



Nonlinear Analysis: Modelling and Control
ISSN: 1392-5113
ISSN: 2335-8963
nonlinear@mii.vu.lt
Vilniaus Universitetas
Lituania

Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

You, Jin; Zhenlai, Han

Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

Nonlinear Analysis: Modelling and Control, vol. 26, núm. 6, 2021

Vilniaus Universitetas, Lituania

Disponible en: <https://www.redalyc.org/articulo.oa?id=694173172005>

DOI: <https://doi.org/10.15388/namc.2021.26.24939>



Esta obra está bajo una Licencia Creative Commons Atribución 4.0 Internacional.

Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

Jin You youjin6666@163.com

Shandong University, China

Han Zhenlai hanzhenlai@163.com

University of Jinan, China

Nonlinear Analysis: Modelling and Control, vol. 26, núm. 6, 2021

Vilniaus Universitetas, Lituania

Recepción: 14 Septiembre 2020

Aprobación: 01 Marzo 2021

Publicación: 01 Noviembre 2021

DOI: <https://doi.org/10.15388/namc.2021.26.24939>

Redalyc: <https://www.redalyc.org/articulo.oa?id=694173172005>

Financiamiento

Fuente: This research is supported by Shandong Provincial Natural Science Foundation

Nº de contrato: (ZR2020MA016)

Beneficiario: Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

Financiamiento

Fuente: Natural Science Foundation of China

Nº de contrato: (62073153)

Beneficiario: Analysis of fractional hybrid differential equations with impulses in partially ordered Banach algebras*

Abstract: In this paper, we investigate a class of fractional hybrid differential equations with impulses, which can be seen as nonlinear differential equations with a quadratic perturbation of second type and a linear perturbation in partially ordered Banach algebras. We deduce the existence and approximation of a mild solution for the initial value problems of this system by applying Dhage iteration principles and related hybrid fixed point theorems. Compared with previous works, we generalize the results to fractional order and extend some existing conclusions for the first time. Meantime, we take into consideration the effect of impulses. Our results indicate the influence of fractional order for nonlinear hybrid differential equations and improve some known results, which have wider applications as well. A numerical example is included to illustrate the effectiveness of the proposed results.

Keywords: impulsive fractional differential equation, hybrid fixed point theorem, Dhage iteration principle, quadratic.

Introduction

Numerous mathematical concepts have been presented to describe or characterize many phenomena in the real world. Differential equation, as one of the most valuable concepts, has been widely used in constructing mathematical models, which can be regarded as an essential tool toward the comprehension of nature. (See, for instance, [21, 23, 27] for applications tied to models from mathematical biology and physics.) On account of the significance of the theory and application of differential equation, it attracts scholars interest extensively to study the qualitative theories of systems constructed by differential equation. Especially, the theory of solvability is one of the fundamental and important topics in practice, such as the existence of the solutions for differential equation; we can refer to [18, 22, 30] for the recent developments.

The existence results about nonlinear systems utilize differential equations of integer order in the majority. As is known to us, fractional dynamic is widely applied in various fields of science, like physics, engineering, biology and modeling infectious diseases, neural networks, as well as anomalous diffusion [17, 26, 32]. For instance, scholars [25] simulated the outbreak of influenza A(H1N1) using a fractional-order SEIR model. Recently, it has a rapid development [1, 2, 19, 24, 25, 28].

One of the methods for solving the existence of solutions of nonlinear systems is relying on fixed point theorems. If the fixed point theory involves the mixed hypotheses of algebra, topology, and geometry, then it is called hybrid fixed point theory. Hybrid fixed point theorem can do well in solving existence of hybrid differential equations, which is also useful for the theory of nonlinear quadratic differential equations; see in literature [6, 20]. Recently, many researchers like Dhage [9, 11], Lakshmikantham [7], Bashiri [5] give several hybrid fixed point theorems to help perfecting the theories. Achievements have been made in this area.

In 2011, Zhao [31] studied fractional quadratic differential equations involving Rie- mann–Liouville differential operators of order $0 < \alpha \leq 1$

$$D^\alpha \left[\frac{x(t)}{f(t, x(t))} \right] = g(t, x(t)), \quad t \in J,$$

$$x(0) = 0.$$

The existence theorem for fractional quadratic differential equations was proved under mixed Lipschitz and Caratheodory conditions.

Recently, Dhage [8] introduced a notion of partially condensing mappings in a partially ordered normed linear space and proved some hybrid fixed point theorems under certain mixed conditions of algebra, analysis, and topology. The author combined the compactness with some algebraic arguments to prove some hybrid fixed point theorems for the mappings in ordered spaces and apply the results to obtain the solutions of integral equations under some mixed compactness and monotonic conditions, which was the main contribution of the works; see Dhage [10, 12–15] for recent development.

