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Infinitely many solutions for a gauged nonlinear Schrödinger equation with a perturbation*

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Abstract: In this paper, we use the Fountain theorem under the Cerami condition to study the gauged nonlinear Schrödinger equation with a perturbation in R.. Under some appropriate conditions, we obtain the existence of infinitely many high energy solutions for the equation.

Keywords: gauged Schrödinger equation, infinitely many solutions, Fountain theorem.

1 Introduction

In this paper, we study the existence of infinitely many high energy solutions for the following gauged nonlinear Schrödinger equation with a perturbation in R2:

$$-\Delta u + \lambda V(x)u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\,\mathrm{d}s\right)u$$
$$= f(u) - \mu g(x)|u|^{q-2}u. \tag{1}$$

We first list our assumptions for our problem (1):

(V1) V ∈ C(R2, R), and infx∈R2 V (x) ≥ V0 > 0, where V0 is a positive constant.

(V2) There exists b > 0 such that meas {x # R2: V (x) ≤ b} is finite; here meas denotes the Lebesgue measure.

(H1) f ∈ C(R, R), and f (u) = o(|u|) as |u| → 0.

(H2) There exists R0 ≥ 0 such that F (u) = ∫u0 f (t) dt ≥ 0 and F(u) = f (u)u/6 – u F (u) ≥ 0 for |u| ≥ R0.

(H3) f (u)u/ |u|6 → +∞ as |u| → ∞.

(H4) There exist $\alpha 0$, R1 > 0, and $\tau \in (1, +\infty)$ such that $|f(u)|\tau \leq \alpha 0F(u)|u|\tau$ for $|u| \geq R1$.

(H5) $f(-u) = -f(u)$ for $u \in R$.

(g) $g \in Lq'(R2)$, and $g(x) \geq 0$ ($/\equiv 0$) for $x \in R2$, where $q' \in (1, 2/(2-q))$, $q \in (1, 2)$.

Problem (1) arises in the study of standing wave solutions for the gauged nonlinear Schrödinger equation

$$
\begin{aligned}
&\mathrm{i}D_0\phi + (D_1D_1 + D_2D_2)\phi + g(\phi) = 0,\\
&\partial_0 A_1 - \partial_1 A_0 = -\operatorname{Im}(\overline{\phi}D_2\phi),\\
&\partial_0 A_2 - \partial_2 A_0 = \operatorname{Im}(\overline{\phi}D_1\phi),\\
&\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2,
\end{aligned}
$$

where i denotes the imaginary unit, $\partial 0 = \partial/\partial t$, $\partial 1 = \partial/\partial x1$, $\partial 2 = \partial/\partial x2$ for (t, x1, x2) R1+2, ϕ : R1+2 → C is the complex scalar field, $A\kappa$: R1+2 → R is the gauge field, and $D\kappa = \partial\kappa + iA\kappa$ is the covariant derivative for κ = 0, 1, 2. From the initial study in [8, 9] many papers on this system appeared in the literature; we refer the reader to [1, 2, 4,5,6,7,8, 10, 11, 13, 14, 18,19,20,21, 25, 26, 28, 29] and the references therein.

When $\lambda = 1$, the authors [12] obtained the existence and multiplicity of solutions for (1) with concave-convex nonlinearities $\mu g(x, u) + \nu f(x, u)$, where g has sublinear growth, and f has asymptotically linear or superlinear growth. In [20], the authors studied the existence, nonexistence, and multiplicity of standing waves for (1) ($\lambda = 1$, $\mu = 0$) with asymptotically linear nonlinearities and external potential, and in [1, 2, 4,5,6,7, 11, 13, 14, 18, 19, 21, 25, 26, 28, 29], the authors studied the existence and multiplicity of solutions (including sign-changing solutions and ground state solutions) for gauged nonlinear Schrödinger equation

$$
-\Delta u + \omega u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\,\mathrm{d}s \right) u = f(u), \quad x \in \mathbb{R}^2.
$$

Moreover, in [26], the authors also discussed the energy doubling property, i.e., the energy of sign-changing solutions is strictly larger than two times the least energy. In [10], the authors studied the existence and multiplicity of the positive standing wave with f (u) + #k(x), where the nonlinearity f behaves like exp(α ∣ u ∣ 2) as ∣ u ∣ →∞. Moreover, they obtained a mountain-pass type solution when # = 0.

There also are some papers in the literature, which consider perturbation terms; see [15, 17, 22, 23, 27] and the references therein. In [15, 17], the authors used the famous Ambrosetti–Rabinowitz conditions to study the existence of solutions for the following fractional equations:

$$M\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\right)(-\Delta)_p^s u + V(x)|u|^{p-2}u$$
$$= f(x,u) + g(x) \quad \text{in } \mathbb{R}^N,$$

and

$$(I-\Delta)^s u + \lambda V(x) u = f(x,u) + \mu\xi(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where (-Δ)s is the fractional p-Laplacian operator, and (I Δ)s is the fractional Bessel operator. Moreover, [17] also considered the effect of the parameter λ, μ on the existence of solutions for their problem.

Motivated by the aforementioned works, in this paper, we study the existence of infinitely many high energy solutions under some appropriate conditions, which are weaker than the Ambrosetti–Rabinowitz conditions, and also consider the effect of the parameters and the perturbation terms on the existence of solutions.

Now, we state our main result:

Theorem 1. Suppose that (V1), (V2), (H1)–(H5), and (g) hold. Then for any μ > 0, there exists Λ > 0 such that system (1) possesses infinitely many high energy solutions when λ ≥ Λ.

