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Abstract: Based on properties of Green's function and by Avery–Peterson fixed point theorem, the existence of multiple positive solutions are obtained for singular p-Laplacian fractional differential equation with infinite-point boundary conditions, and an example is given to demonstrate the validity of our main results.

Keywords: fractional differential model, multiple positive solutions, Avery–Peterson fixed point.

Introduction

In this paper, we will devote to considering the following infinite-point singular p-Laplacian fractional differential equation:

$$D_{0+}^{\beta} (\varphi_p({}^c D_{0+}^{\gamma} u))(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t <_1 1$$

with infinite-point boundary condition

$$\begin{aligned} u(0) &= u'(0) = \dots = u^{(i-1)}(0) = u^{(i+1)}(0) = \dots = u^{(n-1)}(0) = 0, \\ u^{(i)}(1) &= \sum_{j=1}^{\infty} \eta_j u(\xi_j), \quad {}^c D_{0+}^{\gamma} u(0) = 0, \\ \varphi_p({}^c D_{0+}^{\gamma} u(1)) &= \sum_{i=1}^{\infty} \zeta_i \varphi_p({}^c D_{0+}^{\gamma} u(\xi_i)), \end{aligned} \quad (2)$$

Where $\beta, \gamma \in \mathbb{R}^+ = [0, +\infty)$, $1 < \beta \leq 2$, $n-1 < \gamma \leq n$ ($n \geq 3$), $\gamma > i$, $p \geq 1$ Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q \geq 1$, $1/p + 1/q = 1$, and $0 < \eta_i, \zeta_i, \xi_i < 1$ ($i = 1, 2, \dots, \infty$), $f \in C([0, 1] \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \mathbb{R}_+^1)$ ($\mathbb{R}_+^1 = [0, +\infty)$), and $f(t, x_1, x_2)$

has singularity at $t = 0, 1$, $D_{0+}^{\beta} u$ is the standard Riemann-Liouville derivative, ${}^c D_{0+}^{\gamma} u$ is the standard Caputo derivative.

Fractional-order system may have additional attractive feature over the integer-order system. For example, the analytical solutions of the systems

$$\frac{d}{dt}x(t) = at^{a-1}, \quad {}^c D_t^{\alpha} x(t) = at^{a-1}, \quad 0 < \alpha < 1,$$

are $t^a + x(0)$ and $a\Gamma(a)t^{a+\alpha-1}/\Gamma(a+\alpha) + x(0)$, respectively. Obviously, the integer-order system is unstable for $a \in (0, 1)$, the fractional dynamic system is stable as $0 < a \leq 1$. Moreover, fractional-order systems have been shown to be more accurate and realistic than integer-order models, and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly, in viscoelasticity, electrochemistry, porous media, and so on. As a result, there has been a significant development in the study of fractional differential equations in recent years, readers can refer to [2, 4–10, 15–17, 21–24]. Jong [12] studied the following p -Laplacian fractional differential equations:

$$D_{0+}^{\beta} (\varphi_p(D_{0+}^{\alpha} u))(t) = f(t, u(t)), \quad 0 < t < 1, \quad (3)$$

with m -point boundary condition

$$\begin{aligned} u(0) &= 0, \quad D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma} u(\eta_i), \\ D_{0+}^{\alpha} u(0) &= 0, \quad \varphi_p(D_{0+}^{\alpha} u(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0+}^{\alpha} u(\eta_i)), \end{aligned} \quad (4)$$

Where $1 < \alpha, \beta \leq 2$, $3 < \alpha + \beta \leq 4$, $0 < \gamma \leq 1$, $\alpha - \gamma - 1 > 0$, $0 < \eta_i, \zeta_i, \xi_i < 1$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \dots$ permits singularities with respect to both the time and space variables. According to introducing height functions, the author obtained the existence and multiplicity of positive solution theorems, and Zhang and Zhai obtained the existence and uniqueness

of positive solution for this equation in [18]. In [19], Zhang and Liu investigated the following infinite-point fractional differential equation $C_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$):

$$(i = 1, 2, \dots, \infty), \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} < 1, \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} < 1 < 1, p-$$

Laplacian operator ϕ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q > 1$, $1/p + 1/q = 1$, and $f \in C([0, 1] \times (0, +\infty), [0, +\infty))$.

The authors obtained the existence and uniqueness of solutions by using the fixed point theorem for mixed monotone operators. Jong [11] obtained the existence and uniqueness of positive solutions by the Banach contraction mapping principle for equation (3), (4). In [20], the author considered the following fractional differential equation:

$$D_{0+}^{\alpha} u(t) = f(t, x(t), D_{0+}^{\alpha-2} u(t), D_{0+}^{\alpha-1} u(t)), \quad 0 < t < 1,$$

with infinite-point boundary condition

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j),$$

where $2 < \alpha \leq 3$, $f \in [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a Carathéodory function, $\xi_i, \gamma_i \in (0, 1)$, and $\{\xi_i\}_{i=1}^{+\infty}, \{\gamma_i\}_{i=1}^{+\infty}$ are two monotonic sequences with $\lim_{i \rightarrow +\infty} \xi_i = a$, $\lim_{i \rightarrow +\infty} \gamma_i = b$, $a, b \in (0, 1)$, $\alpha_i, \beta_i \in \mathbb{R}$, $D_{0+}^{\alpha} u$ is the standard Riemann–Liouville derivative. The authors established the existence of at least one solution for this equation by Mawhin’s continuation theorem. Motivated by the excellent results above, in this paper, the existence of multiple positive solutions are obtained for a singular infinite-point p -Laplacian boundary value problems. Compared with [19], the equation in this paper is p -Laplacian fractional differential equation, and the method which we used in this paper is Avery–Peterson fixed point theorem. Compared with [12], fractional derivative is involved in the nonlinear terms for BVP (1), (2), and multiple positive solutions are obtained for the BVP (1), (2).