In 2015, Dhage [16] considered the periodic boundary value problem of first-order nonlinear quadratic ordinary differential equations

$$\frac{d}{dt} \left[\frac{x(t)}{f(t, x(t))} \right] + \lambda \left[\frac{x(t)}{f(t, x(t))} \right] = h(t, x(t)), \quad t \in J,$$

$$x(0) = x(T),$$

for $\lambda > 0$, $J = [0, T]$. They applied iteration principle embodied in hybrid fixed point theorem in partially ordered normed linear spaces to obtain the existence as well as approximations of the positive solutions for this system.

In 2020, Ardjouni [3] investigated the nonlinear third-order hybrid differential equation

$$\left(\frac{x(t)}{p(t) + \int_{t_0}^t g(s, x(s)) dt}\right)''' = h(t, x(t)), \quad t \in J, \quad J = [0, T],$$

$$\left(\frac{x(t)}{p(t) + \int_{t_0}^t g(s, x(s)) dt}\right)^{(k)} \Big|_{t=t_0} = a_k, \quad k = 0, 1, 2.$$

By employing Dhage iteration principle in a partially ordered normed linear space they proved the existence and approximation of solutions of the initial value problems of this system.

The previous works were mostly based on differential equations of integer order. We intend to study a class of hybrid fractional differential equations with impulses in a partially ordered linear algebra, which contains linear perturbations and quadratic perturbations in the meantime. Our results not only rely on iteration principles and related hybrid fixed point theorems, which have been proposed, but also extend some known theories to fractional-order systems. Furthermore, the conclusions may be subject to the influence of impulses to some extent. These advantages inspire us to continue current works.

In this paper, we discuss the following hybrid fractional (in the sense of Caputo) differential equation with impulses, which is given by

$${}^c D^\alpha \left[\frac{x(t) - f(t, x(t))}{g(t, x(t))} \right] = h(t, x(t)), \quad t \in J',$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, \dots, m,$$

$$x(0) = x_0, \tag{1}$$

where ${}^c D^\alpha$ denotes the Caputo fractional derivative of order α , $0 < \alpha \leq 1$, $f, h \in$

$C(J \times \mathbb{R}, \mathbb{R}^+)$, $g \in C(J \times \mathbb{R}, \mathbb{R}^+ \setminus \{0\})$, $I_k(x(t_k)) \in C(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, m$, $J =$

$[0, 1]$, $0 = t_0 < t_1 < \dots < t_k < \dots < t_m < t_{m+1} = 1$, $J' = J \setminus \{t_1, \dots, t_m\}$,

$J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, \dots , $J_m = (t_m, 1]$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$,
 $k = 1, \dots, m$.

The plan of this paper is as follows. In Section 2, we recall some necessary notions, definitions, and lemmas that used in the rest. In Section 3, we apply hybrid fixed point theorem to solve the initial value problems of (1) and give rigorous proof in the meantime. In Section 4, an example is given to illustrate our conclusions.

Preliminaries

In this section, we recall some basic and essential definitions of fractional calculus. Meanwhile, we introduce some definitions, lemmas, as well as

relative theories in linear partially ordered spaces for better obtaining our main results.

Definition 1. (See [26].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : [0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0,$$

where $\Gamma(\alpha)$ is the Gamma function, provided the right side is pointwise defined on $[0, +\infty)$.

Definition 2. (See [26].) The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n [0, 1]$ is defined by.

$${}^c D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds,$$

$$n-1 < \alpha < n, n = [\alpha] + 1.$$

Lemma 1. (See [26].) Let $\alpha > 0$. Then the solution of the fractional differential equation

$${}^c D^{\alpha} f(t) = 0$$

is

$$f(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1},$$

$$C_i \in \mathbb{R}, i = 0, 1, 2, \dots, n, n = [\alpha] + 1.$$

Unless otherwise specified, we assume that $(S, \preceq, \|\cdot\|)$ denotes a partially ordered Banach space with an order relation \preceq and the norm $\|\cdot\|$. Two elements x and y in S are said to be comparable if either the relation $x \preceq y$ or $y \preceq x$ holds. A nonempty subset C of S is called a chain or totally ordered if all the elements of C are comparable. The set S is said to be regular if each sequence $\{x_n\}$ is nondecreasing (resp. nonincreasing) in S such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The details can be found in [8] and the reference therein.

Definition 3. (See [16].) A mapping $F : S \rightarrow S$ is called isotone or nondecreasing if it preserves the order relation \preceq that is, if $x \preceq y$ implies $Fx \preceq Fy$ for all $x, y \in S$.

Definition 4. (See [16].) A mapping $F : S \rightarrow S$ is called partially continuous at a point $a \in S$ if for $\epsilon > 0$, there exists a $\delta > 0$ such that $\|Fx - Fa\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. F is called partially continuous on S if it is partially continuous at every point of S .

