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On the boundary value problems of piecewise differentialequations with left-right fractional derivatives and delay

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Abstract: In this paper, we study the multi-point boundary value problems for a new kind of piecewise differential equations with left and right fractional derivatives and delay. In this system, the state variables satisfy the different equations in different time intervals, and they interact with each other through positive and negative delay. Some new results on the existence, no-existence and multiplicity for the positive solutions of the boundary value problems are obtained by using Guo–Krasnoselskii’s fixed point theorem and Leggett–Williams fixed point theorem. The results for existence highlight the influence of perturbation parameters. Finally, an example is given out to illustrate our main results.

Keywords: boundary value problem, piecewise differential equation, left and right fractional derivative, delay, disturbance parameter, fixed point theorem.

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Introduction

In recent decades, fractional calculus has been widely used in various fields of science and technology, and the theoretical research of fractional differential equations has also received extensive attention, see [5–8, 10, 12–15, 17, 18, 22, 23, 25, 27, 30–34, 36] and the references therein. And the differential equation with left and right fractional derivatives have been studied extensively due to the wide application [2–4, 16, 24, 29]. In [2], the following nonlocal boundary value problems of integro-differential equations involving mixed left and right fractional derivatives and left and right fractional integrals are studied

$${}^c D_{1-}^{\alpha} {}^{RL} D_{0+}^{\beta} y(t) + \lambda I_{1-}^p I_{0+}^q h(t, y(t)) = f(t, y(t)), \quad t \in J := [0, 1],$$

$$y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \quad 0 < \xi < \mu < 1,$$

where $1 < \alpha \leq 2, 0 < \beta \leq 1$ and $p, q > 0, f, h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, and $\delta, \lambda, \mu \in \mathbb{R}$ are constants. At the same time, the differential equations with delay have many successful applications in the fields of communication engineering, population control and so on; see [1, 9, 20, 21, 26, 28, 35].

Motivated by above works, we discuss the multi-point boundary value problems for piecewise differential equations with left and right fractional derivatives and delay

$$\begin{aligned} {}^c D_{\xi^-}^\alpha u(t) + f(t, u(t), u(t + \tau_1)) &= 0, \quad t \in [0, \xi], \\ {}^c D_{\xi^+}^\beta u(t) + g(t, u(t), u(t - \tau_2)) &= 0, \quad t \in (\xi, 1], \\ u'(\xi^-) &= \rho_2 u'(0) + a = -(\rho_1 u'(\xi^+) + b), \\ u(0) &= \gamma_1 u(\xi^-), \quad u(1) = \gamma_2 u(\xi^+), \end{aligned}$$

where ${}^c D_{\xi^-}^\alpha$ is the right Caputo fractional derivative, ${}^c D_{\xi^+}^\beta$ is the left Caputo fractional derivative, $1 < \alpha, \beta \leq 2$, $\xi \in (0, 1)$, $u(\xi^-) = \lim_{\varepsilon \rightarrow 0^-} u(\xi + \varepsilon)$, $u(\xi^+) = \lim_{\varepsilon \rightarrow 0^+} u(\xi + \varepsilon)$.

$\gamma_i, \rho_i, \tau_i \in \mathbb{R}$ and $0 < \gamma_i < 1$, $\rho_1 > 0$, $0 \leq \rho_2 < 1$, $a, b \geq 0$, $0 \leq \tau_1 \leq 1 - \xi$, $0 \leq \tau_2 \leq \xi$.

$f \in C([0, \xi] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $g \in C([\xi, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

In boundary value problem (1), the state variable $u = u(\tau)$ satisfies the different equations in different time intervals, and they interact with each other through positive delay τ_1 and negative delay τ_2 . The parameters a and b in the boundary conditions represent the error in certain measurement. Some new results on the existence, no-existence and multiplicity for the positive solutions of the boundary value problems are obtained by using Guo–Krasnoselskii’s fixed point theorem and Leggett–Williams fixed point theorem. The results for existence highlight the influence of perturbation parameters. Finally, an example is given out to illustrate our main results.

Preliminaries

For convenience of reading, in this section, we give out some definitions about the fractional calculus and some lemmas.

Definition 1. (See [17].) Let $\alpha > 0$, $a < b \in \mathbb{R}$, and the left and right Riemann–Liouville fractional integral of $u : [a, b] \rightarrow \mathbb{R}$ are defined as

$$\begin{aligned} {}_{a^+} I_t^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s) \, ds, \\ {}_t I_{b^-}^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha-1} u(s) \, ds, \end{aligned}$$

respectively, for $t \in [a, b]$.

Definition 2. (See [17].) Let $\alpha > 0$, $a < b \in \mathbb{R}$, and the left Caputo fractional derivative and right Caputo fractional derivative of function $u : [a, b] \rightarrow \mathbb{R}$ are defined as

$${}_a^c D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

$${}_a^c D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

respectively, provided the right-sided integral converges, where $n \in \mathbb{N}$.

Lemma 1. (See [17]). If $\alpha > 0$, then

$${}_a^+ I_t^\alpha ({}_a^c D_t^\alpha u(t)) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1},$$

$${}_t I_b^\alpha ({}_t^c D_b^\alpha u(t)) = u(t) + d_0 + d_1(b-t) + d_2(b-t)^2 + \cdots + d_{n-1}(b-t)^{n-1},$$

where $c_i, d_i \in \mathbb{R}, i = 0, 1, \dots, n-1, n \in \mathbb{N}$.

Lemma 2 2. (See [11].) Let E be a Banach space and $P \# E$ is a cone. Assume that Ω_1, Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be a completely continuous operator such that either.

- (i) $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1$, and $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2$, or
- (ii) $\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_1$, and $\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_2$.

Then the operator T has at least one fixed point on $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 3. (See [19].) Assume P is a cone in Banach space, ω is a nonnegative continuous concave functional on P , the constants $0 < d < q < c \leq r$. Denote $\overline{P}_r = \{u \in P: \|u\| \leq r\}$ and $P(\omega, q, c) = \{u \in P: q \leq \omega(u) \text{ and } \|u\| \leq c\}$. Let $T : \overline{P}_r \rightarrow \overline{P}_r$ be a completely continuous operator such that $\omega(u) \leq \|u\|$ for $x \in \overline{P}_r$ such that

- (i) $\{u \in P(\omega, q, c)P: \omega(u) > q\} \neq \emptyset$ and $\omega(Tu) > q$ for $u \in P(\omega, q, c)$;
- (ii) $\|Tu\| < d$ for $u \in \overline{P}_d$;
- (iii) $\omega(Tu) > q$ for any $u \in P(\omega, q, r)$ and $\|Tu\| > c$.

Then T has at least three fixed points $u_1, u_2, u_3 \in \overline{P}_r$ such that $\|u_1\| < d, \omega(u_2) > q$ and

$\|u_3\| > d$ with $\omega(u_3) < q$.