Preliminaries and lemmas

In this section, we introduce definitions and preliminary results, which are used throughout this paper. First, we let $E = C^1[0, 1]$ be the Banach space with the maximum norm

$$\|u\| = \max\{\|u\|_0, \|u'\|_0\}, \text{ where } \|u\|_0 = \max_{t \in [0, 1]} |u(t)|, \|u'\|_0 = \max_{t \in [0, 1]} |u'(t)|,$$

In this section, we introduce definitions and preliminary results, which are used throughout this paper. First, we let $E = C^1[0, 1]$ be the Banach space with the maximum

norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{t \in [0,1]} |u(t)|$, $\|u'\|_0 = \max_{t \in [0,1]} |u'(t)|$, then we list a condition below to be used later in the paper.

(HO) $f: (0,1) \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \rightarrow \mathbb{R}_+^1$, and for all $t \in (0,1)$, there exists a function $\theta(t) \in L^1(0,1)$ and $f(t, x_0, x_1) \leq \theta(t)$, such that $\theta(t) \in L^1(0,1)$ and $f(t, x_0, x_1) \leq \theta(t)$.

Now, we state some lemmas, which are basic and used in this paper.

(See [13, 16].) Assume that $\alpha \in C^n[0,1]$, then $I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) - C_1 - C_2 t - \dots - C_n t^{n-1}$, $\alpha, C_i \in \mathbb{R}^1$ ($i = 1, 2, \dots, n$).

Lemma 2. Assume that $\alpha \in C^n[0,1]$, then $u \in C^n[0,1]$.

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n},$$

where n is the least integer greater than or equal to α , $C_i \in \mathbb{R}^1$ ($i = 1, 2, \dots, n$).

Lemma 3. See [3, Thm. 1.2.7].) Let $H \subset C^1[J, E]$, then H is a relatively compact set if and only if

a) H' is equicontinuous, and $H'(t)$ is a relatively compact set for any $H: [0,1] \times [0,1] \rightarrow \mathbb{R}_+^1$.

There exists $t_0 \in J$ such that $H(t_0)$ is a relatively compact set on E .

i) H is equicontinuous, and $H'(t)$ is a relatively compact set for any $t \in J$ on E ;

(ii) There exists $t_0 \in J$, such that $H(t_0)$ is a relatively compact set on E .

Lemma 4. (See [1, 14].) Let P be a cone of E , Φ and θ be nonnegative continuous concave functional on P , ϕ be nonnegative continuous concave functional on P , and ψ be a nonnegative continuous functional on P , $\psi(\mu x) \leq \mu \psi(x)$ for $0 \leq \mu \leq 1$ such that for some positive numbers L , h , $\phi(x) \leq \psi(x)$ and $\|x\| \leq L\Phi(x)$ for all $x \in P(\Phi, h)$. Let $A: P(\Phi, h) \rightarrow P(\Phi, h)$ is completely continuous and there exist positive numbers e, c, d , with $e < c$ such that following conditions are satisfied:

- (S1) $\{x \in P(\Phi, \theta, \phi, c, d, h): \phi(x) > c\} \neq \emptyset$, $\phi(Ax) > c$ for $x \in P(\Phi, \theta, \phi, c, d, h)$;
- (S2) $\phi(Ax) > c$ for $x \in P(\Phi, \phi, c, h)$, and $\theta(Ax) > d$;
- (S3) $0 \notin R(\Phi, \psi, e, h)$ and $\psi(Ax) < e$ for $x \in R(\Phi, \psi, e, h)$ with $\psi(x) = e$.

Then A has at least three fixed points x_1, x_2, x_3 such that $\Phi(x_i) \leq h$ for $i = 1, 2, 3$ and

$$c < \phi(x_1), e < \psi(x_2), \phi(x_2) < c, \psi(x_3) < e.$$

Lemma 5. Given $y \in L^1[0,1] \cap C(0,1)$, then the solution of the BVP

$${}^c D_{0+}^\gamma u(t) + y(t) = 0, \quad 0 < t < 1, \quad (5)$$

with boundary condition (2) can be expressed by

$$u(t) = \int_0^1 G(t,s)y(s) ds, \quad t \in [0,1], \quad (6)$$

where

$$G(t, s) = \frac{1}{\Delta \Gamma(\gamma)} \begin{cases} t^i \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1} - \Delta (t-s)^{\gamma-1}, & 0 \leq s \leq t \leq 1, \\ t^i \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (7)$$

in which

$$P(s) = \frac{1}{\Gamma(\gamma-i)} - \frac{1}{\Gamma(\gamma)} \sum_{s \leq \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s} \right)^{\gamma-1} (1-s)^i, \quad (8)$$

$$\Delta = i! - \sum_{j=1}^{\infty} \eta_j \xi_j^i. \quad (9)$$

Proof: By Means of Lemma I we reduce(5) to an equivalent equation

$$u(t) = -I_{0+}^{\gamma} y(t) + C_1 + C_2 t + \cdots + C_i t^{i-1} + C_{i+1} t^i + \cdots + C_n t^{n-1}$$

for $C_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$). From $u(0) = u'(0) = \cdots = u^{(i-1)}(0) = u^{(i+1)}(0) = \cdots = u^{(n-1)}(0) = 0$ we have $C_j = 0$ ($j \neq i+1$). Consequently, we get.

$$u(t) = C_{i+1} t^i - I_{0+}^{\gamma} y(t),$$

hence, we have

$$u^{(i)}(t) = i! C_{i+1} - I_{0+}^{\gamma-i} y(t). \quad (10)$$

On the other hand, $u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, combining with (10), we get

$$\begin{aligned} C_{i+1} &= \int_0^1 \frac{(1-s)^{\gamma-i-1}}{\Gamma(\gamma-i)\Delta} y(s) ds - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\gamma-1}}{\Gamma(\gamma)\Delta} y(s) ds \\ &= \int_0^1 \frac{(1-s)^{\gamma-i-1} P(s)}{\Delta} y(s) ds, \end{aligned}$$