It is clear that if F is partially continuous on S , then it is continuous on every chain C contained in S

Definition 5. (See [16].) A mapping $F : S \rightarrow S$ is called partially bounded if $F(C)$ is bounded for every chain C in S . F is called uniformly partially bounded if $F(C)$ is bounded for every chain C in S by a unique constant. F is called bounded if $F(S)$ is a bounded subset of S

Definition 6. (See [4].) The set S is said to be quasi-equicontinuous in $[0, c]$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $u \in S$, $k \in \mathbb{Z}^+$, $t', t'' \in (t_{k-1}, t_k] \cap [0, c]$, and

Lemma 2 [Compactness criterion]. (See [4].) The set $S \subset PC([0, c], \mathbb{R}^n)$ ($c < +\infty$) is relatively compact if and only if

- (i) S is bounded,
- (ii) S is quasi-equicontinuous in $[0, c]$

Definition 7. (See [16].) A mapping $F : S \rightarrow S$ is called partially compact if $F(C)$ is a relatively compact subset of S for all totally ordered sets or chains C in S .

Definition 8. (See [13,14].) The order relation \preceq and the metric d on a nonempty set S are said to be D -compatible if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in S and if subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x^\#$ implies that the original sequence $\{x_n\}$ converges to $x^\#$. Similarly, given a partially ordered normed linear space $(S, \preceq, \|\cdot\|)$, the order relation \preceq and the norm $\|\cdot\|$ are said to be D -compatible if \preceq and the metric d defined through the norm $\|\cdot\|$ are D -compatible. A subset E of S is called a Janhavi set if the order relation \preceq and the metric d or the norm $\|\cdot\|$ are D -compatible in it. In particular, if $E = S$, then S is called a Janhavi metric space or Janhavi Banach space

Definition 9. (See [16].) A upper semicontinuous and nondecreasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a D -function, provided $\varphi(0) = 0$. Let $(S, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $T : S \rightarrow S$ is called partially nonlinear D -Lipschitz if there exists a D -function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|Tx - Ty\| \leq \varphi(\|x - y\|)$ for all comparable elements, $x, y \in S$. If $\varphi(x) = kx$, $k > 0$, then T is called a partially Lipschitz with a Lipschitz constant k

We recall that a nonempty closed and convex subset \mathcal{K} of the Banach algebra S is called a cone if

- (i) $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$
- (ii) $\lambda \mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathbb{R}$, $\lambda > 0$ and
- (iii) $\{-\mathcal{K}\} \cap \mathcal{K} = \{\theta\}$ where θ is a zero element of S .

The cone \mathcal{K} in S is called positive if

- (i) $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$, where “ \circ ” is a multiplicative composition in S ; see Dhage [10] and the reference therein.

Now we assume that \mathcal{K} denotes a positive cone.

Lemma 3. (See [16].) If $x_1, x_2, y_1, y_2 \in \mathcal{K}$ are such that $x_1 \preceq y_1$ and $x_2 \preceq y_2$, then $x_1 x_2 \preceq y_1 y_2$.

Lemma 4. (See [14].) Every ordered Banach space (S, \mathcal{K}) is regular.

Lemma 5. (See [14].) Every partially compact subset E of an ordered Banach space (S, \mathcal{K})

Lemma 6. (See [9].) Let $(S, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra, and let every compact chain C in S be Janhavi set. Let $A, B : S \rightarrow \mathcal{K}$ and $\mathcal{C} : S \rightarrow S$ be three nondecreasing operators such that:

- (i) A and B are partially bounded and partially nonlinear D -Lipschitz with D -functions ψ and φ , respectively;
- (ii) B is partially continuous and compact;
- (iii) $L\psi(r) + \varphi(r) < r$, $r > 0$, where $L = \sup\{\|B(C)\| : C \text{ is a chain in } S\}$; and
- (iv) there exists an element $x_0 \in S$ such that $x_0 \preceq Ax_0 \cdot Bx_0 + \mathcal{C}x_0$ or $x_0 \succeq Ax_0 \cdot Bx_0 + \mathcal{C}x_0$.

Then the operator equation $Ax \cdot Bx + \mathcal{C}x = x$ has a solution $x^* \in S$, and the sequence x_n of successive iterations defined by $x_{n+1} = Ax_n \cdot Bx_n + \mathcal{C}x_n$, $n = 0, 1, \dots$, converges monotonically to x^* .

We consider the space $PC(J, \mathbb{R})$ defined on with norm $\|\cdot\|$, which is given by

$$PC(J, \mathbb{R}) := \{x : J \rightarrow \mathbb{R} \mid x \in C(J_k), x(t_k^+), x(t_k^-) \text{ exist}, \\ x(t_k^-) = x(t_k), k = 0, 1, \dots, m\},$$

$\|x\| = \sup_{t \in J} |x(t)|$. Clearly $PC(J, \mathbb{R})$ is a Banach space with the supremum norm. We define the order cone κ in $PC(J, \mathbb{R})$ by

$$K = \{x \in PC(J, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\},$$

which is a positive cone in $PC(J, \mathbb{R})$, and define order relation \preceq in $PC(J, \mathbb{R})$ by

$$x \preceq y \iff x(t) \leq y(t), \quad t \in J.$$

Clearly, $(PC(J, \mathbb{R}), \kappa)$ is a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$, and every compact chain C in $PC(J, \mathbb{R})$ is Janhavi set in view of the Lemmas 4 and 5. For the sake of convenience, we denote $f(0, 0) := f_0$, $g(0, 0) := g_0$ in the sequel.