Lemma 4. Let $h \in C([0, \xi], \mathbb{R}^+), y \in C([\xi, 1], \mathbb{R}^+)$, then the boundary value problem

$$\begin{aligned} {}_t^c D_{\xi^-}^\alpha u(t) + h(t) &= 0, \quad t \in [0, \xi], \\ {}_{\xi^+}^c D_t^\beta u(t) + y(t) &= 0, \quad t \in (\xi, 1], \\ u'(\xi^-) &= \rho_2 u'(0) + a = -(\rho_1 u'(\xi^+) + b), \\ u(0) &= \gamma_1 u(\xi^-), \quad u(1) = \gamma_2 u(\xi^+) \end{aligned} \tag{2}$$

has a unique solution given by

$$u(t) = \begin{cases} \int_0^\xi G_1(t, s)h(s) \, ds + \frac{1}{1-\rho_2} \left(\frac{\xi\gamma_1}{1-\gamma_1} + t \right) \left(\frac{\rho_2}{\Gamma(\alpha-1)} \int_0^\xi s^{\alpha-2}h(s) \, ds + a \right), & t \in [0, \xi], \\ \int_\xi^1 G_2(t, s)y(s) \, ds - \frac{1}{\rho_1(1-\rho_2)} \left(\frac{\gamma_2\xi-1}{1-\gamma_2} + t \right) \left(\frac{\rho_2}{\Gamma(\alpha-1)} \int_0^\xi s^{\alpha-2}h(s) \, ds + a \right) \\ + (1-\rho_2)b, & t \in (\xi, 1], \end{cases}$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{1}{1-\gamma_1} s^{\alpha-1}, & 0 \leq s < t \leq \xi, \\ -(s-t)^{\alpha-1} + \frac{1}{1-\gamma_1} s^{\alpha-1}, & 0 \leq t \leq s \leq \xi; \end{cases} \quad (4)$$

$$G_2(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} \frac{1}{1-\gamma_2} (1-s)^{\beta-1} - (t-s)^{\beta-1}, & \xi \leq s < t \leq 1, \\ \frac{1}{1-\gamma_2} (1-s)^{\beta-1}, & \xi \leq t \leq s \leq 1, \end{cases} \quad (5)$$

Proof. From Lemma 1 the general solution of the linear differential equation ${}_t^C D_{\xi-}^\alpha u(t) + h(t) = 0$ is given by

$$\begin{aligned} u(t) &= -{}_t I_{\xi-}^\alpha h(t) - c_0 - c_1(\xi - t) \\ &= -\frac{1}{\Gamma(\alpha)} \int_t^\xi (s-t)^{\alpha-1} h(s) \, ds - c_0 - c_1(\xi - t), \quad t \in [0, \xi], \end{aligned} \quad (6)$$

and $u'(t) = (1/\Gamma(\alpha-1)) \int_t^\xi (s-t)^{\alpha-2} h(s) \, ds + c_1$.

The general solution of the linear differential equation ${}_{\xi+}^C D_t^\beta u(t) + y(t) = 0$ is given

$$\begin{aligned} u(t) &= -{}_{\xi+} I_t^\beta y(t) - c_2 - c_3 t \\ &= -\frac{1}{\Gamma(\beta)} \int_\xi^t (t-s)^{\beta-1} y(s) \, ds - c_2 - c_3 t, \quad t \in (\xi, 1], \end{aligned} \quad (7)$$

and $u'(t) = -(1/\Gamma(\beta-1)) \int_\xi^t (t-s)^{\beta-2} y(s) \, ds - c_3$.

By the boundary value conditions $u'(\xi^-) = \rho_2 u'(0) + a = -(\rho_1 u'(\xi^+) + b)$ we can easily get that

$$c_1 = \frac{1}{1 - \rho_2} \left(\frac{\rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} h(s) \, ds + a \right),$$

$$c_3 = \frac{1}{\rho_1(1 - \rho_2)} \left(\frac{\rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} h(s) \, ds + a + (1 - \rho_2)b \right). \quad (8)$$

By (6)–(8) and the boundary conditions $u(0) = \gamma_1 u(\xi^-)$, $u(1) = \gamma_2 u(\xi^+)$ we ca also get that

$$c_0 = \frac{1}{\gamma_1 - 1} \left(\int_0^\xi \frac{1}{\Gamma(\alpha)} s^{\alpha-1} h(s) \, ds + \frac{\xi}{1 - \rho_2} \left(\int_0^\xi \frac{\rho_2}{\Gamma(\alpha - 1)} s^{\alpha-2} h(s) \, ds + a \right) \right),$$

$$c_2 = \frac{1}{\gamma_2 - 1} \left(\frac{1}{\Gamma(\beta)} \int_\xi^1 (1 - s)^{\beta-1} y(s) \, ds \right. \\ \left. + \frac{1 - \gamma_2 \xi}{\rho_1(1 - \rho_2)} \left(\frac{\rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} h(s) \, ds + a + (1 - \rho_2)b \right) \right).$$

Thus, by substituting c_0 and c_1 into (6) we can get that for $t \in (\xi, 1]$,

$$u(t) = \int_0^\xi G_1(t, s) h(s) \, ds + \frac{1}{1 - \rho_2} \left(\frac{\xi \gamma_1}{1 - \gamma_1} + t \right) \left(\frac{\rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} h(s) \, ds + a \right),$$

and by substituting c_2 and c_3 into (7) we can get that for $t \in [0, \xi]$,

$$u(t) = \int_\xi^1 G_2(t, s) y(s) \, ds \\ - \frac{1}{\rho_1(1 - \rho_2)} \left(\frac{\gamma_2 \xi - 1}{1 - \gamma_2} + t \right) \left(\frac{\rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} h(s) \, ds + a + (1 - \rho_2)b \right).$$

Hence, $u(t)$ satisfies equation (3) if it is the solution of the boundary value problem (2) and vice versa.

Lemma 5. Suppose $G_i(t, s)$ ($i = 1, 2$) are defined by (4), (5), then $G_i(t, s)$ has the following properties, respectively

- (i) $G_1(t, s)$ is continuous and $0 \leq \gamma_1 G_1(\xi, s) \leq G_1(t, s) \leq G_1(\xi, s)$, $\partial G_1(t, s) / \partial t \geq 0$ on $(t, s) \in [\xi, 1] \times [\xi, 1]$.
- (ii) $G_2(t, s)$ is continuous and $0 \leq \gamma_1 G_1(\xi, s) \leq G_1(t, s) \leq G_1(\xi, s)$, $\partial G_1(t, s) / \partial t \geq 0$ on $(t, s) \in [\xi, 1] \times [\xi, 1]$.