Therefore, $G(t; s)$ is as (7). By simple calculation we have

$$\frac{\partial G(t, s)}{\partial t} = \frac{1}{\Delta \Gamma(\gamma)} \begin{cases} it^{i-1} \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1} - (\gamma-1) \Delta(t-s)^{\gamma-2}, & 0 \leq s \leq t \leq 1, \\ it^{i-1} \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 6. let $u(t) = -I_{0+}^{\gamma} y(t) + C_1 + C_2 t + \dots + C_i t^{i-1} + C_{i+1} t^i + \dots + C_n t^{n-1}$ then the BVP (1), (2) has a unique solution.

$$u(t) = \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds, \quad (11)$$

where $G(t; s)$ is as (6), and

$$H(t, s) = \frac{1}{\overline{\Delta} \Gamma(\beta)} \begin{cases} \Gamma(\beta) \overline{P}(s) t^{\beta-1} (1-s)^{\beta-1} - \overline{\Delta} (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ \Gamma(\beta) \overline{P}(s) t^{\beta-1} (1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (12)$$

in which

$$\overline{\Delta} = 1 - \sum_{i=1}^{\infty} \zeta_i \xi_i^{\beta-1}, \quad (13)$$

$$\overline{P}(s) = \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta)} \sum_{s \leq \xi_j} \zeta_j \left(\frac{\xi_j - s}{1-s} \right)^{\beta-1}. \quad (14)$$

Proof. Let $\overline{y} \in C[0, 1]$, $v(t) = \varphi_p({}^c D_{0+}^{\gamma} u)(t)$. Consider the boundary value problem

$$D_{0+}^{\beta} v(t) + \overline{y}(t) = 0, \quad 0 < t < 1, \quad v(0) = 0, \quad v(1) = \sum_{j=1}^{\infty} \zeta_j v(\xi_j). \quad (15)$$

By means of the Lemma 2 we reduce (15) to an equivalent integral equation

$$v(t) = -I_{0+}^{\beta} \overline{y}(t) + C_1 t^{\beta-1} + C_2 t^{\beta-2} \quad [16]$$

for $C_i \in \mathbb{R}$ ($i = 1, 2$). From $v(0) = 0$ we have $C_2 = 0$. Consequently, we get

$$v(t) = C_1 t^{\beta-1} - I_{0+}^{\beta} \overline{y}(t).$$

for $i \in \mathbb{R} (i = 1, 2)$. From $v(0) = 0$ we have $C_2 = 0$. Consequently, we get $v(t) = C_1 t^{\beta-1} - I_{0+}^{\beta} \bar{y}(t)$.

On the other hand, $v(1) = \sum_{j=1}^{\infty} \zeta_j v(\xi_j)$, and combining with (16), we get $v(t) = C_1 t^{\beta-1} - I_{0+}^{\beta} \bar{y}(t)$.

$$\begin{aligned} C_1 &= \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\Delta} \bar{y}(s) ds - \sum_{j=1}^{\infty} \zeta_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\beta-1}}{\Gamma(\beta)\Delta} \bar{y}(s) ds \\ &= \int_0^1 \frac{(1-s)^{\beta-1} \bar{P}(s)}{\Delta} \bar{y}(s) ds, \end{aligned}$$

where $\bar{P}$ is as (13), $P(s)$ is as 14. Hence,

$$\begin{aligned} v(t) &= C_1 t^{\beta-1} - I_{0+}^{\beta} \bar{y}(t) \\ &= - \int_0^t \frac{\bar{\Delta}(t-s)^{\beta-1}}{\Gamma(\beta)\Delta} \bar{y}(s) ds + \int_0^1 \frac{(1-s)^{\beta-1} t^{\beta-1} \bar{P}(s)}{\Delta} \bar{y}(s) ds. \end{aligned}$$

Therefore, $H(t, s)$ is as 12

Lemma 7. Take $j, \ell \in (0, 1)$ with $j^i \geq \ell^{i-1}$, $ij^{i-1} > (\gamma-1)\ell^{\gamma-2}$, then $0 \leq G(t, s) \leq \frac{i}{\Gamma(\gamma-i)} g(s)$, $0 \leq \frac{\partial G(t, s)}{\partial t} \leq \frac{i}{\Gamma(\gamma-i)} g(s)$, $t, s \in [0, 1]$, $G(t, s) \geq h g(s)$, $\frac{\partial G(t, s)}{\partial t} \geq h g(s)$, $t \in [j, \ell]$, $s \in [0, 1]$, where $g(s) = (1-s)^{\gamma-i-1}/\Delta$, $h_1 = \Delta(j^i - \ell^{\gamma-1})/\Gamma(\gamma) < 1$, $h_2 = \Delta(ij^{i-1} - (\gamma-1) \times \ell^{\gamma-2})/\Gamma(\gamma) < 1$, $h = \min\{h_1, h_2\} < 1$.

Proof. By direct calculation we get $P'(s) \geq 0$, $s \in [0, 1]$, $P(s)$ and so is nondecreasing with respect to s . For $s \in [0, 1]$, $\gamma-1 > i$, we get $P(s) = \frac{1}{\Gamma(\gamma-i)} \sum_{k=0}^{\infty} \binom{\gamma-i}{k} (1-s)^{\gamma-i-k} \geq \frac{1}{\Gamma(\gamma-i)}$, $s \in [0, 1]$.

