\int_a^b K(t,s)h(s,x(s))\,\mathrm{d}s\right|\leqslant\epsilon,$$

there exists a solution $x^*\in C([a,b])$ of the integral equation (19) such that

$$\sup_{t\in[a,b]}\left|u^*(t)-x^*(t)\right|\leqslant\phi(\epsilon).$$

Let us consider the following the integral inequality:

$$\sup_{t\in[a,b]}\left|u^*(t)-x^*(t)\right|\leqslant\phi(\epsilon). \tag{24}$$

In the following next theorem, we add a new condition to assure the Ulam–Hyers stability of the integral equation (19):

(I7) For any solution x^* of (19) and any solution u^* of (24), one has $x^*\prec u^*$.

Theorem 8. *Let all the hypothesis of Theorem 7 hold. Then integral equation* (19) *has a unique solution* x^*. *Also suppose that* (I7) *holds. Then Problem . is Ulam–Hyers stable, and for given* $\epsilon>0$ *and for any solution u. of* (24), *we have*

$$d(x^*,u^*)<\frac{\lambda(b-a)m+2}{2(1-\lambda(b-a)m)}\epsilon=\phi(\epsilon),$$

where is a mapping given by φ(.) = (. + 2)t/(2(1 − .)) for all t ∈ [0, ∞) *and* |*.(t, s)*| ≤ *m.*

Proof. By Theorem 7 the integral equation (19) has a unique solution ... Hence, it is a unique fixed point of the function . : *Z Z* defined by (20). Let .. is a solution of the integral inequation (24), hence, .. is a solution of .(*x, Fx*) ≤ ., and by (I7), ... By the definition of *γ*, *γ*(.., *u.*) ≥ 1, that is, property (A4) holds. By application of Theorem 3 the fixed point problem . = *Fx* is Ulam–Hyers stable. Therefore, the solution of the integral equation (19) is Ulam–Hyers stable, and

$$d(x^*,u^*)<\frac{(k+2)\epsilon}{2(1-k)}=\frac{\lambda(b-a)m+2}{2(1-\lambda(b-a)m)}\epsilon=\phi(\epsilon),$$

where . : [0, ∞) → [0, ∞) is a mapping given by .(.) = (. + 2)*t*/(2(1 − .)) for all . ∈ [0, ∞).

Being motivated by Definition 10, we give definitions of well-posedness for the case of integral equation (19).

Definition 13. Problem I is called well-posed if (i) integral equation (19) has a unique solution .. in .([*a, b*]), (ii) *x. x.* in .([*a, b*]), whenever *x.* is a sequence in .([*a, b*]) satisfying

$$\sup_{t\in[a,b]}\left|x_n(t)-g(t)-\lambda\int_a^b K(t,s)h\big(s,x_n(s)\big)\,\mathrm{d}s\right|\to 0,\quad n\to\infty.$$

In the following theorem, we add a new condition to assure the well-posedness for integral equation (19).

(I8) If .. is a solution of the integral equation (19) and {*x.*} is any sequence in .([*a, b*]) such that sup then *x.* # .. for all ..

Theorem 9. *Let all the hypothesis of Theorem . hold. Then the integral equation* (19) *has a unique solution x.. Also suppose that* (I8) *holds. Then Problem . is well-posed.*

Proof. By Theorem 7 the integral equation (19) has a unique solution ... Hence, it is a unique fixed point of the function . : . → . defined by (20). Let {*x.*} be a sequence in .([*a, b*]) such that sup

Then by assumption (I8) we have *x. x.* for all .. From definition of . we have .(*x., x.*) = 1 for all ., that is, property (A5) holds. By application of Theorem 4 the fixed point Problem P, that is, the problem . = *Fx*, is well-posed. Therefore, Problem I is well-posed.

Being motivated by Definition 11, we give definitions of data dependence for the case of integral equation.