Proof. (i) Obviously, $G_1(t, s)$ is continuous on $[0, \xi] \times [0, \xi]$; For $\xi \leq s < t \leq \xi$,

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)(1 - \gamma_1)} s^{\alpha-1} \geq 0, \quad \frac{\partial G_1(t, s)}{\partial t} = 0,$$

and for $0 \leq t \leq s \leq \xi$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \left(\frac{1}{1 - \gamma_1} s^{\alpha-1} - (s - t)^{\alpha-1} \right),$$

$$\frac{\partial G_1(t, s)}{\partial t} = \frac{1}{\Gamma(\alpha - 1)} (s - t)^{\alpha-2} \geq 0.$$

Hence, $G_1(t, s)$ is monotone increasing for any $t \in [0, \xi]$, and

$$0 \leq G_1(0, s) \leq G_1(t, s) \leq G_1(s, s) = G_1(\xi, s).$$

Because $G_1(0, s) = \frac{1}{\Gamma(\alpha)(1 - \gamma_1)} s^{\alpha-1} = \gamma_1 G_1(\xi, s)$, then

$$0 \leq \gamma_1 G_1(\xi, s) \leq G_1(t, s) \leq G_1(\xi, s), \quad \frac{\partial G_1(t, s)}{\partial t} \geq 0, \quad (t, s) \in [0, \xi] \times [0, \xi].$$

(ii) Similar to the proof of (i), we can prove that (ii) holds.

Let $J = [0, 1]$, $J_0 = J \setminus \{\xi\}$, $E = PC(J, \mathbb{R}) = \{u : J \rightarrow \mathbb{R} : u \text{ is continuous in } J_0, u(\xi^+) \text{ and } u(\xi^-) \text{ exist, and } u(\xi^-) = u(\xi)\}$. Obviously, E is a Banach space with the norm

$$\|u\| = \sup_{t \in [0, 1]} |u(t)|.$$

Denote $\|u\|_{[0, \xi]} = \sup_{t \in [0, \xi]} |u(t)|$, $\|u\|_{(\xi, 1]} = \sup_{t \in (\xi, 1]} |u(t)|$, then $\|u\| = \max\{\|u\|_{[0, \xi]}, \|u\|_{(\xi, 1]}\}$. Set

$$P = \left\{ u \in E : u(t) \geq 0, t \in [0, 1], \inf_{t \in [0, \xi]} u(t) \geq \gamma_1 \|u\|_{[0, \xi]}, \inf_{t \in (\xi, 1]} u(t) \geq \gamma_2 \|u\|_{(\xi, 1]} \right\},$$

Obviously, $u = u(t)$ is a positive solution of (1) if and only if u is a fixed point of the operator T in P

Lemma 6. The operator $T : P \rightarrow P$ is completely continuous.

Proof. By Lemma 5 we can easily obtain that $T : P \rightarrow P$.

Let $\{u_n\} \subset P$, $u \in P$, and $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. There exists a constant $\overline{M}_0 > 0$ such that $\|u_n\| \leq \overline{M}_0$ and $\|u\| \leq \overline{M}_0$.

By the continuity of $f(t, u, v)$, $g(t, u, v)$ we have

$$\lim_{n \rightarrow \infty} (f(t, u_n(t), u_n(t + \tau_1)) - f(t, u(t), u(t + \tau_1))) = 0,$$

$$\lim_{n \rightarrow \infty} (g(t, u_n(t), u_n(t - \tau_2)) - g(t, u(t), u(t - \tau_2))) = 0,$$

and there is a constant $\overline{M}_1 > 0$, which makes $\sup_{(t, u, v) \in A} |f(t, u, v)| \leq \overline{M}_1$ and $\sup_{(t, u, v) \in B} |g(t, u, v)| \leq \overline{M}_1$, where $A = [0, \xi] \times [-\overline{M}_0, \overline{M}_0] \times [-\overline{M}_0, \overline{M}_0]$, $B = [\xi, 1] \times [-\overline{M}_0, \overline{M}_0] \times [-\overline{M}_0, \overline{M}_0]$.

Thus, $\lim_{n \rightarrow \infty} \|Tu_n - Tu\| = 0$, which implies that the operator T is a continuous operator.

Let $\Omega \subset P$ be bounded. By the continuity of f, g , we can get that there is a constant $\overline{M}_2 > 0$ such that $|f(t, u, v)| \leq \overline{M}_2$ for any $t \in [0, \xi], u, v \in \Omega$, and $|g(t, u, v)| \leq \overline{M}_2$ for all $t \in [0, \xi], u, v \in \Omega$,

By Lemma 5 we can show that $(Tu)^\#(t) \geq 0$ for $t \in [0, \xi]$ and $(Tu)'(t) \leq 0$ for $t \in [0, \xi]$. Hence ,

$$\begin{aligned} \|Tu\|_{[0, \xi]} &= Tu(\xi) \\ &\leq \frac{1}{1 - \gamma_1} \left(\frac{\overline{M}_2 \xi^\alpha}{\Gamma(\alpha + 1)} + \frac{\xi}{1 - \rho_2} \left(\frac{\rho_2 \xi^{\alpha-1} \overline{M}_2}{\Gamma(\alpha)} + a \right) \right), \\ \|Tu\|_{(\xi, 1]} &= Tu(\xi^+) \\ &\leq \frac{1}{1 - \gamma_2} \left(\frac{\overline{M}_2 (1 - \xi)^\beta}{\Gamma(\beta + 1)} + \frac{1 - \xi}{\rho_1 (1 - \rho_2)} \left(\frac{\overline{M}_2 \rho_2 \xi^{\alpha-1}}{\Gamma(\alpha)} + a + (1 - \rho_2)b \right) \right). \end{aligned}$$

Consequently, $T(\Omega)$ is uniformly bounded

Since $G_1(t, s)$ is continuous, it is uniformly continuous on $(t, s) \in [0, \xi] \times [0, \xi]$;

Hence, for any $\varepsilon > 0$, there exists a constant

$$0 < \delta_1 < \frac{\varepsilon (1 - \rho_2) \Gamma(\alpha)}{2(\overline{M}_2 \rho_2 \xi^{\alpha-1} + a \Gamma(\alpha) + 1)}$$

such that $|G_1(t_1, s) - G_1(t_2, s)| < \varepsilon / (2\overline{M}_2)$ for all $t_1, t_2, s \in [0, \xi]$ and $|t_1 - t_2| < \delta_1$.

Thus, for $u \in \Omega, t_1, t_2 \in [0, \xi], |t_1 - t_2| < \delta_1$ we have

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \overline{M}_2 \int_0^\xi |G_1(t_1, s) - G_1(t_2, s)| ds \\ &\quad + \frac{1}{1 - \rho_2} \left(\frac{\overline{M}_2 \rho_2 \xi^{\alpha-1}}{\Gamma(\alpha)} + a \right) |t_1 - t_2| < \varepsilon. \end{aligned}$$

Similarly, due to that $G_2(t, s)$ is continuous on $[\xi, 1] \times [\xi, 1]$, for above mentioned $\varepsilon > 0$, there exists a constant $\delta_2 > 0$ such that for $t_3, t_4, s \in [\xi, 1], |t_3 - t_4| < \delta_2$, we have $Tu(t_3) - Tu(t_4) < \varepsilon$.

Hence, $T(\Omega)$ is equicontinuous on $[0, \xi], (\xi, 1]$, respectively.