Remark 1. By virtue of (H1), (H2), and (H4) we can obtain a growth condition for f . Using (H2) and (H4), for |u| ≥ R2 := max{R0, R1}, we have

$$\left|f(u)\right|^{\tau} \leqslant \alpha_0 \mathbf{F}(u)|u|^{\tau} = \alpha_0\left(\frac{1}{6}f(u)u - F(u)\right)|u|^{\tau} \leqslant \frac{\alpha_0}{6}|f(u)||u|^{\tau+1},$$

and

$$\left|f(u)\right| \leqslant \sqrt[\tau-1]{\frac{\alpha_0}{6}}|u|^{\frac{\tau+1}{\tau-1}}.$$

Let p = (τ + 1)/(τ − 1) + 1 = 2τ/(τ − 1). Then from (H4) we have p ∈ (2, +∞), and

$$\left|f(u)\right| \leqslant \sqrt[\tau-1]{\frac{\alpha_0}{6}}|u|^{p-1} \quad \text{for } |u| \geqslant R_2.$$

On the other hand, using (H1), for all ε > 0, we have
.f (u). ≤ ε|u| for |u| ≤ R2.

Therefore, by the above two inequalities we have the growth condition for f :

$$\left|f(u)\right| \leqslant \varepsilon|u| + c_\varepsilon|u|^{p-1}, \quad u \in \mathbb{R},\ c_\varepsilon := \sqrt[\tau-1]{\frac{\alpha_0}{6}}. \tag{2}$$

Note the relation F and f , and we obtain

$$|F(u)| \leqslant \frac{\varepsilon}{2}|u|^2 + \frac{c_\varepsilon}{p}|u|^p, \quad u \in \mathbb{R}. \tag{3}$$

Remark 2. Let f (t) = t5(6 log |t| + 1), t ∈ R, and t 0. Then F (t) = t6 ln |t|, and we can check that f , F satisfy (H1)–(H5). For example, if we take τ ∈ (1, 3/2), we have

$$\lim_{|t|\to+\infty} \frac{6\ln|t|+1}{|t|^{\frac{6-4\tau}{\tau}}} = \lim_{|t|\to+\infty} \frac{6\tau}{6-4\tau}\frac{1}{|t|^{\frac{6-4\tau}{\tau}}} = 0.$$

Consequently, for |t| large, we obtain

$$\frac{(6\ln|t|+1)^\tau}{|t|^{6-4\tau}} \leqslant \frac{\alpha_0}{6}$$

and

$$|f(t)|^\tau = |t^5(6\ln|t|+1)|^\tau \leqslant \frac{\alpha_0}{6}|t|^{6+\tau} = \alpha_0 \mathrm{F}(t)|t|^\tau.$$

This implies that (H4) holds. Moreover, this function also satisfies (H1)–(H3) and (H5). However, this function does not satisfy the Ambrosetti–Rabinowitz condition, namely:

(AR) There exists μ > 6 such that 0 < μF (u) ≤ f (u)u for u ∈ R \ {0}.

2 Preliminaries

Note the parameter λ, and we can consider the work space

$$E := \left\{ u \in H^1(\mathbb{R}^2) \colon \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x < +\infty \right\}.$$

Then E is a Hilbert space with the inner product and norm

$$E := \left\{ u \in H^1(\mathbb{R}^2) \colon \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \lambda V(x)u^2\right) \mathrm{d}x < +\infty \right\}.$$

Moreover, by [24] we have that the embedding E ‹ → Lr(R2) is continuous for r # [2, + ∞) and E ‹ → Lr(R2) is compact for r # (2, + ∞), i.e., there are constants γr > 0 such that #u# r ≤ γr #u# for 2 ≤ r < ∞, where #.# r is the norm in the usual Lebesgue space Lr(R2).

In what follows, we present the energy functional I: E → R for problem (1) defined as

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \lambda V(x) u^2\right) \mathrm{d}x + B(u) - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x + \frac{\mu}{q} \int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x,$$

Where

$$B(u) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{r}{2} u^2(r)\,\mathrm{d}r \right)^2 \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_{B_{|x|}} \frac{u^2}{4\pi} \right)^2 \mathrm{d}x.$$

Note (3) and (g). We obtain that I is of class C1 and its derivative is

$$\langle \mathcal{I}'(u), \varphi \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + \lambda V(x) u \varphi)\,\mathrm{d}x + \langle B'(u), \varphi \rangle - \int_{\mathbb{R}^2} f(u)\varphi\,\mathrm{d}x$$
$$+ \mu \int_{\mathbb{R}^2} g(x)|u|^{q-2} u \varphi\,\mathrm{d}x \quad \forall \varphi \in E,$$

where

$$\langle B'(u), \varphi \rangle = \int_{\mathbb{R}^2} \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\,\mathrm{d}s \right) u(x)\varphi(x)\,\mathrm{d}x \quad \forall \varphi \in E.$$

Lemma 1. (See [1, 13, 14, 29].) Suppose that {un} converges weakly to a function u in E as n → ∞. Then

(i) limn→+∞ B(un) = B(u),

(ii) limn→+∞#B′ (un), un# = #B′ (u), u#,

(iii) limn→+∞#B′ (un), φ# = #B′ (u), φ#,

(iv) #B′ (u), u# = 6B(u),

(v) B(u) ≤ C0#u#4#u#2 ≤ C0γ2γ4#u#6 := C1#u#6 for some C0, C1 > 0.

In order to obtain our main result, we need to introduce the Fountain theorem under the Cerami condition (C).

Definition 1. (See [16].) Assume that X is a Banach space. We say that J satisfies the Cerami condition if

(C) J ∈ C1(X, R), and for all c ∈ R,

(i) any bounded sequence {un} C X satisfying J(un) → c, J′ (un) → 0 possesses a convergent subsequence;

(ii) there exist σ, R, β > 0 such that for any u ∈ J−1([c − σ, c + σ]) with #u# ≥ R, #J′ (u)##u# ≥ β.

Lemma 2. (See [16].) Assume that $X = \overline{\bigoplus_{j=1}^{\infty} X_j}$, where Xj are finite dimensional

subspaces of X. For each k # N, let $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Suppose that J ∈ C1(X, R) satisfies the Cerami condition (C) and J(−u) = J(u). Assume for each k ∈ N, there exist ρk > rk > 0 such that

(i) bk = infu∈Zk ∩Srk J(u) → +∞, k → ∞,

(ii) ak = maxu∈Yk ∩S ρ k J(u) ≤ 0, where Sρ = {u ∈ X: #u# = ρ}.