Definition 14. Let .. .([*a, b*]) be the unique solution of the integral equation (19) and .. be the solution of the integral equation .(.) = .(.) + . ∫ *b K* .*t, s*). (*s, x*(.)) d. for all . ∈ [*a, b*], where . ∈ .[*a, b*] and .. : [*a, b*] ×R→ [0, ∞), .. : [*a, b*] ×[*a, b*] → [0, ∞) are continuous mappings. The problem of data dependence is to find sup. ∈ [*a,b*] |..(.) − ..(.)|.

Theorem 10. *Let all the hypothesis of Theorem . hold and x. be the unique solution of the integral equation* (19)*. Also suppose that if x be any solution of the integral equation*

$$x(t)=p(t)+\lambda\int_a^b K_1(t,s)h_1\big(s,x(s)\big)\,\mathrm{d}s\quad\forall t\in[a,b],\tag{25}$$

where p ∈ .([*a, b*]) *and h.* : [*a, b*] × [*a, b*] → [0, ∞), *K.* : [*a, b*] × [*a, b*] → [0, ∞) *are continuous mappings, then for all t* ∈ [*a, b*],

$$x(t)\leqslant g(t)+\lambda\int_a^b K(t,s)h\big(s,x(s)\big)\,\mathrm{d}s,\quad t\in[a,b].$$

Further suppose that there exist ν, η > . such that

$$\sup_{t\in[a,b]} \left|K_1(t,s)h_1\big(s,x(s)\big) - K(t,s)h\big(s,x(s)\big)\right| \leqslant \eta,$$

And

$$\sup_{t\in[a,b]} \left|p(t) - g(t)\right| \leqslant \nu.$$

Then

$$d(x,x^*) \leqslant \frac{\nu + \lambda\eta(b-a)}{1-\lambda(b-a)m}.$$

Proof. By Theorem 7 the integral equation (19) has a unique solution ... Let us define a map . : . → . by

$$T(x)(t) = p(t) + \lambda \int_a^b K_1(t,s)h_1\big(s,x(s)\big)\,\mathrm{d}s, \quad t\in[a,b]. \tag{26}$$

Since . is a solution of (25), it is a fixed point of the mapping . defined by (26). By the assumption of the theorem we have .(.) ≤ . (.)(.), . ∈ [*a*, *b*], which implies that

$x \prec Fx$. Then $\gamma(x,Fx) = 1$. Also, for any $x \in C([a,b])$, we have

$$\begin{aligned}
&\left|F(x)(t) - T(x)(t)\right| \\
&\quad\leqslant \left|p(t)-g(t)\right| + \left|\lambda \int_a^b \left[K(t,s)h\big(s,x(s)\big) - K_1(t,s)h_1\big(s,x(s)\big)\right]\mathrm{d}s\right| \\
&\quad\leqslant \left|p(t)-g(t)\right| + \lambda \int_a^b \left|\left[K(t,s)h\big(s,x(s)\big) - K_1(t,s)h_1\big(s,x(s)\big)\right]\right|\mathrm{d}s \\
&\quad\leqslant \nu + \lambda \int_a^b \eta\,\mathrm{d}s = \nu + \lambda\eta(b-a) = M \ \ (\text{say}) \quad \forall t\in[a,b],
\end{aligned}$$

which implies that sup. ∈ [*a*,*b*] |. (.)(.) − . (.)(.)| ≤ . for all . ∈ .([*a*, *b*]). So, .(*Fx*, *Tx*) ≤ . for all . ∈ .. Thus, all the hypothesis of Theorem 5 are met. Therefore, we have .(*x*, *x*.) ≤ *M*/(1 − .) = (. + λ*η*(. − .)).(1 − .(. − .).).

7 Conclusion

The result of the Theorem 1 is also valid if we replace the constant . by a Mizoguchi– Takahashi function [7, 14]. Here we have not proceeded with it but this can be taken up in a future work. Also the corresponding problem with multivalued mappings and possible applications to integral inclusion problems is supposed to be of considerable interest. One reason for it is that, in general, the fixed point sets of multivalued mappings are mathematically complicated in their structures. This can also be taken up in future works.

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