By Arzela–Ascoli theorem we know that operator T is a relative compactness operator, and because operator T is a continuous operator, it is a completely continuous operator.

Existence of the positive solutions

Denote

$$f_{\varphi} = \liminf_{u+v \rightarrow \varphi} \inf_{t \in [\xi - \tau_0, \xi]} \frac{f(t, u, v)}{u + v}, \quad g_{\varphi} = \liminf_{u+v \rightarrow \varphi} \inf_{t \in (\xi, \xi + \tau_0]} \frac{g(t, u, v)}{u + v},$$

$$f^{\varphi} = \limsup_{u+v \rightarrow \varphi} \sup_{t \in [0, \xi]} \frac{f(t, u, v)}{u + v}, \quad g^{\varphi} = \limsup_{u+v \rightarrow \varphi} \sup_{t \in (\xi, 1]} \frac{g(t, u, v)}{u + v},$$

$$M_1 = \frac{(1 - \gamma_1)(1 - \rho_2)\Gamma(\alpha + 1)}{\xi^{\alpha}(2 + 2(\alpha - 1)\rho_2 + \Gamma(\alpha + 1))},$$

$$M_2 = \frac{\Gamma(\beta + 1)(1 - \gamma_2)}{2(1 - \xi)^{\beta}} \left(1 - \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1 \xi(1 - \gamma_2)} \right),$$

$$M_3 = \min \left\{ \frac{M_1}{2}, \frac{M_1 \rho_1 \xi(1 - \gamma_2)}{2(1 - \xi)(1 - \gamma_1)} \right\}, \quad M_4 = \frac{\Gamma(\beta + 1)(1 - \gamma_2)}{6(1 - \xi)^{\beta}},$$

$$N_1 = \frac{(1 - \gamma_1)\Gamma(\alpha + 1)}{\gamma_1 \gamma (\xi^{\alpha} - (\xi - \tau_0)^{\alpha})}, \quad N_2 = \frac{(1 - \gamma_2)\Gamma(\beta + 1)}{\gamma_2 \gamma ((1 - \xi)^{\beta} - (1 - \xi - \tau_0)^{\beta})},$$

$$N = \max\{\gamma N_1, \gamma N_2\},$$

where $\varphi = 0^+$ or $+\infty$ and $\gamma = \min\{\gamma_1, \gamma_2\}$, $0 < \tau_0 \leq \min\{\tau_1, \tau_2\}$.

Theorem 1. Assume that $M_2 > 0$ and the following conditions hold:

$$f^0 < M_1 \text{ and } g^0 < M_2; \quad (\text{H1}')$$

$$f_{\infty} > N_1 \text{ or } g_{\infty} > N_2. \quad (\text{H2}')$$

Then there exist constants $a_0, b_0 \geq 0$ such that boundary value problem (1) has at least one positive solution for the parameters a and b with $0 \leq a \leq a_0, 0 \leq b \leq b_0$

Proof. Because $f^0 < M_1$, there exists a constant $r_1 > 0$ such that $f(t, u, v) < M_1(u+v)$ for any $t \in [0, \xi]$, $u + v \in (0, r_1)$. Similarly, by $g^0 < M_2$ there is a constant $r_2 > 0$ such that $g(t, u, v) < M_2(u + v)$ for any $t \in (\xi, 1]$, $u, v \in \Omega(0, r_2)$.

Let

$$r = \min \left\{ \frac{r_1}{2}, \frac{r_2}{2} \right\}, \quad \Omega_1 = \{u \in P: \|u\| \leq r\},$$

$u \in P \cap \partial \Omega_1$ and $a_0 = \xi^{\alpha-1} M_1 r, b_0 = 2 \xi^{\alpha-1} M_1 r / \Gamma(\alpha + 1)$. For any $u \in P \cap \partial \Omega_1$, we have $\|u\| = r$. When $0 \leq a \leq a_0, 0 \leq b \leq b_0$, for any $u \in P \cap \partial \Omega_1$, we have $0 \leq u(s) + u(s + \tau_1) \leq 2r \leq r_1$ for $s \in [0, \xi]$, and

$$\begin{aligned} \|Tu\|_{[0,\xi]} &= Tu(\xi) \\ &< M_1 \int_0^\xi G_1(\xi, s)(u(s) + u(s + \tau_1)) \, ds \\ &\quad + \frac{\xi}{(1 - \rho_2)(1 - \gamma_1)} \left(\frac{M_1 \rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} (u(s) + u(s + \tau_1)) \, ds + a_0 \right) \\ &< \frac{\xi^\alpha (2 + 2(\alpha - 1)\rho_2) + \Gamma(\alpha + 1)}{(1 - \gamma_1)(1 - \rho_2)\Gamma(\alpha + 1)} M_1 r = r = \|u\|. \end{aligned}$$

Similarly,

$$\|Tu\|_{(\xi,1]} = Tu(\xi^+) < \left(\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1 \xi (1 - \gamma_2)} \right) r.$$

In view of

$$\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1 \xi (1 - \gamma_2)} = 1,$$

we have $\#Tu\#_{(\xi,1]} < r = \#u\#$.

Then for any $u \in P \cap \partial\Omega_1$, we get $\#Tu\# \leq \#u\#$.

If $f_\infty > N_l$, there exists a constant $R_1 > 0$ such that $f(t, u, v) > N_l(u + v)$ for any $t \in [\xi - \tau_0, \xi]$, $u + v \in [R_1, +\infty)$.

Let

$$R = \frac{R_1}{\gamma}, \quad \Omega_2 = \{u \in P: \|u\| \leq R\}.$$

For $u \in P \cap \partial\Omega_2$, we have $\#u\# = R$, and

$$\inf_{t \in [\xi - \tau_0, \xi]} u(t) \geq \inf_{t \in [0, \xi]} u(t) \geq \gamma_1 \|u\|_{[0, \xi]}, \quad \inf_{t \in (\xi, \xi + \tau_0]} u(t) \geq \inf_{t \in (\xi, 1]} u(t) \geq \gamma_2 \|u\|_{(\xi, 1]}.$$

Because $\#u\# = \max\{\#u\#_{[0, \xi]}, \#u\#_{(\xi, 1]}\}$ for $t \in (\xi - \tau_0, \xi] \cup [0, \xi]$, then $t + \tau_1 \in (\xi + \tau_1 - \tau_0, \xi + \tau_1] \subset (\xi, 1]$ and

$$\begin{aligned} u(t) + u(t + \tau_1) &\geq \inf_{t \in (\xi - \tau_0, \xi]} u(t) + \inf_{t \in (\xi + \tau_1 - \tau_0, \xi + \tau_1]} u(t) \\ &\geq \inf_{t \in [0, \xi]} u(t) + \inf_{t \in (\xi, 1]} u(t) \geq \gamma_1 \|u\|_{[0, \xi]} + \gamma_2 \|u\|_{(\xi, 1]} \\ &\geq \gamma (\|u\|_{[0, \xi]} + \|u\|_{(\xi, 1]}) \geq \gamma \|u\|. \end{aligned}$$