Lemma 7. (See [29].) Let $x(t) \in C^n[0, 1]$ be a solution of the following impulsive hybrid fractional differential equation:

$$\begin{aligned} {}^c D^\alpha \left(\frac{x(t) - f(t, x(t))}{g(t, x(t))} \right) &= h(t), \quad t \in J', \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) &= x_0. \end{aligned} \tag{2}$$

Then it satisfies the following impulsive hybrid fractional integral equation:

$$x(t) = g(t, x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s) ds \right. \\ \left. + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0} \right) + f(t, x(t)), \quad t \in J_k, \quad k = 0, 1, \dots, m.$$

Definition 10. Let $x(t) \in PC(J, \mathbb{R})$ satisfy

$$x(t) = g(t, x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s) ds \right. \\ \left. + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0} \right) + f(t, x(t)), \quad t \in J.$$

Then the function x is called a mild solution of IVPs (2).

Definition 11. Let $u \in PC(J, \mathbb{R})$ satisfy

$$u(t) \leq g(t, u(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s) ds \right. \\ \left. + \sum_{i=1}^k \frac{I_i(u(t_i))}{g(t_i, u(t_i))} + \frac{x_0 - f_0}{g_0} \right) + f(t, u(t)), \quad t \in J.$$

Then the function u is called a mild lower solution of IVPs (2).

Main results

In this section, we attempt to obtain the existence and approximation for a mild solution of IVPs for impulsive hybrid fractional differential equation (1).

In order to obtain our main results, hypotheses (H1)–(H5) are given by:

(H1) There exist D -positive constants M and N such that

$$0 < f(t, x) \leq M, \quad 0 < g(t, x) \leq N,$$

and $(x_0 - f_0)/g_0 \geq 0$, where $t \in J, x \in \mathbb{R}$.

(H2) There exist D -functions φ and ψ such that

$$0 \leq f(t, x) - f(t, y) \leq \varphi(x - y), \quad 0 \leq g(t, x) - g(t, y) \leq \psi(x - y),$$

where $t \in J, x, y \in \mathbb{R}, x \geq y$.

(H3) $h(t, x)$ is nondecreasing in x for each $t \in J$, and there exists a constant κ such that

$$0 < h(t, x) \leq K,$$

where $t \in J, x, y \in \mathbb{R}, x \geq y$.

(H4) $I_k(x)/g(t, x)$ is nondecreasing in x for each $t \in J$, and there exists a constant H such that

$$0 \leq \frac{I_k(x)}{g(t, x)} \leq H,$$

where $t \in J, x \in \mathbb{R}, k = 1, \dots, m$.

(H5) IVPs (1) has a mild lower solution $t \in J, (J, \mathbb{R})$.

Theorem 1. Under assumptions (H1)–(H5) and further, we assume that

$$\bar{L}\psi(r) + \varphi(r) < r, \quad r > 0,$$

where we denote

$$\bar{L} = \frac{(m+1)K}{\Gamma(\alpha+1)} + \frac{x_0 - f_0}{g_0} + (m+1)H. \quad (3)$$

Then IVPs (1) has a mild solution x^* , and the sequence $\{x_n\}$ of successive approximations defined by

$$\begin{aligned} x_1(t) &= u(t), \\ x_{n+1}(t) &= g(t, x_n(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x_n(s)) \, ds \right. \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x_n(s)) \, ds + \sum_{i=1}^k \frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))} \\ &\quad \left. + \frac{x_0 - f_0}{g_0} \right) + f(t, x_n(t)), \quad t \in J_k, \quad k = 0, 1, \dots, m, \end{aligned}$$

converges monotonically to x^* (4)

Proof. At first, we define $S = {}_{PC}(J, \mathbb{R})$. Then by Lemma 5 we have that every compact chain C in S possesses the compatibility property with

respect to the norm $\|\cdot\|$, and the order relation \preceq so that every compact chain C is a Janhavi set in S

By Lemma 7 a mild solution of impulsive hybrid fractional differential equation (1) is a solution for impulsive hybrid integral equation

$$\begin{aligned} x(t) = & g(t, x(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x(s)) \, ds \right. \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x(s)) \, ds + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0} \Big) \\ & + f(t, x(t)), \quad t \in J_k, \quad k = 0, 1, \dots, m. \end{aligned}$$

Define operators $A, B : S \rightarrow K$ and $C : S \rightarrow S$ by

$$Ax(t) = g(t, x(t)), \quad Cx(t) = f(t, x(t)),$$

$$\begin{aligned} Bx(t) = & \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x(s)) \, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x(s)) \, ds \\ & + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0}, \quad t \in J_k, \quad k = 0, 1, \dots, m. \end{aligned}$$

In this way, the impulsive hybrid fractional integral equation (5) can be written as the operator equation

$$Ax(t) \cdot Bx(t) + Cx(t) = x(t), \quad x \in S, \quad t \in J_k, \quad k = 0, 1, \dots, m.$$

For the sake of showing our main results, we prove that A, B , and C satisfy the conditions of Lemma 6. This is achieved in the following steps.