Then J has a sequence of critical points un such that J(un) → +∞ as n → ∞.

3 Proof of Theorem 1

Lemma 3. Let sequence {un} converge weakly to a function u in E, un(x) → u(x) a.e. in R2 as n → ∞. Then

$$\mathcal{I}(u_n) = \mathcal{I}(u_n - u) + \mathcal{I}(u) + o(1) \quad as\ n \to \infty, \tag{4}$$

$$\langle \mathcal{I}'(u_n), \varphi\rangle = \langle \mathcal{I}'(u_n - u), \varphi\rangle + \langle \mathcal{I}'(u), \varphi\rangle + o(1) \quad \forall \varphi \in E\ as\ n \to \infty. \tag{5}$$

In particular, if

$$\mathcal{I}(u_n) \to c, \quad \mathcal{I}'(u_n) \to 0 \quad as\ n \to \infty,$$

then

$$\mathcal{I}(u_n - u) = c - \mathcal{I}(u) + o(1) \quad as\ n \to \infty, \tag{6}$$

and

$$\langle \mathcal{I}'(u_n - u), \varphi\rangle = o(1) \quad \forall \varphi \in E\ \ as\ n \to \infty. \tag{7}$$

Proof. From the compactness of E <→ Lr(R2), for r ∈ (2, +∞), we have

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } E, \\ u_n &\to u && \text{strongly in } L^p(\mathbb{R}^2) \text{ for } p \in (2, +\infty), \\ u_n &\to u && \text{for a.e. } x \in \mathbb{R}^2. \end{aligned}$$

Let wn = un − u. Then we have

$$\begin{aligned} w_n &\rightharpoonup 0 \quad \text{weakly in } E, \\ w_n &\to 0 \quad \text{strongly in } L^p(\mathbb{R}^2) \text{ for } p \in (2, +\infty), \\ w_n &\to 0 \quad \text{for a.e. } x \in \mathbb{R}^2. \end{aligned}$$

Since un u in E, we have (un − u, u) → 0 as n → ∞, which implies #un#2 = (wn + u, wn + u) = #wn#2 + #u#2 + o(1) as n → ∞.

Note Lemma 1(v), and we have B(un − u) ≤ C0#un − u#4#un − u#2 → 0 as n → ∞.

Consequently, to obtain (4), by Lemma 1(i) we only need to check that

$$\int_{\mathbb{R}^2} \big(F(u_n) - F(w_n) - F(u)\big)\,\mathrm{d}x = o(1) \quad \text{as } n \to \infty \tag{8}$$

And

$$\int_{\mathbb{R}^2} g(x)\big(|u_n|^q - |w_n|^q - |u|^q\big)\,\mathrm{d}x = o(1) \quad \text{as } n \to \infty. \tag{9}$$

Note the definition of (,), for all n # N, we have (un, φ) = (un - u, φ) + (u, φ). Moreover, since wn 0 in E and by Lemma 1(iii), to prove (5), it suffices to show that

$$\sup_{\|\varphi\|=1} \int_{\mathbb{R}^2} \big(f(u_n) - f(w_n) - f(u)\big)\varphi\,\mathrm{d}x = o(1) \quad \text{as } n \to \infty \tag{10}$$

And

$$\sup_{\|\varphi\|=1} \int_{\mathbb{R}^2} g(x)\big(|u_n|^{q-2}u_n - |u_n - u|^{q-2}(u_n - u) - |u|^{q-2}u\big)\varphi\,\mathrm{d}x = o(1) \quad \text{as } n \to \infty. \tag{11}$$

We first prove that (9) and (11). Using the inequality from page 13 in [17] and the Hölder inequality, for qq′ /(q′ − 1) > 2, we have

$$\left|\int_{\mathbb{R}^2} g(x)\big(|u_n|^q - |u|^q\big)\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}^2} g(x)|w_n|^q\,\mathrm{d}x \leqslant \|g\|_{q'}\|w_n\|^q_{\frac{qq'}{q'-1}} \to 0 \quad \text{as } n\to\infty.$$

Hence, (9) holds. From Lemma 1 in [3] there exists Cq > 0 such that #un | q−2un - | u q−2u# ≤ Cq | un - u | q−1. Therefore, from (g) and the Hölder inequality we only need to prove:

$$\begin{aligned}
&\sup_{\|\varphi\|=1}\left|\int_{\mathbb{R}^2} g(x)|w_n|^{q-2}w_n\varphi\,\mathrm{d}x\right| \\
&\leqslant \sup_{\|\varphi\|=1}\int_{\mathbb{R}^2} g(x)|w_n|^{q-1}|\varphi|\,\mathrm{d}x \\
&\leqslant \|g\|_{q'}\left(\int_{\mathbb{R}^2}|\varphi|^{\frac{qq'}{q'-1}}\,\mathrm{d}x\right)^{\frac{q'-1}{qq'}}\left(\int_{\mathbb{R}^2}|w_n|^{\frac{qq'}{q'-1}}\,\mathrm{d}x\right)^{\frac{q-1}{q'-1}qq'} \\
&\leqslant \|g\|_{q'}\gamma_{\frac{qq'}{q'-1}}\|w_n\|^{q-1}_{\frac{qq'}{q'-1}} \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Consequently, (11) is true. Note that we can use similar methods in Lemma 4.7 of [30] to prove (10). In what follows, we prove (8). Using the ideas in [17, 22, 23], we have

$$F(u_n)-F(u_n-u) = -\int_0^1\left(\frac{\mathrm{d}}{\mathrm{d}t}F(u_n-tu)\right)\mathrm{d}t = \int_0^1 f(u_n-tu)u\,\mathrm{d}t.$$