Hence, for $t, s \in [0, 1]$, $i > \mu$, we have $G(t, s) = \frac{1}{\Gamma(\gamma-i)} \sum_{k=0}^{\infty} \binom{\gamma-i}{k} \int_0^t (t-s)^{\gamma-i-k-1} (1-s)^{\gamma-i-k} ds$ furthermore, $0 \leq s \leq t \leq 1$, for we get $G(t, s) = \frac{1}{\Gamma(\gamma-i)} \sum_{k=0}^{\infty} \binom{\gamma-i}{k} \int_0^t (t-s)^{\gamma-i-k-1} (1-s)^{\gamma-i-k} ds$ and obviously, for $0 \leq t \leq s \leq 1$, we get $G(t, s) \geq 0$ on the other hand, for $0 \leq s \leq t \leq 1$, we have $G(t, s) = \frac{1}{\Gamma(\gamma-i)} \sum_{k=0}^{\infty} \binom{\gamma-i}{k} \int_0^t (t-s)^{\gamma-i-k-1} (1-s)^{\gamma-i-k} ds$.

$$\begin{aligned} &\geq \frac{(i(\frac{\Gamma(\gamma)}{\Gamma(\gamma-i)} - \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-1}) - (\gamma-1)(i! - \sum_{j=1}^{\infty} \eta_j \xi_j^i))(1-s)^{\gamma-i-1} t^{i-1}}{\Delta \Gamma(\gamma)} \\ &= \frac{(\frac{\Gamma(\gamma)i}{\Gamma(\gamma-i)} - (\gamma-1)i! + (\gamma-1) \sum_{j=1}^{\infty} \eta_j \xi_j^i - i \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-1})(1-s)^{\gamma-i-1} t^{i-1}}{\Delta \Gamma(\gamma)} \\ &= \frac{1}{\Delta \Gamma(\gamma)} \left((\gamma-1)i((\gamma-2)(\gamma-3) \cdots (\gamma-i) - (i-1)!) \right. \\ &\quad \left. + (\gamma-1) \sum_{j=1}^{\infty} \eta_j \xi_j^i - i \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-1} \right) (1-s)^{\gamma-i-1} t^{i-1} \geq 0, \end{aligned}$$

and $\partial G(t, s)/\partial t \geq 0$ obviously holds for $0 \leq t \leq s \leq 1$. For $t \in [j, \ell]$,

$$\begin{aligned}
 G(t, s) &= \frac{t^i \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1} - \Delta(t-s)^{\gamma-1}}{\Delta \Gamma(\gamma)} \\
 &\geq \frac{j^i \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1} - \Delta(\ell - \ell s)^{\gamma-1}}{\Delta \Gamma(\gamma)} \\
 &\geq \frac{\Delta(j^i (1-s)^{\gamma-i-1} - \ell^{\gamma-1} (1-s)^{\gamma-1})}{\Delta \Gamma(\gamma)} \\
 &\geq \frac{\Delta(j^i - \ell^{\gamma-1}) (1-s)^{\gamma-i-1}}{\Delta \Gamma(\gamma)} = h_1 g(s), \quad s \in [0, 1],
 \end{aligned}$$

and for $t \in [j, \ell]$, we have

$$\begin{aligned}
 \frac{\partial G(t, s)}{\partial t} &= \frac{it^{i-1} \Gamma(\gamma) P(s) (1-s)^{\gamma-i-1} - \Delta(\gamma-1)(t-s)^{\gamma-2}}{\Delta \Gamma(\gamma)} \\
 &\geq \frac{ij^{i-1} \Delta(1-s)^{\gamma-i-1} - \Delta(\gamma-1)(\ell - \ell s)^{\gamma-2}}{\Delta \Gamma(\gamma)} \\
 &\geq \frac{\Delta(ij^{i-1} - (\gamma-1)\ell^{\gamma-2}) (1-s)^{\gamma-i-1}}{\Delta \Gamma(\gamma)} = h_2 g(s), \quad s \in [0, 1].
 \end{aligned}$$

Therefore, the proof of Lemma 7 is completed.

Lemma 8. Let $\Delta > 0$, then the Green functions defined by (12) satisfies:

- i. $H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+^1$ is continuous and $H(t, s) > 0$ for all $t, s \in [0, 1]$
- ii. $t^{\beta-1} H(1, s) \leq H(t, s) \leq H(1, s)$ for all $t, s \in [0, 1]$ in which $(1/\Delta)(\bar{P}(s) - \Delta/\Gamma(\beta))(1-s)^{\beta-1}$.

Proof. The proof is similar to Lemma 3 of [20], we omit it here.

Now we define a cone K on $C^1[0; 1]$ and an operator $A : K^{C^1[0, 1]}$ as follows:

$$K = \left\{ u \in C^1[0, 1] : u(t) \geq 0, u'(t) \geq 0, t \in [0, 1], \min_{t \in [j, \ell]} u^{(j)}(t) \geq \sigma \|u\|, j = 0, 1 \right\},$$

where $0 < \sigma = h\Gamma(\gamma-i)/i < 1, 0 < h < h_1 = (j^i - \ell^{\gamma-1})/\Gamma(\gamma) < i/\Gamma(\gamma-i), j$ and

$$Au(t) = \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds.$$

Problems (1), (2) has a positive solution if and only if u is a fixed point of A in K .

3 Main results

Lemma 9. The operator is continuous.

Proof. First, by the integrability of f and (H0) we get

$$\begin{aligned}
 Au(t) &= \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \frac{i}{\Gamma(\gamma - i) \Delta} \int_0^1 (1 - s)^{\gamma - i - 1} \varphi_q \left(\int_0^1 \frac{1}{\Delta \Gamma(\beta)} (1 - \tau)^{\beta - 1} f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \frac{i}{\Gamma(\gamma - i) \Delta (\Delta \Gamma(\beta))^{q-1}} \int_0^1 (1 - s)^{\gamma - i - 1} \varphi_q \left(\int_0^1 \vartheta(\tau) d\tau \right) ds \\
 &= \frac{i \varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Gamma(\gamma - i) \Delta (\Delta \Gamma(\beta))^{q-1}} \int_0^1 (1 - s)^{\gamma - i - 1} ds \\
 &= \frac{i \varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Gamma(\gamma - i) \Delta (\Delta \Gamma(\beta))^{q-1}} \frac{(1 - s)^{\gamma - i}}{\gamma - i} < +\infty,
 \end{aligned}$$

so we have that A is well defined on K . Moreover, it follows from the uniform continuity of $G(t; s)$ on $[0; 1] \times [0; 1]$ and (H0)

$$\begin{aligned}
 |Au(t_2) - Au(t_1)| &\leq \int_0^1 |G(t_2, s) - G(t_1, s)| \varphi_q \left(\frac{1}{\Delta \Gamma(\beta)} \int_0^1 (1 - \tau)^{\beta - 1} f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \frac{\varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{(\Delta \Gamma(\beta))^{q-1}} \int_0^1 |G(t_2, s) - G(t_1, s)| ds.
 \end{aligned}$$