Step 1. A, B , and C are nondecreasing operators on S

Let $x, y \in S$ be such that $x \succeq y$. Then by hypothesis (H2) we obtain

$$\begin{aligned} Ax(t) = g(t, x(t)) & \geq g(t, y(t)) = Ay(t), \\ Cx(t) = f(t, x(t)) & \geq f(t, y(t)) = Cy(t), \end{aligned}$$

for all $t \in J_k, k = 0, 1, \dots, m$. By the definition of the order relation in S we get $Ax \succeq Ay, Cx \succeq Cy$. This shows that A and C are nondecreasing operators. Similarly,

$$\begin{aligned}
 \mathcal{B}x(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x(s)) \, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x(s)) \, ds \\
 &\quad + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0} \\
 &\geq \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, y(s)) \, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, y(s)) \, ds \\
 &\quad + \sum_{i=1}^k \frac{I_i(y(t_i))}{g(t_i, y(t_i))} + \frac{y_0 - f_0}{g_0} \\
 &= \mathcal{B}y(t), \quad t \in J_k, \quad k = 0, 1, \dots, m.
 \end{aligned}$$

It is shown that the operator \mathcal{B} is also nondecreasing. Thus, \mathcal{A} , \mathcal{B} , and \mathcal{C} are nondecreasing positive operators.

Step 2. We note that \mathcal{A} , \mathcal{C} are partially bounded and partially D -Lipschitz operator on S

Let $x \in S$ be arbitrary. The by (H1)

$$|\mathcal{A}x(t)| = |g(t, x(t))| \leq N, \quad |\mathcal{C}x(t)| = |f(t, x(t))| \leq M$$

for all $t \in J_k, k = 0, 1, \dots, m$. Taking supremum over t , we obtain $\|\mathcal{A}x\| \leq N, \|\mathcal{C}x\| \leq M$. So, \mathcal{A} and \mathcal{C} are bounded. This further implies that \mathcal{A} and \mathcal{C} are partially bounded on S

Let $x, y \in S$ and $x \succeq y$. Then by (H2) we have

$$\begin{aligned}
 |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |g(t, x(t)) - g(t, y(t))| \leq \psi(|x(t) - y(t)|) \leq \psi(\|x - y\|), \\
 |\mathcal{C}x(t) - \mathcal{C}y(t)| &= |f(t, x(t)) - f(t, y(t))| \leq \varphi(|x(t) - y(t)|) \leq \varphi(\|x - y\|).
 \end{aligned}$$

Taking the supremum over t , we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x - y\|), \quad \|\mathcal{C}x - \mathcal{C}y\| \leq \varphi(\|x - y\|).$$

Therefore, \mathcal{A} and \mathcal{C} are partially nonlinear D -Lipschitz on S . Assumption (i) of Lemma 6 is satisfied.

Step 3. We show that \mathcal{B} is a partially continuous operator on S

At first, let $\{x_n\}$ be a sequence in a chain C of S , which satisfies

$$x_n \rightarrow x, \quad n \rightarrow \infty.$$

Then by Lebesgue dominated convergence theorem

$$\lim_{n \rightarrow \infty} \mathcal{B}x_n(t)$$

$$\begin{aligned}
 &= \frac{1}{\Gamma(\alpha)} \lim_{n \rightarrow \infty} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x_n(s)) \, ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \lim_{n \rightarrow \infty} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x_n(s)) \, ds + \lim_{n \rightarrow \infty} \sum_{i=1}^k \frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))} \\
 &\quad + \frac{u_0 - f_0}{g_0} \\
 &= \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, x(s)) \, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, x(s)) \, ds \\
 &\quad + \sum_{i=1}^k \frac{I_i(x(t_i))}{g(t_i, x(t_i))} + \frac{x_0 - f_0}{g_0} \\
 &= \mathcal{B}x(t), \quad t \in J_k, \, k = 0, 1, \dots, m.
 \end{aligned}$$

This indicates that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on $J_k, k = 0, 1, \dots, m$.

Next, we prove that $\mathcal{B}x_n$ is a quasi-equicontinuous sequence of function in S . Let $t', t'' \in J_k$ be arbitrary, $k = 0, 1, \dots, m$. Then by hypothesis

$$\begin{aligned}
 &|\mathcal{B}x_n(t'') - \mathcal{B}x_n(t')| \\
 &\leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t''} (t''-s)^{\alpha-1} h(s, x_n(s)) \, ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t''-s)^{\alpha-1} h(s, x_n(s)) \, ds \right| \\
 &\leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t'-s)^{\alpha-1} h(s, x_n(s)) \, ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t''-s)^{\alpha-1} h(s, x_n(s)) \, ds \right| \\
 &\quad + \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t''-s)^{\alpha-1} h(s, x_n(s)) \, ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t''} (t''-s)^{\alpha-1} h(s, x_n(s)) \, ds \right| \\
 &\leq \frac{K}{\Gamma(\alpha)} \left(\left| \int_{t_k}^{t'} (t'-s)^{\alpha-1} - (t''-s)^{\alpha-1} \, ds \right| + \left| \int_{t'}^{t''} (t''-s)^{\alpha-1} \, ds \right| \right) \\
 &= \frac{K}{\Gamma(\alpha)} \left(\left| \frac{(t''-s)^\alpha - (t'-s)^\alpha}{\alpha} \right|_{t_k}^{t'} + \frac{(t'-t'')^\alpha}{\alpha} \right).
 \end{aligned}$$