Hence, from (2) we obtain

$$\big|F(u_n)-F(u_n-u)\big| \leqslant \varepsilon_1|u_n||u| + \varepsilon_1|u|^2 + C_{\varepsilon_1}|u_n|^{p-1}|u| + C_{\varepsilon_1}|u|^p$$

for some ε1, Cε1 > 0, where p > 2. Therefore, together with (3), using the Young inequality with ε (for all ε > 0), we obtain

$$\begin{aligned}
&\big|F(u_n)-F(w_n)-F(u)\big| \\
&\leqslant C_{\varepsilon_1,C_{\varepsilon_1}}\big[\varepsilon|u_n|^2 + C_{\varepsilon,\varepsilon_1}|u|^2 + \varepsilon|u_n|^p + C_{\varepsilon,C_{\varepsilon_1},c_\varepsilon}|u|^p\big].
\end{aligned}$$

Consequently, we consider the function fn defined as

$$\tilde{f}_n(x) := \max\big\{\big|F(u_n)-F(w_n)-F(u)\big| - C_{\varepsilon_1,C_{\varepsilon_1}}\varepsilon\big(|u_n|^2+|u_n|^p\big),\, 0\big\}.$$

Then

$$0 \leqslant \widetilde{f}_n(x) \leqslant C_{\varepsilon_1,C_{\varepsilon_1}} C_{\varepsilon,\varepsilon_1}|u|^2 + C_{\varepsilon_1,\varepsilon_1} C_{\varepsilon,C_{\varepsilon_1},c_\varepsilon}|u|^p \in L^1(\mathbb{R}^2),$$

and by the Lebesgue dominated convergence theorem we have

$$\int\limits_{\mathbb{R}^2} \widetilde{f}_n(x)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Note that

$$\left|F(u_n) - F(w_n) - F(u)\right| \leqslant \widetilde{f}_n(x) + C_{\varepsilon_1,C_{\varepsilon_1}}\varepsilon\left(|u_n|^2 + |u_n|^p\right).$$

Using (12) shows that (8) holds.

Compare (4), (5) with (6), (7). We only need to prove that #I′ (u), φ# = 0 for all φ ∈ E. Note∫ Lemma 1(iii), (10), (11), and (un − u, φ) → 0 as n → ∞. It suffices to check that ∫R2 f(wn) # dx = 0(1) as n → ∞. Note the arbitrariness of # in (2), and wn → 0 in Lp(R2), p > 2. Therefore, from (2) we have

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^2} f(w_n)\varphi\,\mathrm{d}x\right| &\leqslant \int\limits_{\mathbb{R}^2} \left(\varepsilon|w_n| + c_\varepsilon|w_n|^{p-1}\right)|\varphi|\,\mathrm{d}x \\ &\leqslant \varepsilon\gamma_2^2\|w_n\|\|\varphi\| + c_\varepsilon\gamma_p\|w_n\|_p^{p-1}\|\varphi\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This completes the proof.

Lemma 4. Suppose that all the assumptions in Theorem 1 hold. Then I satisfies the Cerami condition (C).

Proof. For all c ∈ R, suppose that there exists {un}n∈N # E is bounded and

I(un) → c, I′ (un) → 0 as n → ∞.

Using #I′ (u), φ# = 0 for all φ ∈ E in Lemma 3 and noting Lemma 1(iv), we have

$$\begin{aligned}\mathcal{I}(u) &= \mathcal{I}(u) - \frac{1}{6}\langle \mathcal{I}'(u), u\rangle \\ &= \frac{1}{3}\|u\|^2 + \int\limits_{\mathbb{R}^2} \mathrm{F}(u)\,\mathrm{d}x + \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int\limits_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x.\end{aligned}$$

This implies that

$$\frac{1}{3}\|u\|^2 + \int_{\mathbb{R}^2} \mathbf{F}(u)\,\mathrm{d}x = \mathcal{I}(u) - \frac{1}{6}\langle \mathcal{I}'(u), u\rangle - \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x$$
$$\leqslant \mathcal{I}(u) - \frac{1}{6}\langle \mathcal{I}'(u), u\rangle.$$

Recall wn = un − u. From (6) and (7) we have

$$\frac{1}{3}\|u\|^2 + \int_{\mathbb{R}^2} \mathbf{F}(u)\,\mathrm{d}x = \mathcal{I}(u) - \frac{1}{6}\langle \mathcal{I}'(u), u\rangle - \mu\left(\frac{1}{q} - \frac{1}{6}\right)\int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x$$
$$\leqslant \mathcal{I}(u) - \frac{1}{6}\langle \mathcal{I}'(u), u\rangle.$$

As V (x) < b on a set of finite measure and wn → 0 in E, we have

$$\|w_n\|_2^2 = \int_{\mathbb{R}^2} |w_n|^2\,\mathrm{d}x \leqslant \frac{1}{\lambda b}\int_{V\geqslant b} \lambda V(x)|w_n|^2\,\mathrm{d}x + \int_{V<b} |w_n|^2\,\mathrm{d}x$$
$$\leqslant \frac{1}{\lambda b}\|w_n\|^2 + o(1).$$

Combining this and the Hölder inequality, recall p = 2τ/(τ − 1) ∈ (2, +∞), fixed ν ∈ (p, +∞), we have