So that

$$u(t) + u(t + \tau_1) \geq \gamma \|u\| = \gamma R = R_1, \quad t \in (\xi - \tau_0, \xi].$$

By Lemma 5 and (9) we can easily get that

$$\begin{aligned} Tu(0) &= \gamma_1 Tu(\xi) \geq \gamma_1 \int_0^\xi G_1(\xi, s) f(s, u(s), u(s + \tau_1)) ds \\ &> N_1 \gamma_1 \int_{\xi - \tau_0}^\xi G_1(\xi, s) (u(s) + u(s + \tau_1)) ds \geq N_1 \gamma_1 \gamma R \int_{\xi - \tau_0}^\xi G_1(\xi, s) ds \\ &= \frac{\gamma_1 \gamma (\xi^\alpha - (\xi - \tau_0)^\alpha)}{(1 - \gamma_1) \Gamma(\alpha + 1)} N_1 R = R = \|u\|. \end{aligned}$$

Then for $u \in P \cap \partial \Omega_2$, we have $\|Tu\| \geq \|u\|$.

According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Similarly, if $g_\infty > N_2$, there is a constant $R_2 > 0$, which makes $g(t, u, v) > N_2(u + v)$ for any $t \in (\xi, \xi + \tau_0]$, $u + v \in [R_2, +\infty)$.

Let

$$R_0 = \frac{R_2}{\gamma}, \quad \Omega_3 = \{u \in P: \|u\| \leq R_0\}.$$

We have $\#u\# = R_0$ for any $u \in P \cap \partial \Omega_3$ and for $t \in (\xi, \xi + \tau_0] \subset (\xi, 1]$, then $t - \tau_2 \in (\xi - \tau_2, \xi - (\tau_2 - \tau_0)) \subset [0, \xi]$

$$\begin{aligned} u(t) + u(t - \tau_2) &\geq \inf_{t \in (\xi, 1]} u(t) + \inf_{t \in [0, \xi]} u(t) \geq \gamma_1 \|u\|_{[0, \xi]} + \gamma_2 \|u\|_{(\xi, 1]} \\ &\geq \gamma (\|u\|_{[0, \xi]} + \|u\|_{(\xi, 1]}) \geq \gamma \|u\|. \end{aligned}$$

Thus

$$u(t) + u(t - \tau_2) \geq \gamma \|u\| = \gamma R_0 = R_2, \quad t \in (\xi, \xi + \tau_0],$$

and because

$$\begin{aligned} Tu(1) &= \gamma_2 Tu(\xi^+) \\ &\geq \gamma_2 \int_\xi^1 G_2(\xi, s) g(s, u(s), u(s - \tau_2)) ds > N_2 \gamma_2 \gamma R_0 \int_\xi^{\xi + \tau_0} G_2(\xi, s) ds \\ &= \frac{\gamma_2 \gamma ((1 - \xi)^\beta - (1 - \xi - \tau_0)^\beta)}{(1 - \gamma_2) \Gamma(\beta + 1)} N_2 R_0 = R_0 = \|u\|, \end{aligned}$$

then for any $u \in P \cap \partial \Omega_3$, we have $\|Tu\| \geq \|u\|$.

According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$, which implies the boundary value problem (1) has at least one positive solution.

In particular, the following result holds by Theorem 1.

Corollary 1. Assume that the following conditions hold: $(H1')$ $f_0 = g_0 =$

$$f^0 = g^0 = 0; \quad (H1')$$

$$f_\infty = +\infty \text{ or } g_\infty = +\infty. \quad (H2')$$

Then there exist constants $a_0, b_0 \geq 0$ such that boundary value problem (1) has at least one positive solution for $0 \leq a \leq a_0, 0 \leq b \leq b_0$.

Theorem 2. Assume that the following conditions (H3) and (H4) hold: $(H3)$ f

$$f^\infty < M_3 \text{ and } g^\infty < M_4; \quad (H3')$$

$$f_0 > N_1 \text{ or } g_0 > N_2. \quad (H4')$$

Then there exist constants $a_0, b_0 > 0$ such that boundary value problem (1) has at least one positive solution for $0 \leq a \leq a_0, 0 \leq b \leq b_0$.

Proof. Due to $f^\infty < M_3$, then there exists a constant $\lambda_1 > 0$ such that $f(t, u, v) < M_3(u + v)$ for any $t \in [0, \xi], u + v \in [\lambda_1, +\infty)$.

Let

$$D = \{(t, u, v): t \in [0, \xi], u \geq 0, v \geq 0\} := D_1 \cup D_2,$$

where $D_1 = \{(t, u, v) \in D: u + v \leq \lambda_1\}, D_2 = \{(t, u, v) \in D: u + v > \lambda_1\}$.

Since f is bounded on D_1 , then there is $L_1 > 0$, which makes $|f(t, u, v)| \leq L_1$ for any $(t, u, v) \in D_1$. Hence, for all $(t, u, v) \in D$, we have

$$|f(t, u, v)| < L_1 + M_3(u + v).$$

Similarly, because $g^\infty < M_4$, then there is a constant $\lambda_2 > 0$ such that $g(t, u, v) < M_4(u + v)$ for any $t \in (\xi, 1], u + v \in [\lambda_2, +\infty)$.

Let $\tilde{D} = \{(t, u, v): u + v \leq \lambda_2, t \in [\xi, 1], u \geq 0, v \geq 0\}$, then there is $L_2 > 0$ such that $|g(t, u, v)| \leq L_2$ for any $(t, u, v) \in \tilde{D}$. Then for any $t \in [\xi, 1]$ and $u, v \in [0, +\infty]$, we have $|g(t, u, v)| < L_2 + M_4(u + v)$. Denote

$$\lambda = \max \left\{ \lambda_1, \lambda_2, \frac{2\xi^\alpha(1 + (\alpha - 1)\rho_2)L_1}{\Gamma(\alpha + 1)(1 - \gamma_1)(1 - \rho_2)}, \right. \\ \left. 6 \left(\frac{(1 - \xi)\rho_2\xi^{\alpha-1}}{\rho_1(1 - \rho_2)(1 - \gamma_2)\Gamma(\alpha)} L_1 + \frac{(1 - \xi)^\beta}{\Gamma(\beta + 1)(1 - \gamma_2)} L_2 \right) \right\},$$

$$\Omega_4 = \{u \in P: \|u\| \leq \lambda\}, \quad a_0 = \frac{\xi^{\alpha-1}M_3\lambda}{2}, \quad b_0 = \frac{\xi^{\alpha-1}M_3\lambda}{\Gamma(\alpha + 1)}.$$

For any $u \in P \cap \partial\Omega_4$, which implies $\|u\| = \lambda$. Since

$$M_3 = \min \left\{ \frac{M_1}{2}, \frac{M_1\rho_1\xi(1 - \gamma_2)}{2(1 - \xi)(1 - \gamma_1)} \right\}.$$