Note that $0 < \alpha \leq 1$, then we have

$$|(t''-s)^\alpha - (t'-s)^\alpha| \leq |(t''-t')^\alpha|.$$

So

$$|\mathcal{B}x_n(t'') - \mathcal{B}x_n(t')| \leq \frac{2|(t' - t'')^\alpha|}{\alpha} \rightarrow 0$$

as $\tau' \rightarrow \tau' - \epsilon$ for all $n \in \mathbb{N}$, which shows that the convergence $Bx_n \rightarrow Bx$ is uniform, and hence, B is partially continuous on S .

Step 4. B is a partially compact operator on S .

Let C be an arbitrary chain in S . We need to prove that (C) is a uniformly bounded set in S . Let $y \in (C)$ be any element. Then there is an element $x \in C$ such that $y = Bx$. Now by hypotheses (H3), (H4) we have

$$\begin{aligned} |y(t)| &= |Bx(t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} |h(s, x(s))| ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |h(s, x(s))| ds \\ &\quad + \sum_{i=1}^k \frac{|I_i(x(t_i))|}{|g(t_i, x(t_i))|} + \frac{|x_0 - f_0|}{|g_0|} \\ &\leq \frac{(m+1)K}{\Gamma(\alpha+1)} + (m+1)H + \frac{x_0 - f_0}{g_0} := \bar{L}, \end{aligned}$$

where $t \in J_k$, $k = 0, 1, \dots, m$. Taking supremum over t , we obtain $\|y\|_k = \|Bx\|_k \leq \bar{L}$ for

Next, we prove that $B(C)$ is a quasi-equicontinuous set in S . Let $t', t'' \in J_k$ be arbitrary, $k = 0, 1, \dots, m$. Likewise, we discuss above in Step 3:

$$\begin{aligned} |Bx_n(t'') - Bx_n(t')| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t''} (t''-s)^{\alpha-1} h(s, x_n(s)) ds - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t'} (t'-s)^{\alpha-1} h(s, x_n(s)) ds \right| \end{aligned}$$

when $\tau' \rightarrow \tau' - \epsilon$. Hence, $B(C)$ is a quasi-equicontinuous subset of S . Now, $B(C)$ is a uniformly bounded and quasi-equicontinuous set of functions in S . By applying Lemma 2 it is compact. Consequently, B is a partially compact operator on S into itself.

Therefore, assumption (ii) of Lemma 6 is satisfied.

Step 5. There is a $u \in S$ such that satisfies $u \leq Au \cdot Bu + Cu$. IVPs (1) has a mild lower solution u . Then by application of Lemma 7 and Definition 11 we have

$$\begin{aligned} u(t) &\leq g(t, u(t)) \left(\frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s, u(s)) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} h(s, u(s)) ds + \sum_{i=1}^k \frac{I_i(u(t_i))}{g(t_i, u(t_i))} + \frac{x_0 - f_0}{g_0} \right) \\ &\quad + f(t, u(t)), \quad t \in J_k, \quad k = 0, 1, \dots, m. \end{aligned}$$

Then from the definitions of A , B , and C we can deduce that

$$u(t) \leq Au(t) \cdot Bu(t) + Cu(t), \quad t \in J_k, \quad k = 0, 1, \dots, n.$$

Hence, $u \leq Au \cdot Bu + Cu$, which meets (iii) in Lemma 6.

Step 6. ψ and φ satisfy $L\psi(r) + \varphi(r) < r, r > 0$. It is clear that ψ and ϕ are D -functions of operators A and C from condition (H2), L denotes $\sup\{\#B(C)\# : C \text{ is a chain in } S\}$ from the proof given in Step 4. Then, combining with condition (3), we have

$$L\psi(r) + \varphi(r) < \bar{L}\psi(r) + \varphi(r) < r.$$

Assumption (iv) of Lemma 6 is satisfied

In a word, all the conditions of Lemma 6 are satisfied. Thus, we can conclude that the operator equation $Ax \cdot Bx + Cx = x$ has a solution $x^* \in B(C)$. Consequently, IVPs (1) has a mild solution x^* defined on $J_k, k = 0, 1, \dots, m$. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (4) converges monotonically to x^* . This completes the proof