$$\|w_n\|_p^p = \int_{\mathbb{R}^2} |w_n|^p\,\mathrm{d}x = \int_{\mathbb{R}^2} |w_n|^{\frac{2(\nu-p)}{\nu-2}}|w_n|^{p-\frac{2(\nu-p)}{\nu-2}}\,\mathrm{d}x$$
$$\leqslant \left(\int_{\mathbb{R}^2} |w_n|^{\frac{2(\nu-p)}{\nu-2}\frac{\nu-2}{\nu-p}}\,\mathrm{d}x\right)^{\frac{\nu-p}{\nu-2}} \left(\int_{\mathbb{R}^2} |w_n|^{(p-\frac{2(\nu-p)}{\nu-2})\frac{\nu-2}{p-2}}\,\mathrm{d}x\right)^{\frac{p-2}{\nu-2}}$$
$$= \left(\int_{\mathbb{R}^2} |w_n|^2\,\mathrm{d}x\right)^{\frac{\nu-p}{\nu-2}} \left(\int_{\mathbb{R}^2} |w_n|^\nu\,\mathrm{d}x\right)^{\frac{p-2}{\nu-2}}$$
$$\leqslant \left(\frac{1}{\lambda b}\right)^{\frac{\nu-p}{\nu-2}} \gamma_\nu^{\frac{\nu(p-2)}{\nu-2}} \|w_n\|^{\frac{2(\nu-p)}{\nu-2}} \|w_n\|^{\frac{\nu(p-2)}{\nu-2}}$$
$$= \left(\frac{1}{\lambda b}\right)^{\frac{\nu-p}{\nu-2}} \gamma_\nu^{\frac{\nu(p-2)}{\nu-2}} \|w_n\|^p \quad \text{for } \gamma_\nu > 0.$$

From (H1), for all ε > 0, there exists δ = δ(ε) > 0 such that ∣ f (u) ∣ ≤ ε ∣ u ∣ for x # R2 and u ≤ δ. Without loss of generality, we can choose this δ > R1, where R1 is defined in (H4). Therefore, we have

$$\int_{|w_n|\leqslant R_1} f(w_n)w_n\,\mathrm{d}x \leqslant \varepsilon \int_{|w_n|\leqslant R_1} |w_n|^2\,\mathrm{d}x \leqslant \frac{\varepsilon}{\lambda b}\|w_n\|^2 + o(1).$$

On the other hand, when |wn| ≥ R1, from (H4) we have

$$\begin{aligned}\int\limits_{|w_n|\geqslant R_1} f(w_n)w_n\,\mathrm{d}x &= \int\limits_{|w_n|\geqslant R_1} \frac{f(w_n)}{w_n}w_n^2\,\mathrm{d}x\\ &\leqslant \biggl(\int\limits_{|w_n|\geqslant R_1} \biggl|\frac{f(w_n)}{w_n}\biggr|^\tau \mathrm{d}x\biggr)^{1/\tau}\biggl(\int\limits_{|w_n|\geqslant R_1} |w_n|^{\frac{2\tau}{\tau-1}}\,\mathrm{d}x\biggr)^{(\tau-1)/\tau}\\ &\leqslant \biggl(\int\limits_{|w_n|\geqslant R_1} \alpha_0\mathbf{F}(w_n)\,\mathrm{d}x\biggr)^{1/\tau}\|w_n\|_p^2\\ &\leqslant (\alpha_0\widetilde{M})^{1/\tau}\biggl(\frac{1}{\lambda b}\biggr)^{\frac{2(\nu-p)}{p(\nu-2)}}\gamma_\nu^{\frac{2\nu(p-2)}{p(\nu-2)}}\|w_n\|^2+o(1).\end{aligned}$$

Consequently, from (7) we obtain

$$\begin{aligned}o(1) &= \langle J'(w_n), w_n\rangle\\ &= \|w_n\|^2 + \langle B'(w_n), w_n\rangle - \int\limits_{\mathbb{R}^2} f(w_n)w_n\,\mathrm{d}x + \mu\int\limits_{\mathbb{R}^2} g(x)|w_n|^q\,\mathrm{d}x\\ &\geqslant \biggl[1-\frac{\varepsilon}{\lambda b} - (\alpha_0\widetilde{M})^{1/\tau}\biggl(\frac{1}{\lambda b}\biggr)^{\frac{2(\nu-p)}{p(\nu-2)}}\gamma_\nu^{\frac{2\nu(p-2)}{p(\nu-2)}}\biggr]\|w_n\|^2+o(1).\end{aligned}$$

Thus, given the arbitrariness of ε, there exists Λ > 0 such that wn 0 in E when λ > Λ.

This implies that un → u in E, and Definition 1(i) holds.

Finally, we prove that Definition 1(ii) holds. We argue indirectly, i.e., suppose that there exist c ∈ R and {un} n∈N # E such that

$$\mathcal{I}(u_n)\to c,\quad \|u_n\|\to\infty,\quad \bigl\|\mathcal{I}'(u_n)\bigr\|\|u_n\|\to 0\quad \text{as } n\to\infty. \tag{13}$$

Then we have

$$\begin{aligned}c+o(1) &= \mathcal{I}(u_n) - \frac{1}{6}\langle \mathcal{I}'(u_n), u_n\rangle\\ &= \frac{1}{3}\|u_n\|^2 + \int\limits_{\mathbb{R}^2}\mathbf{F}(u_n)\,\mathrm{d}x + \mu\biggl(\frac{1}{q}-\frac{1}{6}\biggr)\int\limits_{\mathbb{R}^2} g(x)|u_n|^q\,\mathrm{d}x\\ &\geqslant \int\limits_{\mathbb{R}^2}\mathbf{F}(u_n)\,\mathrm{d}x.\end{aligned} \tag{14}$$

Using Lemma 1(iv), (13), and (g), we obtain

$$\begin{aligned}1 &= \frac{\|u_n\|^2}{\|u_n\|^2}\\ &= \frac{\langle \mathcal{I}'(u_n), u_n\rangle}{\|u_n\|^2} - \frac{\langle B'(u_n), u_n\rangle}{\|u_n\|^2} + \frac{\int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x}{\|u_n\|^2} - \frac{\mu\int_{\mathbb{R}^2} g(x)|u_n|^q \,\mathrm{d}x}{\|u_n\|^2}\\ &\leqslant \limsup_{n\to\infty}\left[\frac{\langle \mathcal{I}'(u_n), u_n\rangle}{\|u_n\|^2} + \frac{\int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x}{\|u_n\|^2} + \frac{\mu\|g\|_{q'}\gamma^q_{\frac{qq'}{q'-1}}\|u_n\|^q}{\|u_n\|^2}\right]\\ &\leqslant \limsup_{n\to\infty}\frac{\int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x}{\|u_n\|^2}.\end{aligned} \tag{15}$$

Let vn = un/ #un# . Then #vn# = 1, and there exists a function v E such that vn → v weakly in E, vn → v strongly in Lr(R2) with r # (2, + ∞), vn(x) → v(x) for a.e. x # R2. Define a set Ωn(a, b) = {x # R2: a ≤ un(x) < b} with 0 ≤ a < b, and consider the following two possible cases.