Thus, $Au \in C[0, 1]$, $u \in K$. we have that Furthermore, by the uniform continuity of $\partial G(t, s)/\partial t$, for $t, s \in [0, 1]$, we get

$$(Au)'(t) = \int_0^1 \frac{\partial G(t, s)}{\partial t} \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \in C[0, 1].$$

Let $u_n, u \in K$, $u_n \rightarrow u$ in $C^1[0, 1]$. Since $G(t, s), \partial G(t, s)/\partial t$ are uniformly continuous, there exists $M > 0$ such that

$$\max \left\{ G(t, s), \frac{\partial G(t, s)}{\partial t} \right\} \leq M, \quad t, s \in [0, 1].$$

On the other hand, since $u_n \rightarrow u$ in $C^1[0, 1]$, there exists $A > 0$ such that $\|u_n\| \leq A$ ($n = 1, 2, \dots$), and then $\|u\| \leq A$. Furthermore, by (H0) we have

$$\begin{aligned}
 \int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau &\leq \frac{1}{\Delta \Gamma(\beta)} \int_0^1 (1 - \tau)^{\beta - 1} f(\tau, u(\tau), u'(\tau)) d\tau \\
 &= \frac{1}{(\Delta \Gamma(\beta))^{q-1}} \int_0^1 (1 - \tau)^{\beta - 1} \vartheta(\tau) d\tau,
 \end{aligned}$$

so, we have

$$\begin{aligned} \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, x_0, x_1) d\tau \right) &\leq \frac{1}{(\Delta\Gamma(\beta))^{q-1}} \varphi_q \left(\int_0^1 (1-\tau)^{\beta-1} \vartheta(\tau) d\tau \right) \\ &= M_0, \end{aligned} \quad [17]$$

hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such, for any $s_1, s_2 \in [0, 1]$, $x_0^1, x_0^2, x_1^1, x_1^2 \in [0, A]$, $|s_1 - s_2| < \delta$, $|x_0^1 - x_0^2| < \delta$, $|x_1^1 - x_1^2| < \delta$, by (17) and Lebesgue controls convergence theorem, we have

$$\left| \varphi_q \left(\int_0^1 H(s_1, \tau) f(\tau, x_0^1, x_1^1) d\tau \right) - \varphi_q \left(\int_0^1 H(s_2, \tau) f(\tau, x_0^2, x_1^2) d\tau \right) \right| < \varepsilon. \quad [18]$$

By $\|u_n - u\| \rightarrow 0$ for the above $\delta > 0$, there exists N such that for all $n > N$, we get $|u_n(t) - u(t)|, |u'_n(t) - u'(t)| \leq \|u_n - u\| < \delta$. Hence, for any $t \in [0, 1]$, $n > N$, by (18) we obtain

$$\left| \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u_n(\tau), u'_n(\tau)) d\tau \right) - \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) \right| < \varepsilon. \quad [19]$$

Thus, for $n > N, t \in [0, 1]$, by (19) we have

$$\begin{aligned}
 & |(Au_n)(t) - (Au)(t)| \\
 &= \left| \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u_n(\tau), u'_n(\tau)) d\tau \right) ds \right. \\
 &\quad \left. - \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right| \\
 &= \left| \int_0^1 G(t,s) \left(\varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u_n(\tau), u'_n(\tau)) d\tau \right) \right. \right. \\
 &\quad \left. \left. - \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) \right) ds \right| \\
 &\leq M \int_0^1 \varepsilon ds \leq M\varepsilon
 \end{aligned}$$

and

$$\begin{aligned}
 & |(Au_n)'(t) - (Au)'(t)| \\
 &= \left| \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u_n(\tau), u'_n(\tau)) d\tau \right) ds \right. \\
 &\quad \left. - \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right| \\
 &= \left| \int_0^1 \frac{\partial G(t,s)}{\partial t} \left(\varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u_n(\tau), u'_n(\tau)) d\tau \right) \right. \right. \\
 &\quad \left. \left. - \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) \right) ds \right| \\
 &\leq M\varepsilon,
 \end{aligned}$$

and hence, we get $\|Au_n - Au\|_0 \rightarrow 0$, $\|(Au_n)' - (Au)'\|_0 \rightarrow 0$ ($n \rightarrow \infty$). That is, $\|Au_n - Au\| \rightarrow 0$ ($n \rightarrow \infty$), namely, A is continuous in the space E .

Lemma 10. $A: P \rightarrow P$ is completely continuous.