Example

Given the interval $J = [0, 1]$ and the points $t_1 = 1/4, t_2 = 1/2, t_3 = 3/4$, we consider the IVPs of the following impulsive hybrid fractional differential equation:

$$\begin{aligned} {}^cD^{1/2} \left[\frac{x(t) - f(t, x(t))}{g(t, x(t))} \right] &= h(t, x(t)), \quad t \in J := [0, 1] \setminus \{t_1, t_2, t_3\}, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, 3, \\ x(0) &= \frac{1}{15}, \end{aligned} \tag{6}$$

where functions f, g, h are defined as follo

$$\begin{aligned} f(t, x) &= \frac{1}{15\pi} (\arctan x + \pi), & h(t, x) &= \frac{1}{40} (\tanh x + 2), \\ g(t, x) &= \begin{cases} \frac{1}{2}, & x \leq 0, \\ \frac{1+x}{2}, & 0 < x < 1, \\ 1, & x \geq 1, \end{cases} & I_k(x) &= \begin{cases} 0, & x \leq 0, \\ \frac{1}{(k+1)^2} \frac{x}{1+x}, & x > 0, \end{cases} \quad k = 1, 2, 3, \end{aligned}$$

for arbitrary $x \in \mathbb{R}, t \in J$. Through calculation, we hav

$$0 < f(t, x) \leq 1/10, 0 < g(t, x) \leq 1. \tag{i}$$

$$0 < f(t, x) - f(t, y) \leq (x - y)/(15\pi), 0 < g(t, x) - g(t, y) \leq (x - y)/2 \text{ when } x \geq y. \tag{ii}$$

$h(t, x)$ is nondecreasing, and $0 < h(t, x) \leq 3/40$ (iii)

$I_k(x)/g(t, x)$ is nondecreasing, and $0 \leq I_k(x)/g(t, x) \leq 1/4$, $k = 1, 2, 3$. (iv)

It is clear that $u(t) = 0$, $t \in J$, is a mild lower solution of IVP (6) by Definition 11. (v)

Take $\varphi(r) = r/(15\pi)$, $\psi(r) = r/2$, $N = 1$, $M = 1/10$, $K = 3/40$, $H = 3/40$, $H = 1/4$ in Theorem 1. Then we calculate that $L = 1.338$, moreover, $L\psi(r) + \varphi(r) < r$, $r > 0$. Therefore, all the conditions of Theorem 1 are satisfied. Then we can conclude that system (6) has a mild solution x^* , and x_n of successive approximation is defined by where $x_1 = u = 0$, converges monotonically to x^* .

$$\begin{aligned} x_{n+1}(t) = & g(t, x_n(t)) \left(\frac{1}{40\Gamma(\alpha)} \left(\int_{t_k}^t (t-s)^{\alpha-1} (\tanh x_n(s) + 2) ds \right. \right. \\ & + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} (\tanh x_n(s) + 2) ds \Big) + \sum_{i=1}^k \frac{I_i(x_n(t_i))}{g(t_i, x_n(t_i))} \Big) \\ & + \frac{1}{15\pi} (\arctan x_n(t) + \pi), \quad t \in J_k, \quad k = 0, 1, \dots, m, \end{aligned}$$

where $x_1 = u = 0$, converges monotonically to x^* .

Acknowledgments

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript

References

- 1.S. Abbas, M. Benchohra, J. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018, <https://doi.org/10.1515/9783110553819>.
- 2.B. Ahmad, M. Alghanmi, S. Ntouyas, A. Alsaedi, Fractional differential equations involving generalized derivative with stieltjes and fractional integral boundary conditions, *Appl. Math. Lett.*, 84:111–117, 2018, <https://doi.org/10.1016/j.aml.2018.04.024>.
3. A. Ardjouni, A. Djoudim, Approximating solutions of third-order nonlinear hybrid differential equations via Dhage iteration principle, *Khayyam J. Math.*, 6(1):95–103, 2020, <https://doi.org/10.22034/kjm.2019.97175>.