Case 1. The function v is a zero function in E, i.e., v = 0, and vn 0 weakly in E, vn(x) → 0 for a.e. x ∈ R2. From (2) we have

$$\begin{aligned}\int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x &= \int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{|u_n|^2}|v_n|^2\mathrm{d}x \leqslant (\varepsilon + c_\varepsilon R_1^{p-2}) \int_{\Omega_n(0,R_1)} |v_n|^2\,\mathrm{d}x\\ &\leqslant (\varepsilon + c_\varepsilon R_1^{p-2}) \int_{\mathbb{R}^2} |v_n|^2\,\mathrm{d}x \to 0.\end{aligned} \tag{16}$$

On the other hand, by the Hölder inequality, (14), and (H4) we obtain

$$\begin{aligned}\int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x &= \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^2}|v_n|^2\,\mathrm{d}x\\ &\leqslant \left(\int_{\Omega_n(R_1,\infty)} \left(\frac{f(u_n)u_n}{|u_n|^2}\right)^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(R_1,\infty)} |v_n|^{\frac{2\tau}{\tau-1}}\,\mathrm{d}x\right)^{\frac{\tau-1}{\tau}}\\ &\leqslant \left(\int_{\Omega_n(R_1,\infty)} \left|\frac{f(u_n)}{u_n}\right|^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(R_1,\infty)} |v_n|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\\ &\leqslant \left(\int_{\Omega_n(R_1,\infty)} \alpha_0 \mathbf{F}(u_n)\,\mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(R_1,\infty)} |v_n|^p\,\mathrm{d}x\right)^{\frac{2}{p}}\\ &\leqslant [\alpha_0(c+1)]^{\frac{1}{\tau}}\|v_n\|_p^2 \to 0.\end{aligned} \tag{17}$$

Combining (16) and (17), we have

$$\int_{\mathbb{R}^2} \frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x = \int_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x + \int_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{\|u_n\|^2}\,\mathrm{d}x \to 0,$$

which contradicts (15).

Case 2. The function v is not a zero function in E, i.e., v(x) ≡ 0, x ∈ R2. Hence, if we set A = {x ∈ R2: v(x) =/ 0}, then meas A > 0. For x ∈ A, we have limn→∞ un(x) = ∞, and hence A C Ωn(R1, ∞) for large n. By (H3) and Lemma 1(iv), (v), noting the nonnegativity of f (u)u, Fatou's Lemma enables us to obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\langle \mathcal{I}'(u_n), u_n\rangle}{\|u_n\|^6} \\
&= \lim_{n\to\infty} \left[\frac{\|u_n\|^2}{\|u_n\|^6} + \frac{\langle B'(u_n), u_n\rangle}{\|u_n\|^6} - \frac{\int_{\mathbb{R}^2} f(u_n)u_n \,\mathrm{d}x}{\|u_n\|^6} + \frac{\mu \int_{\mathbb{R}^2} g(x)|u_n|^q \,\mathrm{d}x}{\|u_n\|^6} \right] \\
&\leqslant \lim_{n\to\infty} \left[\frac{\|u_n\|^q}{\|u_n\|^6} \mu \|g\|_{q'} \gamma^q_{\frac{qq'}{q'-1}} + 6C_1 \frac{\|u_n\|^6}{\|u_n\|^6} \right. \\
&\quad \left. - \int\limits_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^6} \,\mathrm{d}x - \int\limits_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} |v_n|^6 \,\mathrm{d}x \right] \\
&\leqslant 6C_1 + \limsup_{n\to\infty} \int\limits_{\Omega_n(0,R_1)} \frac{f(u_n)u_n}{\|u_n\|^6} \,\mathrm{d}x - \liminf_{n\to\infty} \int\limits_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} |v_n|^6 \,\mathrm{d}x \\
&\leqslant 6C_1 + \limsup_{n\to\infty} \frac{\varepsilon R_1^2 + c_\varepsilon R_1^p}{\|u_n\|^6} \operatorname{meas}\bigl(\Omega_n(0,R_1)\bigr) \\
&\quad - \liminf_{n\to\infty} \int\limits_{\Omega_n(R_1,\infty)} \frac{f(u_n)u_n}{|u_n|^6} \bigl[\chi_{\Omega_n(R_1,\infty)}(x)\bigr] |v_n|^6 \,\mathrm{d}x \\
&\leqslant 6C_1 - \int\limits_{\Omega_n(R_1,\infty)} \liminf_{n\to\infty} \frac{f(u_n)u_n}{|u_n|^6} \bigl[\chi_{\Omega_n(R_1,\infty)}(x)\bigr] |v_n|^6 \,\mathrm{d}x \\
&\to -\infty.
\end{aligned}$$

This is also a contradiction.

Combining the above two cases, we have that Definition 1(ii) holds. Thus, I satisfies the Cerami condition (C). This completes the proof.