Then for $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$, we have

$$\|T_1 u\|_{(\xi, 1]} = T_1 u(\xi^+) \\ < c \left(M_2 \int_{\xi}^1 G_2(\xi, s) ds + \frac{1 - \xi}{\rho_1(1 - \rho_2)(1 - \gamma_2)} \right. \\ & \quad \times \left. \left(\frac{M_1\rho_2}{\Gamma(\alpha - 1)} \int_0^{\xi} s^{\alpha-2} ds + a_0 + (1 - \rho_2)b_0 \right) \right) \\ < c \left(\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1\xi(1 - \gamma_2)} \right) = c.$$

and

$$\|T_1 u\|_{(\xi, 1]} = T_1 u(\xi^+) \\ < c \left(M_2 \int_{\xi}^1 G_2(\xi, s) ds + \frac{1 - \xi}{\rho_1(1 - \rho_2)(1 - \gamma_2)} \right. \\ & \quad \times \left. \left(\frac{M_1\rho_2}{\Gamma(\alpha - 1)} \int_0^{\xi} s^{\alpha-2} ds + a_0 + (1 - \rho_2)b_0 \right) \right) \\ < c \left(\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1\xi(1 - \gamma_2)} \right) = c.$$

Then for any $u \in P \cap \partial\Omega_4$ $\|Tu\| \leq \|u\|$.

If $f_0 > N_1$, then there exists a constant $\lambda_1 > \mu_1 > 0$ such that $f(t, u, v) > N_1(u + v)$ for any $t \in [\xi - \tau_0, \xi]$, $u + v \in (0, \mu_1]$.

Let

$$\Omega_5 = \{u \in P: \|u\| \leq \mu\}, \quad 0 < \mu \leq \frac{\mu_1}{2}.$$

Hence, for any $u \in \partial\Omega_5$, we have $\|u\| = \mu$. It is similar to (10), for $t \in (\xi - \tau_0, \xi]$

$$\gamma\mu = \gamma\|u\| \leq u(t) + u(t + \tau_1) \leq 2\|u\| \leq \mu_1.$$

Then for any $u \in \partial\Omega_5$

$$\begin{aligned} Tu(0) &= \gamma_1 Tu(\xi) \geq N_1 \gamma_1 \gamma \mu \int_{\xi - \tau_0}^{\xi} G_1(\xi, s) ds \\ &= \frac{\gamma_1 \gamma (\xi^\alpha - (\xi - \tau_0)^\alpha)}{(1 - \gamma_1) \Gamma(\alpha + 1)} N_1 \mu = \mu = \|u\|. \end{aligned}$$

Thus, for any $u \in P \cap \partial\Omega_5$, there is $\|Tu\| \geq \|u\|$. According to Lemma 2, T has at least one fixed point in $P \cap (\overline{\Omega_4} \setminus \Omega_5)$.

Similarly, if $g_0 > N_2$, there is a constant $\lambda_2 > \mu_2 > 0$ that makes $g(t, u, v) > N_2(u + v)$ for any $t \in (\xi, \xi + \tau_0]$, $u + v \in (0, \mu_2]$.

Let

$$\Omega_6 = \{u \in P: \|u\| \leq \bar{\mu}\}, \quad 0 < \bar{\mu} \leq \frac{\mu_2}{2}.$$

Then for any $u \in P \cap \partial\Omega_6$, we have $\|u\| = \bar{\mu}$. Similar to (11), for $t \in (\xi, \xi + \tau_0]$, we have

$$\gamma\bar{\mu} = \gamma\|u\| \leq u(t) + u(t - \tau_2) \leq 2\|u\| \leq \mu_2,$$

and for any $u \in P \cap \partial\Omega_6$,

$$\begin{aligned} Tu(1) &= \gamma_2 Tu(\xi^+) \geq \gamma_2 N_2 \gamma \bar{\mu} \int_{\xi}^{\xi+\tau_0} G_2(\xi, s) ds \\ &= \frac{\gamma_2 \gamma ((1-\xi)^\beta - (1-\xi-\tau_0)^\beta)}{(1-\gamma_2)\Gamma(\beta+1)} N_2 \bar{\mu} = \bar{\mu} = \|u\|. \end{aligned}$$

Thus, for any $u \in P \cap \partial\Omega_6$, we have $\|Tu\| \geq \|u\|$.

According to Lemma 2, T has at least one fixed point in $P \cap (\bar{\Omega}_4 \setminus \Omega_6)$, which implies that boundary value problem (1) has at least one positive solution.

In particular, the following result holds by Theorem 2.

Corollary 2. Assume that the following conditions hold:

$$f^\infty = g^\infty = 0; \quad (\text{H3}')$$

$$f_0 = +\infty \text{ or } g_0 = +\infty. \quad (\text{H4}')$$

Then there exist constants $a_0, b_0 \geq 0$ such that boundary value problem (1) has at least one positive solution for $0 \leq a \leq a_0, 0 \leq b \leq b_0$.

Theorem 3. Assume $f_\infty > N_1$ holds. Then there exists a large enough positive constant $a_1 > 0$ such that boundary value problem (1) has no positive solution for $a > a_1$.

Proof. If $f_\infty > N_1$, there exists a constant $R > 0$ such that for any $t \in [\xi, \xi + \tau_0]$, $u + v \in [\gamma R, +\infty)$, we have $f(t, u, v) > N_1(u + v)$. Assume that for any large enough $a > 0$, boundary value problem (1) has a positive solution $u = u(t)$.

Let

$$a_1 > \frac{(1-\rho_2)(1-\gamma_1)R}{\xi\gamma_1}, \quad a > a_1.$$

In fact, since

$$u(0) \geq \frac{\xi\gamma_1 a}{(1-\rho_2)(1-\gamma_1)} > R.$$

Hace

From (10) of Theorem 1 we have for any

$$u(t) + u(t + \tau_1) \geq \gamma \|u\| \geq \gamma R.$$

Hence

$$\begin{aligned} u(0) &= \gamma_1 u(\xi) \\ &\geq \gamma_1 \int_{\xi-\tau_0}^{\xi} G_1(\xi, s) f(s, u(s), u(s + \tau_1)) \, ds + \frac{\xi \gamma_1}{(1 - \rho_2)(1 - \gamma_1)} a \\ &\geq \frac{\gamma_1 \gamma (\xi^\alpha - (\xi - \tau_0)^\alpha)}{(1 - \gamma_1) \Gamma(\alpha + 1)} N_1 \|u\| + \frac{\xi \gamma_1}{(1 - \rho_2)(1 - \gamma_1)} a_1 \geq \|u\| + R. \end{aligned}$$

So $\|u\| \geq \|u\| + R$, which is a contradiction. Thus, there exists a constant $a_1 > 0$ such that the boundary value problem (1) has no positive ssolution for $a > a_2$

Theorem 4. Assume $g_\infty > N_2$ holds. Then there exist large enough positive constants $a_2, b_1 > 0$ such that boundary value problem (1) has no positive solution for $a > a_2, b > b_1$.