Proof. From Lemma 7 we have $(Au)(t), (Au)'(t) \geq 0, t \in [0,1]$, y

$$\begin{aligned}
 \max_{t \in [0,1]} (Au)(t) &= \max_{t \in [0,1]} \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \int_0^1 \max_{t \in [0,1]} G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds, \\
 \max_{t \in [0,1]} (Au)'(t) &= \max_{t \in [0,1]} \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \int_0^1 \max_{t \in [0,1]} \frac{\partial G(t,s)}{\partial t} \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds,
 \end{aligned}$$

thus, $\|Au\|_0, \|(Au)'\|_0 \leq \int_0^1 (i/\Gamma(\gamma-i)) g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds$,
consequently,

$$\begin{aligned}
 \|Au\| &= \max \{ \|Au\|_0, \|(Au)'\|_0 \} \\
 &\leq \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds.
 \end{aligned}$$

On the other hand, for all $|\mu| \in P, t \in [j, \ell]$, by Lemma 7 we have

$$\begin{aligned}
 (Au)(t) &= \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\geq \int_0^1 h_1 g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\geq \frac{h_1}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\geq \frac{h_1}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds,
 \end{aligned}$$

Hence

$$\begin{aligned} & \min\{Au(t), (Au)'(t)\} \\ & \geq \frac{\min\{h_1, h_2\}}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 \frac{i}{\Gamma(\alpha-i)} g(s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ & \geq \frac{h\Gamma(\alpha-i)}{i} \|Au\| = \sigma \|Au\|. \end{aligned}$$

Thus, $A(P) \subset P$.

Now we will prove that AV is relatively compact for bounded $V \subset K$. Since V is bounded, there exists $D > 0$ such that for any $u \in V$, $\|u\| \leq D$. For $t \in [0, 1]$, $u \in V$, we have

$$\begin{aligned} |Au(t)| &= \left| \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right| \\ &\leq \frac{i}{\Gamma(\gamma-i)\Delta} \int_0^1 (1-s)^{\gamma-i-1} \varphi_q \left(\int_0^1 \frac{1}{\Delta\Gamma(\beta)} (1-\tau)^{\beta-1} f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ &\leq \frac{i}{\Gamma(\gamma-i)\Delta(\Delta\Gamma(\beta))^{q-1}} \int_0^1 (1-s)^{\gamma-i-1} \varphi_q \left(\int_0^1 \vartheta(\tau) d\tau \right) ds \\ &= \frac{i\varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Gamma(\gamma-i)\Delta(\Delta\Gamma(\beta))^{q-1}} \frac{(1-s)^{\gamma-i}}{\gamma-i} < +\infty. \end{aligned}$$

Similarly, for $t \in [0, 1]$, $u \in V$, we derive

$$|(Au)'(t)| \leq \frac{i\varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Gamma(\gamma-i)\Delta(\Delta\Gamma(\beta))^{q-1}} \frac{(1-s)^{\gamma-i}}{\gamma-i},$$

which shows that AV is bounded. Next, we will verify that $(AV)'$ is equicontinuous. Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, $u \in V$, we get

$$\begin{aligned}
 & |(Au)'(t_2) - (Au)'(t_1)| \\
 & \leq |(t_2^{i-1} - t_1^{i-1})| \\
 & \quad \times \int_0^1 \frac{i\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}}{\Delta\Gamma(\gamma)} \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 & \quad + \left| \frac{1}{\Gamma(\gamma-1)} \int_0^{t_2} (t_2-s)^{\gamma-2} \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right. \\
 & \quad \left. - \frac{1}{\Gamma(\gamma-1)} \int_0^{t_1} (t_1-s)^{\gamma-2} \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right| \\
 & \leq |(t_2^{i-1} - t_1^{i-1})| \\
 & \quad \times \frac{i\Gamma(\gamma)}{\Delta\Gamma(\gamma)} \int_0^1 \frac{(1-s)^{\gamma-i-1}}{\Gamma(\gamma-i)} \varphi_q \left(\int_0^1 \frac{1}{\Delta\Gamma(\beta)} (1-\tau)^{\beta-1} f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 & \quad + \frac{M_0}{\Gamma(\gamma-1)} \left[\int_0^{t_2} (t_2-s)^{\gamma-2} ds - \int_0^{t_1} (t_1-s)^{\gamma-2} ds \right] \\
 & \leq \frac{i\varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Delta\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} |(t_2^{i-1} - t_1^{i-1})| + \frac{M_0}{\Gamma(\gamma-1)} \int_0^1 (1-s)^{\gamma-2} ds \\
 & \quad \times |t_2^{\gamma-1} - t_1^{\gamma-1}|.
 \end{aligned}$$

Furthermore,

$$\int_0^t (t-s)^{\gamma-2} ds = t^{\gamma-1} \int_0^1 (1-s)^{\gamma-2} ds.$$

Thus, we obtain

$$\begin{aligned}
 |(Au)'(t_2) - (Au)'(t_1)| & \leq \frac{i\varphi_q(\int_0^1 \vartheta(\tau) d\tau)}{\Delta\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} |(t_2^{i-1} - t_1^{i-1})| \\
 & \quad + \frac{C}{\Gamma(\gamma-1)} \int_0^1 (1-s)^{\gamma-2} ds \cdot |t_2^{\gamma-1} - t_1^{\gamma-1}|, \quad u \in V.
 \end{aligned}$$

From above and the uniform continuity of t^{i-1} , $t^{\alpha-2}$, and together with Lemma 3, we can derive that AV is relatively compact in $C^1[0, 1]$, $C^1[0; 1]$, and so we get that $A : K \rightarrow K$ is completely continuous.

Let φ, θ be nonnegative continuous convex functionals on P , ϕ be a nonnegative continuous concave functional on P , and ψ be a nonnegative continuous functional on P . Then for nonnegative numbers e, c, d, h , we define the following convex sets:

$$\begin{aligned} P(\varphi, h) &= \{x \in P \mid \varphi(x) < h\}, \\ P(\varphi, \phi, c, h) &= \{x \in P \mid \phi(x) \geq c, \varphi(x) \leq h\}, \\ P(\varphi, \theta, \phi, c, d, h) &= \{x \in P \mid c \leq \phi(x), \theta(x) \leq d, \varphi(x) \leq h\}, \\ R(\varphi, \psi, e, h) &= \{x \in P \mid e \leq \psi(x), \varphi(x) \leq h\}. \end{aligned}$$

We will apply the following fixed point theorem of Avery and Peterson to solve problem (1), (2).