Proof of Theorem 1. Note that E is a Hilbert space, and let ej be an orthonomormal basis of E. Then we have

$$Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \quad k \in \mathbb{N},\ X_j = \mathbb{R}_{e_j}.$$

In what follows, we show that for each k ∈ N, there exist ρk > rk > 0 such that

$$b_k = \inf_{u\in Z_k,\ \|u\|=r_k} \mathcal{I}(u) \to +\infty \quad \text{as } k\to\infty \tag{18}$$

and

$$a_k = \max_{u \in Y_k,\ \|u\| = \rho_k} \mathcal{I}(u) \leqslant 0. \tag{19}$$

Note that the compact embedding E <-> Lr(R2) with r ∈ (2, +∞), and by Lemma 3.8 in [24] we have βk(r) = supu Zk, #u# =1 #u# r → 0, k→∞ . This, together with (3), implies that

$$\begin{aligned}\mathcal{I}(u) &= \frac{1}{2}\|u\|^2 + B(u) - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x + \frac{\mu}{q}\int_{\mathbb{R}^2} g(x)|u|^q\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x \geqslant \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|_2^2 - \frac{c_\varepsilon}{p}\|u\|_p^p \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\gamma_2^2\|u\|^2 - \frac{c_\varepsilon}{p}\|u\|_p^p.\end{aligned} \tag{20}$$

Note that p = 2τ/(τ − 1), and if we take ε ≤ 1/(γ22(2τ − 1)) and rk = (cεβkp)1/(2−p), by (20), for u ∈ Zk and #u# = rk, we find

$$\begin{aligned}\mathcal{I}(u) &\geqslant \frac{\tau-1}{2\tau-1}\|u\|^2 - \frac{c_\varepsilon}{p}\beta_k^p\|u\|^p \geqslant \left(\frac{\tau-1}{2\tau-1} - \frac{\tau-1}{2\tau}\right)\left(c_\varepsilon\beta_k^p\right)^{\frac{2}{2-p}} \\ &\to +\infty \quad \text{as } k \to +\infty\end{aligned}$$

with τ > 1, p > 2. Therefore, (18) holds.

On the other hand, for any finite dimensional subspace E # E, we show that

$$\mathcal{I}(u) \to -\infty, \quad \|u\| \to \infty, \quad u \in \widetilde{E}. \tag{21}$$

Arguing indirectly, assume that for some sequence {un} # E with #un# → ∞, there exists M > 0 such that I(un) ≥ −M for all n ∈ N. Let vn = un/ #un#. Then #vn# = 1, and there is a function v ∈ E such that vn v in E. Since dim E < ∞, we have vn → v in E˜, vn(x) → v(x) for a.e. x ∈ R2, and #v# = 1. Let Ω = {x ∈ R2: v(x) /= 0}. Then meas Ω > 0, and limn→∞ | un(x)| → ∞ for a.e. x ∈ Ω. From Lemma 1(v) and (g) we have

$$\begin{aligned}\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x &= \lim_{n\to\infty}\frac{\frac{1}{2}\|u_n\|^2+B(u_n)-\mathcal{I}(u_n)+\frac{\mu}{q}\int_{\mathbb{R}^2}g(x)|u_n|^q\,\mathrm{d}x}{\|u_n\|^6}\\ &\leqslant \lim_{n\to\infty}\frac{\frac{1}{2}\|u_n\|^2+C_1\|u_n\|^6-\mathcal{I}(u_n)+\frac{\mu}{q}\|g\|_{q'}\gamma^q_{\frac{qq'}{q'-1}}\|u_n\|^q}{\|u_n\|^6}\\ &= C_1. \end{aligned}\tag{22}$$

From the L'Hôspital's rule and (H3) we have

$$\lim_{|u|\to\infty}\frac{F(u)}{|u|^6}=+\infty \quad \text{uniformly in } x\in\mathbb{R}^2.$$

Fatou's lemma implies that

$$\begin{aligned}\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}\frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x &\geqslant \lim_{n\to\infty}\int\limits_{\Omega}\frac{F(u_n)}{\|u_n\|^6}\,\mathrm{d}x \geqslant \liminf_{n\to\infty}\int\limits_{\Omega}\frac{F(u_n)}{|u_n|^6}|v_n|^6\,\mathrm{d}x\\ &\geqslant \int\limits_{\Omega}\liminf_{n\to\infty}\frac{F(u_n)}{|u_n|^6}|v_n|^6\,\mathrm{d}x \geqslant \int\limits_{\Omega}\liminf_{n\to\infty}\frac{F(u_n)}{|u_n|^6}\big[\chi_\Omega(x)\big]|v_n|^6\,\mathrm{d}x\\ &= +\infty.\end{aligned}$$

This contradicts (22), and thus (21) holds. As a result, we can take u Yk and large ρk (ρk > rk) such that

$$J(u)\leqslant 0 \quad \text{for } u\in Y_k,\ \|u\|=\rho_k.$$

Thus, (19) holds.

Finally, (H5) implies that I is an even functional on E, and by Lemma 4 I satisfies all the conditions of Lemma 2. Then I has a sequence of critical points un such that (un) →+ ∞ as n →∞. This means that (1) has infinitely many high energy solutions.

This completes the proof.

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References

1 J. Byeon, H. Huh, J. Seok, Standing waves of nonlinear Schrödinger equations with the gauge field, *J. Funct. Anal.*, **263**(6):1575–1608, 2012, https://doi.org/10.1016/j.jfa.2012.05.024

2 S. Chen, B. Zhang, X. Tang, Existence and concentration of semiclassical ground state solutions for the generalized Chern–Simons–Schrödinger

system in ..(R.), *Nonlinear Anal., Theory Methods Appl.*, **185**:68–96, 2019, https://doi.org/10.1016/j.na.2019.02.028

3 S.-S. Chow, Finite element error estimates for nonlinear elliptic equations of monotone type, *Numer. Math.*, **54**(4):373–393, 1989, https://doi.org/10.1007/BF01396320

4 P. d'Avenia, A. Pomponio, Standing waves for a Schrödinger–Chern–Simons–Higgs system, *J. Differ. Equations*, **268**(5):2151–2162, 2020, https://doi.org/10.1016/j.jde.2019.09.017

5 P. d'Avenia, A. Pomponio, T. Watanabe, Standing waves of modified Schrödinger equations coupled with the Chern-Simons gauge theory, *Proc. R. Soc. Edinb., Sect. A, Math.*, **150**(4):1915–1936, 2020, https://doi.org/10.1017/prm.2019.9