4. D. Bainov, P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman, New York, 1993.
5. T. Bashiri, S. Vaezpour, C. Park, Existence results for fractional hybrid differential systems in Banach algebras, *Adv. Difference Equ.*, **2016**:13, 2016, <https://doi.org/10.1186/s13662-016-0784-8>.
6. D. Boccaletti, G. Pucacco, *Theory of Orbits: Perturbative and Geometrical Methods*, Springer, Berlin, Heidelberg, 1999, <https://doi.org/10.1007/978-3-662-09240-8>.
7. B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl.*, **.(4)**:465–486, 2010, <https://doi.org/10.7153/dea-02-28>.
8. B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.*, **5(2)**:155–184, 2013, <https://doi.org/10.7153/dea-05-11>.
9. B.C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.*, **45(4)**:397–426, 2014, <https://doi.org/10.5556/j.tkmj.45.2014.1512>.
10. B.C. Dhage, A coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space with applications, *Differ. Equ. Appl.*, **.(4)**:453–477, 2017, <https://doi.org/10.7153/dea-2017-09-31>.
11. B.C. Dhage, Dhage iteration method in the theory of IVPs of nonlinear first order functional differential equations, *Malaya J. Mat.*, **..**:680–689, 2017, <https://doi.org/10.26637/MJM0504/0011>.
12. B.C. Dhage, Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, *Differ. Equ. Appl.*, **11(1)**:1–85, 2019, <https://doi.org/10.7153/dea-2019-11-01>.
13. B.C. Dhage, Dhage monotone iteration principle for IVPs of nonlinear first order impulsive differential equations, *Jñāna-bha*, **50(1)**:265–278, 2020.
14. B.C. Dhage, Dhage monotone iteration method for PBVPs of nonlinear first order impulsive differential equations, *Electron. J. Math. Anal. Appl.*, **..**:37–51, 2021.
15. B.C. Dhage, S.B. Dhage, K. Buvaneswari, Existence of mild solutions of nonlinear boundary value problems of coupled hybrid fractional integrodifferential equations, *J. Fract. Calc. Appl.*, **10(2)**:191–206, 2019.
16. B.C. Dhage, S.B. Dhage, S. K. Ntouyas, Approximating positive solutions of PBVPs of nonlinear first order ordinary quadratic differential equations, *Appl. Math. Lett.*, **46**:133–142, 2015, <https://doi.org/10.1016/j.aml.2015.02.023>.
17. K. Diethelm, *The Analysis of Fractional Differential Equations*, Lect. Notes Math., Vol. 2004, Springer, Berlin, Heidelberg, 2010, <https://doi.org/10.1007/978-3-642-14574-2>.
18. C.A. Gallegos, H.R. Henríquez, J.G. Mesquita, Growth of solutions for measure differential equations and dynamic equations on time scales, *J. Math. Anal. Appl.*, **479(1)**:941–962, 2019, <https://doi.org/10.1016/j.jmaa.2019.06.059>.

19. G. Iskenderoglu, D. Kaya, Symmetry analysis of initial and boundary value problems for fractional differential equations in Caputo sense, *Chaos Solitons Fractals*, **134**:109684, 2020, <https://doi.org/10.1016/j.chaos.2020.109684>.
20. T. Kato, *A Short Introduction to Perturbation Theory for Linear Operators*, Springer, New York, 1982, <https://doi.org/10.1007/978-1-4612-5700-4>.
21. T. Li, N. Pintus, G. Viglialoro, Properties of solutions to porous medium problems with different sources and boundary conditions, *Z. Angew. Math. Phys.*, **70**(3):91, 2019, <https://doi.org/10.1007/s00033-019-1130-2>
22. T. Li, Y. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, *Appl. Math. Lett.*, **105**:7, 2020, <https://doi.org/10.1016/j.aml.2020.106293>.
23. T. Li, G. Viglialoro, Analysis and explicit solvability of degenerate tensorial problems, *Bound. Value Probl.*, **2018**:2, 2018, <https://doi.org/10.1186/s13661-017-0920-8>.
24. T. Ma, Y. Tian, Q. Huo, Y. Zhang, Boundary value problem for linear and nonlinear fractional differential equations, *Appl. Math. Lett.*, **86**:1–7, 2018, <https://doi.org/10.1016/j.aml.2018.06.010>.
25. G. Parra, A. Arenas, B. Chen-Charpentier, A fractional order epidemic model for the simulation of outbreaks of influenza A(H1N1), *Math. Methods Appl. Sci.*, **37**(15):2218–2226, 2014, <https://doi.org/10.1002/mma.2968>.
26. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng., Vol. 198, Academic Press, San Diego, CA, 1999.
27. G. Viglialoro, T. Woolley, Boundedness in a parabolic-elliptic chemotaxis system with nonlinear diffusion and sensitivity and logistic source, *Math. Methods Appl. Sci.*, **41**(5):1809–1824, 2018, <https://doi.org/10.1002/mma.4707>.
28. M. Xu, Z. Han, Positive solutions for integral boundary value problem of two-term fractional differential equations, *Bound. Value Probl.*, **2018**:100, 2018, <https://doi.org/10.1186/s13661-018-1021-z>.
29. J. You, S. Sun, On impulsive coupled hybrid fractional differential systems in Banach algebras, *J. Appl. Math. Comput.*, **62**(1–2):189–205, 2020, <https://doi.org/10.1007/s12190-019-01280-z>.
30. A. Zemlyanova, A. Machina, A new B-spline collocation method for singular integro- differential equations of higher orders, *J. Comput. Appl. Math.*, **380**:12, 2020, <https://doi.org/10.1016/j.cam.2020.112949>
31. Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid differential equations, *Comput. Math. Appl.*, **62**(3):1312–1324, 2011, <https://doi.org/10.1016/j.camwa.2011.03.041>.
32. Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, Singapore, 2014, <https://doi.org/10.1142/9069>

Notes

- * This research is supported by Shandong Provincial Natural Science Foundation (ZR2020MA016), also supported by the Natural Science Foundation of China (62073152)

supported by the Natural Science Foundation of China (62073153)