Let the convex functions $\psi(u) = \theta(u) = \Phi(u) = \|u\|$ on P , and define a concave function where j, ℓ are the same as in Lemma 7.

Theorem 1: Assume that there exist positive numbers e, c, d, h with $c > e, d > \max\{1/h, e^{1-j/2}\}c, h > rc/(hQ)$, and $h \geq d$ such that

- (H1) $\phi_q(\int_0^1 f(t, x, y) dt) < h/r$ for $(t, x, y) \in [0, 1] \times [0, h]^2$;
 (H2) $\phi_q(\int_0^1 H(1, t)f(t, x, y)) ds \geq c/(hQ)$ for $(t, x, y) \in [j, \ell] \times [c, d]^2$;
 (H3) $\phi_q(\int_0^1 f(t, x, y) dt) < e/r$ for $(t, x, y) \in [0, 1] \times [0, e]^2$, where $r = i \int_0^1 g(s) ds / (\Gamma(\gamma - i)(\Delta\Gamma(\beta))^{q-1})$, $Q = \int_j^\ell s^{(q-1)(\beta-1)} g(s) ds$.

Then problem (1), (2) has at least three fixed points u_1, u_2, u_3 satisfying

$$\|u_i\| \leq h, \quad i = 1, 2, 3, \quad [20]$$

and

$$c < \min \left\{ \min_{t \in [j, \ell]} |u_1(t)|, \min_{t \in [j, \ell]} |u'_1(t)| \right\}, \quad e < \|u_2\|, \quad [21]$$

$$\min \left\{ \min_{t \in [j, \ell]} |u_2(t)|, \min_{t \in [j, \ell]} |u'_2(t)| \right\} < c, \quad \|u_3\| < e. \quad [22]$$

Proof. Let $u \in P(\varphi, h)$. By condition H1 we get

$$\begin{aligned} \|Au\|_0 &= \max_{t \in [0, 1]} |Au(t)| \\ &\leq \frac{i}{\Gamma(\gamma - i)} \int_0^1 g(s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ &= \frac{i}{\Gamma(\gamma - i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) ds \cdot \frac{h}{r} \leq h, \end{aligned}$$

$$\begin{aligned}
 \|(Au)'\|_0 &= \max_{t \in [0,1]} |(Au)'(t)| \\
 &= \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &\leq \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) \varphi_q \left(\int_0^1 f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &= \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) ds \cdot \frac{h}{r} \leq h.
 \end{aligned}$$

Consequently, we obtain $\phi(Au) = \|Au\| \leq h$. This together with Lemmas 9 and 10, means that: $A : \overline{P(\Phi, h)} \rightarrow \overline{P(\Phi, h)}$ is completely continuous.

Take $u(t) = ce^{t-0.5j}$, $t \in [0, 1]$. By simple calculation we have $u \in P$, $\|u\| < d$, and $\phi(u) > c$, and so $\{u \in P(\Phi, \theta, \phi, c, d, h) : c < \phi(u)\} \neq \emptyset$. For $u \in P(\Phi, \theta, \phi, c, d, h)$, by (H2)

$$\begin{aligned}
 \phi(Au) &= \min \left\{ \min_{t \in [j, \ell]} |Au(t)|, \min_{t \in [j, \ell]} |(Au)'(t)| \right\} \\
 &\geq h \int_0^1 g(s) s^{(q-1)(\beta-1)} \varphi_q \left(\int_0^1 H(1, \tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\
 &> h \int_j^\ell g(s) s^{(q-1)(\beta-1)} ds \frac{c}{hQ} = c,
 \end{aligned}$$

which shows that condition (S1) is satisfied.

Take $u \in P(\Phi, \phi, c, h)$ and $\|Au\| > d$. Since $Au \in P$, we obtain

$$\phi(Au) = \min \left\{ \min_{t \in [j, \ell]} |Au(t)|, \min_{t \in [j, \ell]} |(Au)'(t)| \right\} \geq \sigma \|Au\| \geq \sigma d > c,$$

which implies that condition (S2) holds.

Next, we will verify that condition (S3) holds. For $\psi(0) = 0$ we have $0 \in R(\Phi, \psi, e, h)$. Let $u \in R(\Phi, \psi, e, h)$ and $\psi(u) = \|u\| = e$, by (H3) we get

$$\begin{aligned}
 \|Au\|_0 &= \max_{t \in [0,1]} |Au(t)| \\
 &\leq \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau \right) ds
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) \, ds \cdot \varphi_q \left(\int_0^1 f(\tau, u(\tau), u'(\tau)) \, d\tau \right) \\ &< \frac{e}{r} \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) \, ds \leq e \end{aligned}$$

and

$$\begin{aligned} \|(Au)'\|_0 &= \max_{t \in [0,1]} |(Au)'(t)| \\ &\leq \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, u(\tau), u'(\tau)) \, d\tau \right) \, ds \\ &= \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) \, ds \cdot \varphi_q \left(\int_0^1 f(\tau, u(\tau), u'(\tau)) \, d\tau \right) \\ &< \frac{e}{r} \frac{i}{\Gamma(\gamma-i)(\Delta\Gamma(\beta))^{q-1}} \int_0^1 g(s) \, ds \leq e. \end{aligned}$$

Consequently, we have $\psi(Au) = \|Au\| < e$. Thus, condition (S3) holds. By Lemma 4 we get that (1), (2) has at least three positive solutions u_1, u_2, u_3 satisfying (20)–(22).