6 Y. Deng, S. Peng, W. Shuai, Nodal standing waves for a gauged nonlinear Schrödinger equation in R., *J. Differ. Equations*, **264**(6):4006–4035, 2018, https://doi.org/10.1016/j.jde.2017.12.003

7 H. Huh, J. Seok, Equivalence and nonexistence of standing waves for coupled Schrödinger equations with Chern–Simons gauge fields, *Topol. Methods Nonlinear Anal.*, **46**(1):495–504, 2015, https://doi.org/10.12775/TMNA.2015.056

8 R. Jackiw, S.-Y. Pi, Classical and quantal nonrelativistic Chern–Simons theory, *Phys. Rev. D (3)*, **42**(10):3500–3513, 1990, https://doi.org/10.1103/PhysRevD.42.3500

9 R. Jackiw, S.-Y. Pi, Soliton solutions to the gauged nonlinear Schrödinger equation on the plane, *Phys. Rev. Lett.*, **64**(25):2969–2972, 1990, https://doi.org/10.1103/PhysRevLett.64.2969

10 C. Ji, F. Fang, Standing waves for the Chern–Simons–Schrödinger equation with critical exponential growth, *J. Math. Anal. Appl.*, **450**(1):578–591, 2017, https://doi.org/10.1016/j.jmaa.2017.01.065

11 G. Li, X. Luo, W. Shuai, Sign-changing solutions to a gauged nonlinear Schrödinger equation, *J. Math. Anal. Appl.*, **455**(2):1559–1578, 2017, https://doi.org/10.1016/j.jmaa.2017.06.048

12 W. Liang, C. Zhai, Solutions to a gauged Schrödinger equation with concave-convex nonlinearities without (AR) condition, *Appl. Anal.*, 2019, https://doi.org/10.1080/00036811.2019.1639046

13 Z. Liu, Z. Ouyang, J. Zhang, Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in R., *Nonlinearity*, **32**(8):3082–3111, 2019, https://doi.org/10.1088/1361-6544/ab1bc4

14 X. Luo, Existence and stability of standing waves for a planar gauged nonlinear Schrödinger equation, *Comput. Math. Appl.*, **76**(11-12):2701–2709, 2018, https://doi.org/10.1016/j.camwa.2018.09.003

15 P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional .-Laplacian in R , *Calc. Var. Partial Differ. Equ.,* **54**(3):2785–2806, 2015, https://doi.org/10.1007/s00526-015-0883-5

16 A. Qian, C. Li, Infinitely many solutions for a Robin boundary value problem, *Int. J. Differ. Equ.,* p. 548702, 2010, https://doi.org/10.1155/2010/548702

17 S. Secchi, Concave-convex nonlinearities for some nonlinear fractional equations involving the Bessel operator, *Complex Var. Elliptic Equ.*, **62**(5):654–669, 2017, https://doi.org/10.1080/17476933.2016.1234465

18 J. Seok, Infinitely many standing waves for the nonlinear Chern–Simons–Schrödinger equations, *Adv. Math. Phys.*, p. 519374, 2015, https://doi.org/10.1155/2015/519374

19 X. Tang, J. Zhang, W. Zhang, Existence and concentration of solutions for the Chern–Simons– Schrödinger system with general nonlinearity, *Results Math.*, **71**(3-4):643–655, 2017, https://doi.org/10.1007/s00025-016-0553-8

20 Y. Wan, J. Tan, Standing waves for the Chern–Simons–Schrödinger systems without (AR) condition, *J. Math. Anal. Appl.*, **415**(1):422–434, 2014, https://doi.org/10.1016/j.jmaa.2014.01.084

21 Y. Wan, J. Tan, The existence of nontrivial solutions to Chern–Simons–Schrödinger systems, *Discrete Contin. Dyn. Syst., Ser. A*, **37**(5):2765–2786, 2017, https://doi.org/10.3934/dcds.2017119

22 L. Wang, S. Chen, Two solutions for nonhomogeneous Klein–Gordon–Maxwell system with sign-changing potential, *Electron. J. Differ. Equ.,* p. 124, 2018.

23 L. Wang, S. Ma, N. Xu, Multiple solutions for nonhomogeneous Schrödinger–Poisson equations with sign-changing potential, *Acta Math. Sci., Ser. B, Engl. Ed.*, **37**(2):555–572, 2017, https://doi.org/10.1016/S0252-9602(17)30021-8

24 M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhäuser, Basel, 1996, https://doi.org/10.1007/978-1-4612-4146-1

25 A. Xia, Existence, nonexistence and multiplicity results of a Chern–Simons–Schrödinger system, *Acta Appl. Math.*, **166**:147–159, 2020, https://doi.org/10.1007/s10440-019-00260-6

26 W. Xie, C. Chen, Sign-changing solutions for the nonlinear Chern–Simons–Schrödinger equations, *Appl. Anal.*, **99**(5):880–898, 2020, https://doi.org/10.1080/00036811.2018.1514020

27 J. Xu, Z. Wei, D. O'Regan, Y. Cui, Infinitely many solutions for fractional Schrödinger– Maxwell equations, *J. Appl. Anal. Comput.,* .(3):1165–1182, 2019, https://doi.org/10.11948/2156-907X.20190022

28 J. Zhang, X. Tang, F. Zhao, On multiplicity and concentration of solutions for a gauged nonlinear Schrödinger equation, *Appl. Anal.*, **99**(12):2001–2012, 2020, https://doi.org/10.1080/00036811.2018.1553033

29 J. Zhang, W. Zhang, X. Xie, Infinitely many solutions for a gauged nonlinear Schrödinger equation, *Appl. Math. Lett.,* **88**:21–27, 2019, https://doi.org/10.1016/j.aml.2018.08.007

30 J. Zhang, W. Zou, Solutions concentrating around the saddle points of the potential for critical Schrödinger equations, *Calc. Var. Partial Differ. Equ.,* **54**(4):4119–4142, 2015, https:// doi.org/10.1007/s00526-015-0933-z

Notes

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