Proof. Similarly, if $g_\infty > N_2$, there exists a constant $R_0 > 0$ such that for any $(\xi, \xi + \tau_0), u + v \in [\gamma R_0, +\infty)$, we have $g(t, u, v) > N_2(u + v)$. Assume that for any large enough $a > 0, b > 0$, the boundary value problem (1) has a positive solution $u = u(t)$.

Let

$$a_2 + (1 - \rho_2)b_1 > \frac{(1 - \rho_2)(1 - \gamma_2)\rho_1 R_0}{(1 - \xi)\gamma_2}, \quad a > a_2, \quad b > b_1.$$

Since $Tu = u$, we have

$$u(1) \geq \frac{(1 - \xi)\gamma_2((a_2 + (1 - \rho_2)b_1))}{\rho_1(1 - \rho_2)(1 - \gamma_2)} > R_0.$$

Hence, $\#u\# > R_0$. Since for any $t \notin (\xi, \xi + \tau_0]$,

$$u(t) + u(t - \tau_2) \geq \gamma \|u\| \geq \gamma R_0.$$

Then

$$\begin{aligned}
 u(1) &= \gamma_2 u(\xi^+) \\
 &\geq \gamma_2 \int_{\xi}^{\xi+\tau_0} G_2(\xi, s) g(s, u(s), u(s-\tau_1)) \, ds + \frac{(1-\xi)\gamma_2}{\rho_1(1-\rho_2)(1-\gamma_2)} (a_2 + (1-\rho_2)b_1) \\
 &\geq \frac{\gamma_2 \gamma ((1-\xi)^\beta - (1-\xi-\tau_0)^\beta)}{(1-\gamma_2)\Gamma(\beta+1)} N_2 \|u\| + \frac{(1-\xi)\gamma_2}{\rho_1(1-\rho_2)(1-\gamma_2)} (a_2 + (1-\rho_2)b_1) \\
 &\geq \|u\| + R_0.
 \end{aligned}$$

So $\|u\| \geq \|u\| + R_0$, which is a contradiction. Thus, there exist constants $a_2, b_1 > 0$ such that the boundary value problem (1) has no positive solution for $a > a_2, b > b_1$.

Multiplicity of the positive solutions

In this section, we consider the multiplicity of solutions for boundary value problem (1) by using Lemma 3.

Let $\bar{P}_c = \{u \in P : \|u\| \leq c\}$. Define a nonnegative continuous concave functional $\omega : P \rightarrow [0, +\infty)$ by $\omega(u) = \inf_{t \in [\xi-\tau_0, \xi+\tau_0]} u(t)$. Obviously, $\omega(u) \leq \|u\|$ for any $u \in P$. Set $\bar{P}(\omega, q, c) = \{u \in P : q \leq \omega(u), \|u\| \leq c\}$.

Theorem 5. Suppose there are three constants d, q, c with $0 < d < q < c$, where $\min\{M_1c, M_2c\} \geq Nq > 0$, and the following hypotheses hold:

$$f(t, u, v) < M_1d \text{ for } (t, u, v) \in [0, \xi] \times [0, d] \times [0, d]; \quad g(t, u, v) < M_2d \text{ for } (t, u, v) \in [\xi, 1] \times [0, d] \times [0, d]; \quad (\text{H5})$$

$$f(t, u, v) > Nq \text{ for } (t, u, v) \in [\xi - \tau_0, \xi] \times [q, c] \times [0, c]; \quad g(t, u, v) > Nq \text{ for } (t, u, v) \in (\xi, \xi + \tau_0] \times [q, c] \times [0, c]; \quad (\text{H6})$$

$$f(t, u, v) \leq M_1c, \quad g(t, u, v) \leq M_2c \text{ for } (t, u, v) \in [0, \xi] \times [0, c] \times [0, c]; \quad g(t, u, v) \leq M_2c \text{ for } (t, u, v) \in [\xi, 1] \times [0, c] \times [0, c]. \quad (\text{H7})$$

Then there exist constants $a_0, b_0 \geq 0$ such that boundary value problem (1) has at least three positive solutions u_1, u_2, u_3 on P for $0 \leq a \leq a_0, 0 \leq b \leq b_0$, where $\#u_1\# < d, \omega(u_2) > q, \#u_3\# > d, \omega(u_3) < q$.

Proof. First of all, for any $u \in \bar{P}_c$, we have $0 \leq a \leq a_0, 0 \leq b \leq b_0$. Let $a_0 = \xi^{\alpha-1} M_1 c$.

Define a operator $T_1 : \bar{P}_c \rightarrow P$ by

$$T_1 u(t) = \begin{cases} \int_0^\xi G_1(t, s) f(s, u(s), u(s+\tau_1)) \, ds + \frac{1}{1-\rho_2} \left(\frac{\xi\gamma_1}{1-\gamma_1} + t \right) \\ \quad \times \left(\frac{\rho_2}{\Gamma(\alpha-1)} \int_0^\xi s^{\alpha-2} f(s, u(s), u(s+\tau_1)) \, ds + a \right), & t \in [0, \xi], \\ \int_\xi^1 G_2(t, s) g(s, u(s), u(s-\tau_2)) \, ds - \frac{1}{\rho_1(1-\rho_2)} \left(\frac{\gamma_2\xi-1}{1-\gamma_2} + t \right) \\ \quad \times \left(\frac{\rho_2}{\Gamma(\alpha-1)} \int_0^\xi s^{\alpha-2} f(s, u(s), u(s+\tau_1)) \, ds + a + (1-\rho_2)b \right), & t \in (\xi, 1]. \end{cases}$$

Then $u = u(t)$ is a solution of (1) if and only if u is a fixed point of the operator T_1 on P_c . By (H5)

$$\begin{aligned} \|T_1 u\|_{[0, \xi]} &= T_1 u(\xi) \\ &\leq \int_0^\xi G_1(\xi, s) ds + \frac{\xi}{(1 - \rho_2)(1 - \gamma_1)} \left(\frac{\rho_2 M_1 c}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} ds + a_0 \right) \\ &\leq M_1 c \left(\frac{\xi^\alpha (2 + 2(\alpha - 1)\rho_2 + \Gamma(\alpha + 1))}{(1 - \gamma_1)(1 - \rho_2)\Gamma(\alpha + 1)} \right) = c. \end{aligned}$$

Since $\min \{M_1 c, M_2 c\} > Nq > 0$, then $M_2 > 0$

$$\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1 \xi(1 - \gamma_2)} = 1$$

and

$$\begin{aligned} \|T_1 u\|_{(\xi, 1]} &= T_1 u(\xi^+) \\ &< c \left(M_2 \int_\xi^1 G_2(\xi, s) ds + \frac{1 - \xi}{\rho_1(1 - \rho_2)(1 - \gamma_2)} \right. \\ &\quad \times \left. \left(\frac{M_1 \rho_2}{\Gamma(\alpha - 1)} \int_0^\xi s^{\alpha-2} ds + a_0 + (1 - \rho_2)b_0 \right) \right) \\ &< c \left(\frac{2(1 - \xi)^\beta M_2}{\Gamma(\beta + 1)(1 - \gamma_2)} + \frac{(1 - \xi)(1 - \gamma_1)}{\rho_1 \xi(1 - \gamma_2)} \right) = c. \end{aligned}$$

He, we have $\|T_1 u\| \leq c$, which implies $T_1 : \overline{P}_c \rightarrow \overline{P}_c$.