4 An example

Consider the following infinite-point p-Laplacian fractional differential equations:

$$f(t, x, y) = \begin{cases} \frac{1}{4\pi t^{1/2}(1-t)^{1/2}}(x^2 + y^2)^2, & (t, x, y) \in (0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\ \frac{499^2}{4\pi t^{1/2}(1-t)^{1/2}}(\sqrt[6]{x} + \sqrt[6]{y})^2, & (t, x, y) \in (0, 1] \times [1, 20] \times [1, 20], \\ \frac{499^2}{4\pi t^{1/2}(1-t)^{1/2}}, & (t, x, y) \in (0, 1] \times [100, \infty) \times [100, \infty). \end{cases}$$

where

$$\begin{aligned} D_{0+}^{3/2}(\varphi_3({}^c D_{0+}^{7/2}u))(t) + f(t, u(t), u'(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'''(0) &= 0, \quad u''(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \\ {}^c D_{0+}^{7/2}u(0) &= 0, \quad \varphi_3({}^c D_{0+}^{7/2}u(1)) = \sum_{j=1}^{\infty} \zeta_j \varphi_3({}^c D_{0+}^{7/2}u(\xi_j)), \end{aligned}$$

Clearly

$$f(t, x, y) \leq \frac{499^3}{4\pi t^{1/2}(1-t)^{1/2}} = \vartheta(t)$$

and

$$\int_0^1 \vartheta(t) dt = \frac{499^3}{4\pi} \int_0^1 t^{-1/2}(1-t)^{-1/2} dt = \frac{499^3}{4\pi} B\left(\frac{1}{2}, \frac{1}{2}\right),$$

so $\vartheta(t)$ is integrable, condition (H0) holds.

We take $\zeta = 3/15$, $\ell = 4/15$, $\zeta_j = 2/(3j^{7/2})$, $\eta_j = 1/(2j^2)$, $\xi_j = 1/j$, by simple calculation we have

$$\Delta = i! - \sum_{j=1}^{\infty} \eta_j \xi_j^i \approx 1.4589,$$

$$\overline{\Delta} = 1 - \sum_{i=1}^{\infty} \zeta_i \xi_i^{\beta-1} = 1 - \frac{2}{3} \sum_{j=1}^{\infty} \frac{1}{j^{7/2} j^{1/2}} \approx 0.7215,$$

$$g(s) = \frac{(1-s)^{\alpha-i-1}}{\Delta} = \frac{1}{1.4589} (1-s)^{1/2}.$$

Hence, we have

$$\begin{aligned}
 r &= \frac{i}{\Gamma(\gamma - i)(\overline{\Delta}\Gamma(\beta))^{q-1}} \int_0^1 g(s) \, ds \\
 &= \frac{2}{\Gamma(\frac{7}{2} - 2)(0.7215\Gamma(\frac{3}{2}))^{1/2}} \int_0^1 \frac{1}{1.4589} (1-s)^{1/2} \, ds \\
 &= \frac{8}{3 \times 1.4589 \times (0.7215 \times 0.5)^{1/2}} \frac{1}{\pi^{3/4}} \approx \frac{4.3039}{\pi^{3/4}}, \\
 L &= \int_j^\ell g(s) \, ds = \frac{1}{1.4589} \frac{2}{3} \left(\left(\frac{4}{5}\right)^{3/2} - \left(\frac{11}{15}\right)^{3/2} \right), \\
 Q &= \int_j^\ell s^{(q-1)(\beta-1)} g(s) \, ds = \int_{3/15}^{4/15} s^{1/2 \cdot 1/2} \frac{1}{1.4589} (1-s)^{1/2} \, ds \\
 &= 0.6854 \int_{3/15}^{4/15} s^{1/4} (1-s)^{1/2} \, ds = 0.0278, \\
 h_1 &= \Delta \frac{j^i - \ell^{\alpha-1}}{\Gamma(\gamma)} \approx 0.1938 \frac{1}{\sqrt{\pi}} \approx 0.1093 < 1, \\
 h_2 &= \frac{\Delta(ij^{i-1} - (\gamma-1)\ell^{\alpha-2})}{\Gamma(\gamma)} \approx 0.4336 \frac{1}{\sqrt{\pi}} \approx 0.2446 < 1,
 \end{aligned}$$

and as $h_1 < h_2 < 1$, $h = h_1$. Let $e = 1/2$, $c = 1$, $d = 20$, $h = 400$.
Then for $(t, x, y) \in [0, 1] \times [0, 400]^2$, we for

$$\begin{aligned}
 \varphi_q \left(\int_0^1 f(t, x, y) \, dt \right) &= \varphi_{3/2} \left(\int_0^1 \frac{499^2}{4\pi t^{1/2} (1-t)^{1/2}} \, dt \right) = \sqrt{\frac{499^2}{4\pi} B\left(\frac{1}{2}, \frac{1}{2}\right)} \\
 &= 124.75 < \frac{h}{r} \approx 219.23,
 \end{aligned}$$

so condition (H1) of Thorem 1 hold. For $(t, x, y) \in [0, 1] \times [1, 20] \times [1, 20]$, by MATLAB software we have

$$\begin{aligned}
 &\varphi_q \left(\int_0^1 H(1, t) f(t, x, y) \, dt \right) \\
 &\geq \varphi_{3/2} \left(\int_0^1 \frac{1}{\overline{\Delta}} \left(\overline{P}(t) - \frac{\overline{\Delta}}{\Gamma(\beta)} \right) (1-t)^{\beta-1} \frac{499^2}{\pi t^{1/2} (1-t)^{1/2}} \, dt \right) \approx 345.56 \\
 &\geq \frac{1}{0.1093 \times 0.6854 \int_{3/15}^{4/15} s^{1/4} (1-s)^{1/2} \, ds} = \frac{c}{hQ} \approx 333.33,
 \end{aligned}$$

therefore, condition (H2) of Theorem 1 hold. By the same method with proofing (H1) we get (H3) hold, so all the conditions of Theorem 1 hold. Hence, the BVP (23) has at least three positive solutions u_1, u_2, u_3 satisfying $u_i \leq 600, i = 1, 2, 3$, and $1 < \phi(u_1), u_i \leq 600, i = 1, 2, 3$, and $1 < \phi(u_1)$.

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