Similarly, by (H5), we can get that $\|T_1 u\| < d$ for $u \in P_d$. Therefore, the condition (ii) in Lemma 3 is satisfied.

Select $u(t) = (q + c)/2$, $0 \leq t \leq 1$. Obviously, $u(t) = (q + c)/2 \in P(\omega, q, c)$ and $\omega(u) = \inf_{t \in [\xi - \tau_0, \xi + \tau_0]} u(t) = (q + c)/2 > q$, then $\{u \in P(\omega, q, c) : \omega(u) > q\} \neq \emptyset$. For any $u \in P(\omega, q, c)$, we can get that $u(t) > q$ for $t \in [\xi - \tau_0, \xi + \tau_0]$ and $0 \leq u(t) \leq c$ for $t \in [0, 1]$. By Lemma 5, we can easily get that $T_1 u$ is monotone increasing on $[0, \xi]$ and $T_1 u$ is monotone decreasing on $(\xi, 1]$. From (H6), we get $f(t, u, v) > Nq$, $t \in [\xi - \tau_0, \xi]$ and $g(t, u, v) > Nq$, $t \in (\xi, \xi + \tau_0]$

Hence

$$\begin{aligned} \inf_{t \in (\xi, \xi + \tau_0]} T_1 u(t) &= T_1 u(\xi + \tau_0) \geq T_1 u(1) = \gamma_2 T_1 u(\xi^+) \\ &> \gamma_2 \int_{\xi}^{\xi + \tau_0} G_2(\xi, s) N q \, ds \\ &\geq \frac{\gamma_2 \gamma ((1 - \xi)^\beta - (1 - \xi - \tau_0)^\beta)}{(1 - \gamma_2) \Gamma(\beta + 1)} N_2 q = q. \end{aligned}$$

and

$$\begin{aligned} \inf_{t \in [\xi - \tau_0, \xi]} T_1 u(t) &= T_1 u(\xi - \tau_0) \geq T_1 u(0) = \gamma_1 T_1 u(\xi) \\ &> \gamma_1 \int_{\xi - \tau_0}^{\xi} G_1(\xi, s) N q \, ds \\ &\geq \frac{\gamma_1 \gamma (\xi^\alpha - (\xi - \tau_0)^\alpha)}{(1 - \gamma_1) \Gamma(\alpha + 1)} N_1 q = q, \end{aligned}$$

Therefore, for $u \in P(\omega, q, c)$, we have $\omega(T_1 u(t)) > q$. Hence, condition (i) in Lemma 3 holds.

Due to Lemma 3 involves parameters d, q, c, r with $0 < d < q < c < r$. Let $c = r$, then by condition (i) in Lemma 3 it is clearly that for $u \in P(\omega, q, c)$ and $\|Tu\| > c$, we have $\omega(Tu) > q$.

Therefore, condition (iii) in Lemma 3 also satisfied. Then Lemma 3 implies that the boundary value problem (1) has at least three solutions u_1, u_2, u_3 on P_c and $\|u_1\| < d, \omega(u_2) > q, \|u_3\| > d, \omega(u_3) < q$.

Illustration

In order to illustrate the applicability of our main results, the following boundary value problem is considered in this section.

Example 1. For the following boundary value problem

$$\begin{aligned} {}^c D_{\pi/8-}^{3/2} u(t) + u(t) &+ \left(u(t) + u\left(t + \frac{3}{5}\right) \right) \left(\frac{1}{100} \sin t + \frac{u(t) + u\left(t + \frac{3}{5}\right)}{\cos t} \right) = 0, \quad t \in \left[0, \frac{\pi}{8}\right], \\ {}^c D_{\pi/8+}^{7/4} u(t) &+ \left(u(t) + u\left(t - \frac{1}{3}\right) \right) \left(\frac{1}{4} \cos t + \left(u(t) + u\left(t - \frac{1}{3}\right) \right)^2 \sin t \right) = 0, \quad t \in \left(\frac{\pi}{8}, 1\right], \\ u'\left(\left(\frac{\pi}{8}\right)^-\right) &= -\left(2u'\left(\left(\frac{\pi}{8}\right)^+\right) + b\right) = a, \\ u(0) &= \frac{1}{2}u\left(\left(\frac{\pi}{8}\right)^-\right), \quad u(1) = \frac{1}{2}u\left(\left(\frac{\pi}{8}\right)^+\right), \end{aligned}$$

we can establish the following results:

(i) If $a \in [0, 0.019191]$, $b \in [0, 0.028873]$, then boundary value problem (12) has at least one positive solution.

(ii) If $a \in (2.46995 \cdot 10^6, +\infty)$, $b \in (2.22295 \cdot 10^7, +\infty)$, then boundary value problem (12) has no positive solutions.

Proof. Boundary value problem (12) can be regarded as boundary value problem (1), where $\alpha = 3/2$, $\beta = 7/4$, $\xi = \pi/8$, $\rho_1 = 2$, $\rho_2 = 0$, $\gamma_1 = \gamma_2 = 1/2$, $\tau_1 = 3/5$, $\tau_2 = 1/3$, $f(t, u, v) = (u + v)(1/100) \sin t + (u + v) \cos t$ and $g(t, u, v) = (u + v)((1/4) \cos t + (u + v)^2 \sin t)$.

Let $\tau_0 = 1/4 < \min\{\tau_1, \tau_2\}$, we can easily obtain that $M_1 \approx 0.811256$, $M_2 \approx 0.218239 > 0$, $N_1 \approx 13.8342$ and $N_2 \approx 12.7312$, $f_0 \approx 0.00382783 < M_1$, $g_0 \approx 0.23097 < M_2$ and $g_\infty = +\infty > N_2$

Then there exist constants $r_1 = 0.151$, $r_2 = 0.234$, $r = \min\{r_1/2, r_2/2\} = 0.0755$, $R_0 = 7.5 \times 10^6$, It is easy to get that $a_0 = (\xi \alpha - 1 M_1 r)/2 = 0.019191$, $b_0 = (\xi^{\alpha-1} M_{1r})/\Gamma(\alpha + 1) = 0.028873$, $a_2 = 2.46995 \times 10^6$ and $b_1 = 2.22295 \times 10^7$.

(i) According to Theorem 1, if $a \in [0, a_0]$ and $b \in [0, b_0]$, then boundary value problem (12) has at least one positive solution.

(ii) According to Theorem 4, if $a \in (a_2, +\infty)$ and $b \in (b_1, +\infty)$, then boundary value problem (12) has no positive solutions.

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